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THE QUANTUM FIELD THEORY APPROACH TO THE COMBINED FIELD MODEL

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Abstract: Using the *combined field* concept and the relations between the *fictive sources* and the real sources, it is possible to find a condition which allows us to consider the combined field as a *free field*, even in the presence of a perfect conducting fluid medium. This paper shows that this approach gives us permission to consider a photon interacting with an ideal charged fluid as a free combined field quantum, which can be modeled with a similar mathematical pattern, like any other field analyzed in the *free space*.

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1. Introduction

The theoretical investigation of the electromagnetic field in the presence of a material medium, from the point of view of the quantum field theory, represents a complex of specific problems. One of these problems is to describe the interaction between electromagnetic field and its sources, i.e. the current density $\overrightarrow{\mathrm{J}}$ and the electric charge density ρ_e. In general terms, this means that we need to find an invariant expression for the interaction Lagrangean density, and on this basis to calculate the elements of the diffusion matrix.

This complication can be avoided using the definition of the *combined field* [8], which includes the effects of the interactions by the *pseudo-polarization field*

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[1]. This approach proved to be very useful in the theory of an infinite conducting magnetofluid, placed in external electromagnetic field. Making allowance for the applications of the combined field model in the field treatment of MHD theory, we will analize the problem in the quantum approach.

In our opinion, it is important to pass from the classical picture of the *combined field* to the relativistic quantum reprezentation, by using the formalism of the quantum field theory. The aim of this paper is – among other things – to give the quantized expressions of the main physical quantities, like the energy, the momentum, the spin, etc. This formalism is a *sine qua non* condition in a quantum approach to any field.

2. Basic Equations of the Combined Field Model

As we have already shown, the physical processes in an ideal magnetofluid can be described in terms of *generalized antipotentials* (see [8], [7]), denoted as $\overrightarrow{M}$, ψ. This formalism permits us to define a new field, by the vector expressions:

$$\overrightarrow{D} = \epsilon_0 \overrightarrow{E} + \overrightarrow{P} \ ; \quad \overrightarrow{H} = \mu_0^{-1} \overrightarrow{B} - \overrightarrow{P} \times \overrightarrow{v} \ , \tag{1a,b}$$

where $\overrightarrow{P}$ represents the pseudopolarization vector, related to real sources by the equations [1]:

$$\overrightarrow{\mathrm{J}} = \partial_t \overrightarrow{P} + \nabla \times (\overrightarrow{P} \times \overrightarrow{v}) + \overrightarrow{v} \nabla \cdot \overrightarrow{P} \ ; \quad \rho_e = -\nabla \cdot \overrightarrow{P}. \tag{2a,b}$$

The vector field $(\overrightarrow{D}, \overrightarrow{H})$ can be also described in connection with the two antipotentials:

$$\overrightarrow{D} = \nabla \times \overrightarrow{M} \ ; \quad \overrightarrow{H} = \nabla \psi + \partial_t \overrightarrow{M}. \tag{3a,b}$$

Using this motivation, we will consider the generalized antipotentials as *combined potentials*. It is already proved [8] that we can impose a *Lorentz-type condition* $(\nabla \cdot \overrightarrow{M} + \epsilon_0 \mu_0 \partial_t \psi = 0)$ and, in this context, these two potentials satisfy the following equations:

$$\triangle \overrightarrow{M} - \epsilon_0 \mu_0 \partial_{tt} \overrightarrow{M} = -\nabla \times \overrightarrow{P} + \epsilon_0 \mu_0 \partial_t (\overrightarrow{P} \times \overrightarrow{v}); \tag{4a}$$

$$\triangle \psi - \epsilon_0 \mu_0 \partial_{tt} \psi = -\nabla \cdot (\overrightarrow{P} \times \overrightarrow{v}). \tag{4b}$$

It is obvious that the r.h.s. of equation (4a) and equation (4b) describe two analytical sources, which we called *fictive sources*, defined by:

$$\nabla \times \overrightarrow{P} - \epsilon_0 \mu_0 \partial_t (\overrightarrow{P} \times \overrightarrow{v}) = \overrightarrow{S} \ ; \quad \nabla \cdot (\overrightarrow{P} \times \overrightarrow{v}) = (\epsilon_0 \mu_0)^{-1} Q. \tag{5a,b}$$

If we describe the combined field in Maxwellian terms, by taking the *curl* of (3a), and the div of (3b), we obtain Maxwell's source equations, for the new field introduced by this formalism:

$$\nabla \times \overrightarrow{D} = \overrightarrow{S} - \epsilon_0 \mu_0 \partial_t \overrightarrow{H} \ ; \quad \nabla \cdot \overrightarrow{H} = -c^2 Q \quad (c^2 \overset{not.}{=} \epsilon_0^{-1} \mu_0^{-1}). \qquad \text{(6a,b)}$$

So, we can consider that the fictive sources represent, in fact, the sources of the combined field.

