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MAPPINGS WHICH PRESERVE REGULAR ICOSAHEDRONS

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Abstract: In this paper, we prove that if a one-to-one mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ preserves regular icosahedrons, then f is a linear isometry up to translation.

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1. Introduction

Let us begin with the properties of an isometry. For normed spaces X and Y, a mapping $f : X \to Y$ is called an isometry if f satisfies the equality

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $r > 0$ is said to be preserved by a mapping $f : X \to Y$ if $\|f(x) - f(y)\| = r$ for all $x, y \in X$ whenever $\|x - y\| = r$.

If f is an isometry, then every distance $r > 0$ is preserved by f, and conversely. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, A.D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $r > 0$ is an isometry, which is now known to us as the Aleksandrov problem.

F.S. Beckman and D.A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = \mathbb{R}^n$ (see also [3, 4, 5, 6, 11, 12, 13, 14, 15, 16, 17]).

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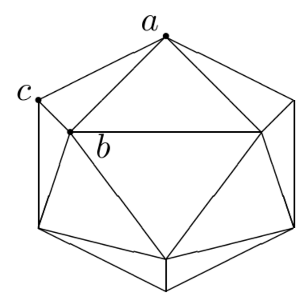


icosahedron A

Figure 1:

Theorem (see Beckman and Quarles [2]) *If a mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ $(2 \leq n < \infty)$ *preserves a distance* $r > 0$*, then* f *is a linear isometry up to translation.*

It is an interesting question whether the 'distance $r > 0$' in the above theorem can be replaced by some properties characterized by 'geometrical figures' without loss of its validity.

In [8, 9, 10], the authors proved that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ maps every unit circle (or unit sphere, tetrahedron) onto a unit circle (or a unit sphere, tetrahedron, respectively), then f is a linear isometry up to translation.

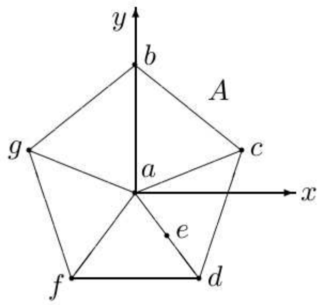
In this connection, we will extend these results to the more general 3 dimensional objects, i.e., we prove in this paper that if a one-to-one mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ maps every regular icosahedron onto a regular icosahedron, then f is a linear isometry up to translation.

2. Main Theorem

In the following by a regular icosahedron we mean a regular icosahedron with its side length one. We first make our terms precise as follows. In Figure 1, we will call the point a a 'vertex' and the line $\overline{ab}$ an 'edge' and the plane bounded by the three edges $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ 'face abc' or simply a 'face'. Further by an icosahedron we will mean the 20 faces only and not the three dimensional open set bounded by those 20 faces. Let us denote the three dimensional open set bounded by the icosahedron A as 'Inside of A' or simply as Ins$A$.

Now we begin with the following lemma.

Lemma. *Suppose that a one-to-one mapping* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *maps every*



$$a = (0, 0, 0)$$
$$b = (0, \tfrac{1}{2\cos 54^\circ}, -z_o)$$
$$c = (\tfrac{1+2\cos 72^\circ}{2}, \sin 72^\circ - \tfrac{1}{2}\tan 54^\circ, -z_o)$$
$$d = (0.5, -\tfrac{1}{2}\tan 54^\circ, -z_o)$$
$$f = (-0.5, -\tfrac{1}{2}\tan 54^\circ, -z_o)$$
$$g = (-\tfrac{1+2\cos 72^\circ}{2}, \sin 72^\circ - \tfrac{1}{2}\tan 54^\circ, -z_o)$$

Figure 2:

icosahedron onto an icosahedron. Then for any icosahedrons A *and* B, *if* $\mathrm{Ins}A \cap \mathrm{Ins}B = \phi$, $\mathrm{Ins}f(A) \cap \mathrm{Ins}f(B) = \phi$.

Proof. First, we show that if $q \notin \mathrm{Ins}A$, then $f(q) \notin \mathrm{Ins}f(A)$. In other words we show that if $f(q) \in \mathrm{Ins}f(A)$, then $q \in \mathrm{Ins}A$. Suppose that $q \in A$. Then $f(q) \in f(A)$ and so $f(q) \notin \mathrm{Ins}f(A)$. Suppose that $q \notin \mathrm{Ins}A$ and $q \notin A$. Then choose another icosahedron B such that $q \in B$ and $B \cap A = \phi$. Then $f(B) \cap f(A) = \phi$ and therefore $f(q) \notin \mathrm{Ins}f(A)$.