From the analytical point of view, a very important tool of theoretical investigation of an ideal, perfect conducting magnetofluid, is the Lagrangean density of the field. This can be constructed by using the previous equations, on the basis of variational considerations [3]:

$$\mathfrak{L} = \frac{1}{2\epsilon_0}(\nabla \times \overrightarrow{M}) \cdot (\nabla \times \overrightarrow{M}) - \frac{1}{2}\mu_0(\nabla\psi + \partial_t \overrightarrow{M}) \cdot (\nabla\psi + \partial_t \overrightarrow{M}) - \frac{1}{\epsilon_0}\overrightarrow{S} \cdot \overrightarrow{M} + \frac{1}{\epsilon_0} Q\psi. \qquad (7)$$

Using this model, we can rewrite the whole MHD system of equations, in terms of combined field, combined potentials, fictive sources and pseudopolarization definition.

3. The Four-Dimensional Representation

Before going further, an important step of our representation is to formulate a four-dimensional image of our model. To do this, we will briefly present both the hyperbolic and Euclidean representations.

3.1. Hyperbolic Representation

Let us consider a four-dimensional space, characterized by $x^1 = x,\ x^2 = y,\ x^3 = z\ , x^4 = ct$, and the metric $ds^2 = -g_{ik} dx^i dx^k$, where g_{ik} is the *metric tensor*. For a complete description of the model, we need to reconsider all notions previously defined in our formalism. Let us first analyze the physical units for the combined field potentials. Since for the Lagrangean density we have $[\mathfrak{L}]_{SI} = J/m^3 = V \cdot A \cdot s/m^3$, then (7) yields:

$$[\psi]_{SI} = \frac{V \cdot A \cdot s/m^3}{V \cdot s/m^3} = A \ ; \quad [M]_{SI} = \frac{A}{m/s} = \frac{[\psi]_{SI}}{[v]_{SI}}. \qquad \text{(8a,b)}$$

The connection between the units of the generalized antipotentials expressed by previous relations, suggests us to define *the combined potential four-vector* as

$$(M^i)_{i=\overline{1,4}} \stackrel{def}{=} \Big(M^\alpha = M_\alpha,\ \alpha = \overline{1,3};\quad M^4 = -M_4 = \frac{1}{c}\psi\Big). \tag{9}$$

In the same way, for the fictive sources we find:

$$[S]_{SI} = C \cdot m^{-3}\ ;\quad [c^2 Q]_{SI} = C \cdot s^{-1} \cdot m^{-2} = [S]_{SI} \cdot [v]_{SI}\ , \tag{10a,b}$$

We are now able to define the *fictive source four-vector* by:

$$(S^i)_{i=\overline{1,4}} \stackrel{def}{=} \Big(S^\alpha = S_\alpha,\ \alpha = \overline{1,3};\ S^4 = -S_4 = \frac{c^2 Q}{c} = cQ\Big). \tag{11}$$

It is obvious that the combined field is suitable to the classical mathematical pattern, so, to go further, similar to the electromagnetic field [5], it is necessary to define the *combined field four-tensor*:

$$C_{ik} \stackrel{def}{=} \frac{\partial M_k}{\partial x^i} - \frac{\partial M_i}{\partial x^k}. \tag{12}$$

As one can observe, this is an antisymmetric tensor, i.e. all the diagonal components are zero, while the non-diagonal elements are:

$$C_{\alpha\beta} = C^{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\partial_\beta M_\gamma = \epsilon_{\alpha\beta\gamma} D_\gamma\ ; \tag{13}$$

$$C_{\alpha 4} = -C^{\alpha 4} = -\frac{1}{c}\Big(\frac{\partial\psi}{\partial x^\alpha} + \frac{\partial M_\alpha}{\partial t}\Big) = -\frac{1}{c}(\nabla\psi + \partial_t \overrightarrow{M})_\alpha \stackrel{(3b)}{=} -\frac{1}{c}H_\alpha. \tag{14}$$

Using the three quantities previously defined, we can rewrite the Lagrangean density. First of all, we observe that the expression (7) can be considered as a sum between the Lagrangean density of the free combined field

$$\mathfrak{L}_0 = \frac{1}{2\epsilon_0}(\epsilon_{\alpha\beta\gamma}\partial_\beta M_\gamma)(\epsilon_{\alpha\beta\gamma}\partial_\beta M_\gamma) - \frac{1}{2}\mu_0(\psi_{,\alpha} + \partial_t M_\alpha)(\psi_{,\alpha} + \partial_t M_\alpha)\ , \tag{15}$$

and the Lagrangean density describing the interaction with the fictive sources

$$\mathfrak{L}_i = -\frac{1}{\epsilon_0}S_\alpha M_\alpha + \frac{1}{\epsilon_0}Q\psi. \tag{16}$$

Making use of (13) and (14), we find for the free Lagrangean density:

$$\mathfrak{L}_0 = \frac{1}{4\epsilon_0}(C^{\alpha\beta}C_{\alpha\beta} + 2C^{\alpha 4}C_{\alpha 4}) = \frac{1}{4\epsilon_0}C^{ij}C_{ij},\quad i,j = \overline{1,4}. \tag{17}$$

In an analogous manner, by using the definitions (9) and (11), for the other term of the Lagrangean density we have:

$$\mathfrak{L}_i = -\frac{1}{\epsilon_0}(S_\alpha M_\alpha - Q\psi) = -\frac{1}{\epsilon_0}S^i M_i. \tag{18}$$

To summarize, we will write the general expression of the Lagrangean density of the combined field as:

$$\mathfrak{L} = \mathfrak{L}_0 + \mathfrak{L}_i = \frac{1}{\epsilon_0}\Big(\frac{1}{4}C^{ij}C_{ij} - S^j M_j\Big). \tag{19}$$

It is easy to prove that, if we also define the *combined field dual-tensor* ($\tilde{C^{ij}} = \frac{1}{2}\epsilon^{ijkm}C_{km}$), we are able to put in a covariant form the whole MHD system of equations for our magnetofluid model, but this is not the purpose of the paper.

3.2. Euclidian Representation

Another possibility is to choose a four-dimensional space with $x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = it$. As well-known, there is no difference between covariant and contravariant forms of vectors in this representation. To simplify the calculations, we have considered the so-called *system of natural coordinates* ($\hbar = c = 1$), without restricting the generality of the problem. In this context, the defintion of the combined potential four-vector, and of the fictive source four-vector, becomes:

$$M_i \stackrel{def}{=} (M_\alpha = M_{\{x,y,z\}}\ ;\ M_4 = i\psi)\,; \tag{20a}$$

$$S_i \stackrel{def}{=} (S_\alpha = S_{\{x,y,z\}}\ ;\ S_4 = iQ). \tag{20b}$$

In this case, the combined field four-tensor can be defined as:

$$C_{ij} = \begin{pmatrix} 0 & D_z & -D_y & -iH_x \\ -D_z & 0 & D_x & -iH_y \\ D_y & -D_x & 0 & -iH_z \\ iH_x & iH_y & iH_z & 0 \end{pmatrix}. \tag{21}$$

Using (7), (20a) and (21), we find for the Lagrangean density the general expression:

$$\mathfrak{L} = \frac{1}{4}C_{ij}C_{ij} - S_j M_j. \tag{22}$$

4. The Lagrangean Formalism

Let us consider a peculiar case of our magnetofluid model, e.g. suppose that the fluid charged medium is traversed by irrotational currents, and the analysis is performed in its self-reference frame. These restrictions can be expressed by the mathematical conditions:

$$\nabla \times \overrightarrow{\mathrm{J}} = 0 \; ; \; \overrightarrow{v} \cong 0. \tag{23a,b}$$

If the analysis is performed in the self-reference frame, on the basis of (2a) we find that $\nabla \times \overrightarrow{P} = \int (\nabla \times \overrightarrow{\mathrm{J}} dt) = 0$ in accordance with (23a). Introducing these results in (5a,b), it is obvious that the fictive sources become null ($\overrightarrow{S} = 0,\ Q = 0$), which means that we may consider our magnetofluid model as a *vacuum* for the combined field.