Suppose now that $\mathrm{Ins}f(A) \cap \mathrm{Ins}f(B) \neq \phi$. Then $\mathrm{Ins}f(A) \cap f(B) \neq \phi$, which means that for some $b \in B$, $f(b) \in \mathrm{Ins}f(A)$. Therefore $b \in \mathrm{Ins}A$ and $(\mathrm{Ins}A) \cap B \neq \phi$ by which we conclude that $\mathrm{Ins}A \cap \mathrm{Ins}B \neq \phi$. □

We show now that if any one-to-one mapping preserves regular icosahedrons, then it is actually an isometry. More precisely,

Theorem. *If a one-to-one mapping* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *maps every regular icosahedron onto a regular icosahedron, then* f *is a linear isometry up to translation.*

Proof. We show f preserves the distance 1. Then the theorem of Beckman and Quarles implies that f is an isometry.

We will use solid angle arguments. Let A be a regular icosahedron. For any $p \in A$ let us denote the solid angle that $\mathrm{Ins}A$ subtends with respect to $p \in A$ as $\Omega(A, p)$. We first find the solid angles at a vertex and at an edge point. To do so let us choose suitable coordinate axes (as in Figure 2) so that the vertex a is at the origin. Here $z_o = \sqrt{1 - (\frac{1}{2\cos 54^\circ})^2}$ and one can easily check that the distance between any two vertices is 1. Let $e = \frac{1}{2}d$ which belongs to an edge of A. We find $\Omega(A, e)$ first.

Numerically these points are $b = (0, 0.851, -0.525)$, $c = (0.809, 0.263, -0.525)$, $d = (0.5, -0.688, -0.525)$, $f = (-0.5, -0.688, -0.525)$, $g = (-0.809, 0.263, -0.525)$.

The two faces adc and adf meet at the edge $\overline{ad}$ and the lines $\overline{ec}$ and $\overline{ef}$ are both perpendicular to $\overline{ad}$. Therefore to find the (ordinary) angle between the two planes adc and adf, we have only to find the angle between the two vectors $\vec{ec}$ and $\vec{ef}$. Since $\vec{ec} = \vec{ac} - \vec{ae} = \vec{ac} - \frac{1}{2}\vec{ad}$ and $\vec{ef} = \vec{af} - \vec{ae} = \vec{af} - \frac{1}{2}\vec{ad}$, the angle θ between these two vectors is given by

$$\cos\theta = \frac{\vec{ec}\cdot\vec{ef}}{|\vec{ec}||\vec{ef}|} = -0.7452$$

and we get $\theta = 138.18^\circ$ or 2.412 rad. This angle is often called the dihedral angle of the icosahedron. Now we are ready to find the solid angle $\Omega(A,e)$. We use the unit of the solid angle such that the solid angle of the whole sphere with respect to its center is 4π. Then the solid angle that Ins$A$ subtends with respect to $e \in A$ is

$$\Omega(A,e) = 4\pi\frac{\theta}{2\pi} = 2\theta = 4.824 \equiv \Omega_e\,.$$

Having found the solid angle Ω_e at the edge point of the regular icosahedron, let us now find the solid angle that Ins$A$ subtends at the vertex a. For that we use the result of [7], where it is shown that given any three vectors $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, all starting from the origin, the solid angle Ω that these three vectors subtend with respect to the origin (i.e., the solid angle that the (not necessarily regular) tetrahedron whose vertices are the end tips of these three vectors and the origin subtends with respect to the origin) is

$$\tan(\frac{1}{2}\Omega) = \frac{[\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3]}{T_1T_2T_3 + (\mathbf{T}_1\cdot\mathbf{T}_2)T_3 + (\mathbf{T}_2\cdot\mathbf{T}_3)T_1 + (\mathbf{T}_3\cdot\mathbf{T}_1)T_2}\,. \tag{1}$$

Here $[\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3] \equiv |\mathbf{T}_1\cdot\mathbf{T}_2\times\mathbf{T}_3|$ and $T_i = |\mathbf{T}_i|$ $(i = 1,2,3)$. Note that this expression does not depend on the lengths of the three vectors. For example when $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ are mutually orthogonal we get $\Omega = \frac{\pi}{2}$.