This consideration leads to the conclusion that the combined field is a *free field* in the presence of a such a material medium. In this case, the Lagrangean density (22) reduces to:

$$\mathfrak{L} = \frac{1}{4} C_{ij} C_{ij}. \tag{24}$$

To obtain the result of our investigation, we must modify the Lagrangean using a *Dirac-Fock type method* [6], by taking:

$$\mathfrak{L} = \frac{1}{4} C_{ij} C_{ij} + \frac{1}{2} M_{i,i} M_{j,j}. \tag{25}$$

Introducing the definition of the combined field tensor in (25) and performing some simple mathematical manipulation, we obtain:

$$\mathfrak{L} = \frac{1}{2} M_{j,i} M_{j,i} + \frac{1}{2} \frac{\partial}{\partial x_i} (M_i M_{j,j} - M_j M_{i,j}). \tag{26}$$

Since the second term of (26) represents a *four-divergence*, it has no impact on our Lagrangean formulation, so, according to the usual procedure (see [6], [9], [4], [2]), we may consider as the Lagrangean density the following expression:

$$\mathfrak{L} = \frac{1}{2} M_{j,i} M_{j,i}. \tag{27}$$

This relation allows us to express the quantities characterising the combined field quantum. First of all, we can define the *generalized momentum* of the quantum by:

$$\Pi_{jk} = \frac{\partial \mathfrak{L}}{\partial M_{k,j}} = M_{k,j} \ , \tag{28}$$

which is very useful in writing the *basic condition* of quantization in the quantum theory approach to our model:

$$[\Pi^{\dagger}_{4i}(\overrightarrow{r}', t), M_k(\overrightarrow{r}, t)] = -i\delta_{ik}\delta(\overrightarrow{r}' - \overrightarrow{r}). \tag{29}$$

Another important theoretical tools are the *canonical energy tensor*

$$T_{ik} = \Pi_{kj}M_{j,i} - \mathfrak{L}\delta_{ik} = M_{j,k}M_{j,i} - \frac{1}{2}M_{l,j}M_{l,j}\delta_{ik}\ , \tag{30}$$

as well as the *spin density*

$$\mathcal{S}_{ijk} = i(M_j\Pi_{ki} - M_i\Pi_{kj}) = i(M_jM_{i,k} - M_iM_{j,k}). \tag{31}$$

Applying the Lagrangean formalism [6], we are now able to calculate the momentum, the energy and the spin tensor of the combined field quantum:

$$P_\alpha = i\int T_{\alpha 4}d\overrightarrow{r}\ ; \quad H = \int T_{44}d\overrightarrow{r}\ ; \quad I_{ij} = \int \mathcal{S}_{ijk}d\Sigma_k\ , \tag{32a,b,c}$$

where $\alpha = \overline{1,3}$, $i, j, k = \overline{1,4}$ and $d\Sigma_k$ is *the elementary hypersurface*, orthogonal to x_k axis.

5. The Momentum Representation

First of all, we have to mention some basic characteristics of the combined field quantum. Since our field model was defined by using the frame of the electromagnetic field, it is necessary to consider that the rest-mass of the quantum is null, and it possesses an integer spin, like the photon. The motive of these observations is that the interactions between a photon and the real sources cannot modify these fundamental characteristic quantities. So, using the *de Broglie relation* ($\overrightarrow{p} = \hbar\overrightarrow{k}$) and keeping in mind that we work in natural coordinate system, we can define the coordinates in the *momentum space* as: $k_\alpha = k_{\{x,y,z\}}, k_4 = i\omega \equiv ik_0$.

5.1. The Combined Field Potential

The connection between the two represenions of the combined field potential is given by the *Fourier transform*:

$$M_j(x) = \frac{1}{(2\pi)^{3/2}}\Big[\int \frac{1}{(2\omega)^{1/2}}M_{j+}(\vec{k})e^{ikx}d\overrightarrow{k}$$

$$+\int \frac{1}{(2\omega)^{1/2}} M_{j-}(\vec{k}) e^{-ikx} d\overrightarrow{k} \Big] \, . \quad (33)$$

It is easy to observe that we may consider the represention (33) as a sum between two terms $M_j(x) \overset{not.}{=} M_{j+}(x) + M_{j-}(x)$, where

$$M_{j\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{(2\omega)^{1/2}} M_{j\pm}(\vec{k}) e^{\pm ikx} d\overrightarrow{k} \ , \quad (34)$$

express the so-called *positive frequency part*, and the *negative frequency part* of the generalized antipotential operator, respectively.