Now we are ready to find the solid angle that Ins$A$ subtends at the vertex a, $\Omega(A,a)$. We first find $\Omega' = \frac{1}{5}\Omega(A,a)$. To find Ω', let $\mathbf{T}_1 = \vec{ad} = (0.5, -\frac{1}{2}\tan 54^\circ, -z_o)$, $\mathbf{T}_2 = \vec{af} = (-0.5, -\frac{1}{2}\tan 54^\circ, -z_o)$ and $\mathbf{T}_3 = (0,0,-1)$. Then using (1) we can compute the solid angle we want. We find $[\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3] = 0.688$, $\mathbf{T}_1\cdot\mathbf{T}_2 = 0.499$, $\mathbf{T}_2\cdot\mathbf{T}_3 = \mathbf{T}_3\cdot\mathbf{T}_1 = 0.525$ and $T_1 = T_2 = T_3 = 1$. Therefore we get

$$\Omega' = 2\tan^{-1}(\frac{0.688}{1 + 0.525 + 0.499 + 0.525}) = 2\tan^{-1}0.2699$$

and

$$\Omega(A,a) = 5\Omega' = 10\tan^{-1} 0.2699 = 2.636 \equiv \Omega_v\,.$$

Let us summarize our results above. Suppose that $p \in A$, where p is a point and A is a regular icosahedron. If p is a vertex, say $p = a$, then the solid angle that Ins$A$ subtends with respect to p is $\Omega(A,p) = \Omega_v = 2.636$. If p is a point which belongs to an edge and is not a vertex, then $\Omega(A,p) = \Omega_e = 4.824$. If $p \in A$ is neither a vertex nor an edge point, then $\Omega(A,p) = 2\pi$.

Suppose now that a is a vertex of a regular icosahedron $A = A_1$. We show then that $f(a)$ is a vertex of $f(A)$ too. Or we have only to show that $\Omega(f(A), f(a)) = \Omega_v$. Construct 3 more regular icosahedrons A_i $(i = 2,3,4)$ such that a is the common vertex of all four regular icosahedrons A_i $(i = 1, \cdots, 4)$ and $\mathrm{Ins}A_i \cap \mathrm{Ins}A_j = \phi$ for any $i \neq j$ (see Appendix). Then $f(a)$ belongs to $f(A_i)$ for $i = 1, \cdots, 4$ and by the above lemma $\mathrm{Ins}f(A_i) \cap \mathrm{Ins}f(A_j) = \phi$ for $i \neq j$. Now the solid angle that $\mathrm{Ins}f(A_i)$ subtends with respect to $f(a)$ can be Ω_v, Ω_e, or 2π. Further

$$\sum_{i=1}^{4} \Omega(f(A_i), f(a)) \leq 4\pi.$$

Since $3\Omega_v + \Omega_e = 3 \times (2.636) + 4.824 = 12.732 > 4\pi = 12.566$, we conclude that $\Omega(f(A_i), f(a)) = \Omega_v$, $(i = 1, \cdots, 4)$ and that $f(a)$ is a vertex of $f(A) = f(A_1)$.

Now we prove the statement of the theorem. Given any two points a and b which are distanced by the unit distance from each other, form two regular icosahedrons A and B such that the following three conditions are met. (1) Both a and b are common vertices of A and B. (2) $\mathrm{Ins}A \cap \mathrm{Ins}B = \phi$. (3) No other vertices are common vertices of A and B. This means that the two icosahedrons A and B share exactly one edge which has end points a and b. Now it is obvious that the two icosahedrons $f(A)$ and $f(B)$ also satisfy the above three conditions with the common vertices $f(a)$ and $f(b)$. Therefore the distance between $f(a)$ and $f(b)$ is again one too. Since f preserves the unit distance, by the theorem of Beckman and Quarles, we conclude that f is a linear isometry up to translation. $\square$

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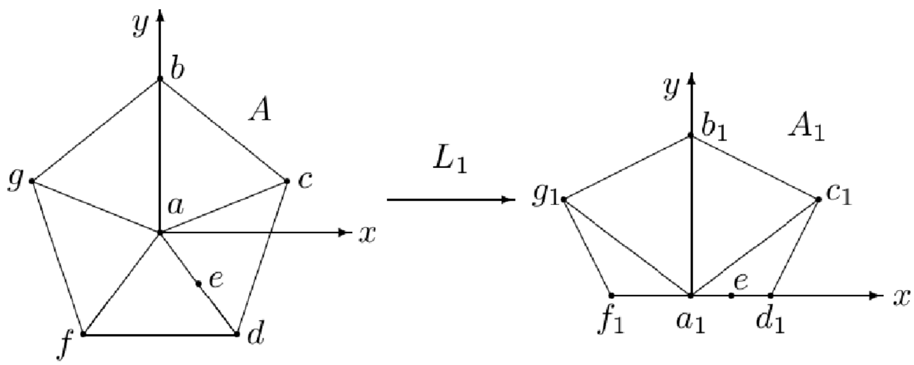

Figure 3:

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Appendix

We show that we can form four regular icosahedrons A_i $(i=1,\cdots,4)$ such that $a \in A_i$ is a common vertex of A_i $(i=1,\cdots,4)$ and $\mathrm{Ins}A_i \cap \mathrm{Ins}A_j = \phi$ for any $i \neq j$, $(i,j=1,\cdots,4)$. Let A be a regular icosahedron some part of which is shown as in Figure 2. Let $L_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation around the x-axis by the amount $\alpha = 52.65°$. If we call h the midpoint of the line $\overline{fd}$, then $\vec{ah} = (0, -\frac{1}{2}\tan 54°, -z_o) = (0, -0.688, -0.525)$ and $\tan^{-1}(\frac{0.525}{0.688}) = 37.35°$. Therefore α is the angle such that the face $a_1d_1f_1$ which is the image of the face adf under L_1 is completely contained in the xz plane (see Figure 3). Let us call $L_1(A) = A_1$. The coordinates are $b_1 = (0, 0.934, 0.358)$, $c_1 = (0.809, 0.577, -0.110)$, $d_1 = (0.5, 0, -0.866)$ and $a_1 = a = (0,0,0) = o$.

Let L_2 be a reflection through the xz plane, i.e., $L_2(x,y,z) = (x,-y,z)$. Call $L_2(A_1) = A_2$. Let L_3 be a $90°$ rotation around z axis followed by the reflection through the xy plane, i.e., $L_3(x,y,z) = (-y,x,-z)$. Call $L_3(A_1) = A_3$ and $L_3(A_2) = A_4$ (see Figure 4). Note that $A_4 = L_3L_2(A_1)$ and $L_3L_2(x,y,z) = (y,x,-z)$. Therefore $b_4 = (0.934, 0, -0.358)$, $c_4 = (0.577, 0.809, 0.110)$, $d_4 = (0, 0.5, 0.866)$. We claim that $\mathrm{Ins}A_i \cap \mathrm{Ins}A_j = \phi$ for any $i \neq j$ $(i,j = 1,\cdots,4)$.

By symmetry we have only to check the part where $x \geq 0$ and $y \geq 0$. We use the fact that if we think of z coordinate as a height function, then in Figure 4, for any vector k, if $k \cdot c_1 \times b_1 > 0$, then k is above the plane spanned by c_1 and b_1 and if $k \cdot c_1 \times b_1 < 0$, k is below the plane. Now since $d_4 \cdot c_1 \times b_1 = 0.5095$

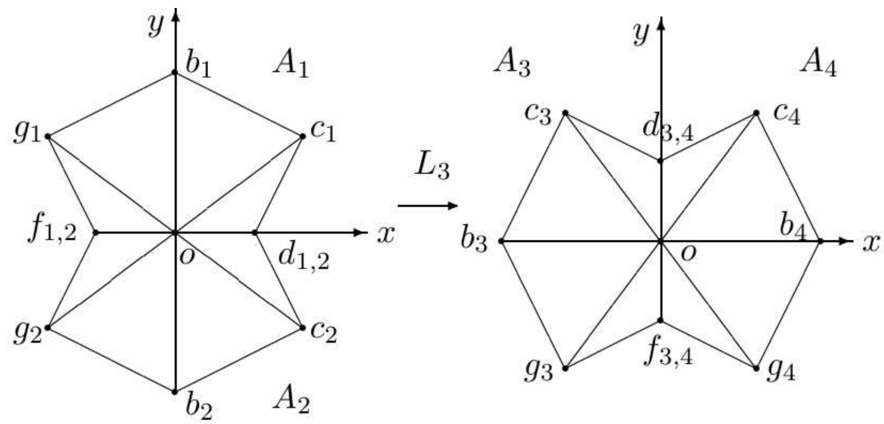

Figure 4:

and $c_4 \cdot c_1 \times b_1 = 0.0273$, both d_4 and c_4 are above the face oc_1b_1. Similarly since $d_1 \cdot b_4 \times c_4 = -0.5095$ and $c_1 \cdot b_4 \times c_4 = -0.0273$, both d_1 and c_1 are below the face ob_4c_4. Therefore we conclude that $\text{Ins}A_i \cap \text{Ins}A_j = \phi$ for any $i \neq j$ $(i, j = 1, \cdots, 4)$.