The components of the combined potential can be considered as functions of the *polarization unit* four-vectors, defined by: $e^i_j e^k_j = \delta_{ik}$. We will suppose that the combined field propagates parallel to the third axis in the space reference frame ($\overrightarrow{k} = (0, 0, k = \omega)$), and its direction is identical to the third axis of the polarization system. Then we may write:

$$e^3_j = \frac{\overrightarrow{k}}{\mid \overrightarrow{k} \mid} = \frac{\overrightarrow{k}}{\omega}. \quad (35)$$

In this context, the versors e^1_j and e^2_j characterize the quantum polarization in a plane orthogonal to the direction of propagation, while the fourth versor e^4_j is related to the temporal dimension of the wave. Using these assumptions, the operators $M_{j\pm}(\overrightarrow{k})$ can be expressed in a new form, in the polarization frame:

$$M_{j\pm}(\overrightarrow{k}) = e^i_j c_{i\pm}(\overrightarrow{k}) \ , \quad (36)$$

where the significance of the two operators c_{i+} and c_{i-} will be explained later.

5.2. The Space Components of the Momentum

Introducing the expression (30) of the canonical energy tensor in (32a), we can put the momentum components in the following form:

$$P_\alpha = i \int M_{j,\alpha} M_{j,4} d\overrightarrow{r} \ . \quad (37)$$

Using the representation (34) of combined potential, we can consider that the space components of the momentum are $P_\alpha = P_{\alpha 1+} + P_{1\alpha -} + P_{\alpha 2+} + P_{\alpha 2-}$, where

$$P_{\alpha 1\pm} = i \int M_{j\pm,\alpha} M_{j\pm,4} d\overrightarrow{r} \ ; \quad (38a)$$

$$P_{\alpha 2\pm} = i \int M_{j\pm,\alpha} M_{j\mp,4} d\overrightarrow{r} \ . \quad (38b)$$

Taking the derivatives of $M_{j\pm}$, we can calculate the expressions of each term of (38a). For $P_{\alpha 1+}$, for example, we find:

$$P_{\alpha 1+} = -\frac{i}{(2\pi)^3} \times \int \frac{ik_\alpha}{(2\omega)^{1/2}} \frac{\omega'}{(2\omega')^{1/2}} M_{j+}(\overrightarrow{k}) M_{j+}(\overrightarrow{k}') e^{i(k+k')x} d\overrightarrow{r} d\overrightarrow{k} d\overrightarrow{k}' \ . \quad (39)$$

Since $i(k+k')x = i(\overrightarrow{k}+\overrightarrow{k}')\cdot\overrightarrow{r} - i(\omega+\omega')t$ and $\delta(\overrightarrow{k}+\overrightarrow{k}') = (2\pi)^{-3}\int e^{i(\overrightarrow{k}+\overrightarrow{k}')\cdot\overrightarrow{r}} d\overrightarrow{r}$, the integral (39) becomes:

$$P_{\alpha 1+} = \int \frac{k_\alpha}{2} M_{j+}(\overrightarrow{k}) \times \left[\int \frac{\omega'}{(\omega\omega)^{1/2}} M_{j+}(\overrightarrow{k}') \delta(\overrightarrow{k} + \overrightarrow{k}') e^{-i(\omega+\omega')t} d\overrightarrow{k}' \right] d\overrightarrow{k} \ . \quad (40)$$

Using the properties of the *Dirac distribution*, we finally obtain the expression:

$$P_{\alpha 1+} = \int \frac{k_\alpha}{2} M_{j+}(\overrightarrow{k}) M_{j+}(-\overrightarrow{k}) e^{-2i\omega t} d\overrightarrow{k} \ . \quad (41)$$

It is obvious that, if we make the substitution $\overrightarrow{k} \to -\overrightarrow{k}$, we find $P_{\alpha 1+} = -P_{\alpha 1+}$, and the result is:

$$P_{\alpha 1+} = 0 \ . \quad (42)$$

In a similar way, we find $P_{\alpha 1-} = 0$. For the last two terms, it is easy to show that $P_{\alpha 2+} = P_{\alpha 2-} = -\int \frac{k_\alpha}{2} M_{j+}(\overrightarrow{k}) M_{j-}(\overrightarrow{k}) d\overrightarrow{k}$, and the final formula for the space components of the momentum is:

$$P_\alpha = -\int k_\alpha M_{j+}(\overrightarrow{k}) M_{j-}(\overrightarrow{k}) d\overrightarrow{k} \ . \quad (43)$$

Using representation in the polarization frame, we have

$$M_{j+}(\overrightarrow{k}) M_{j-}(\overrightarrow{k}) = e^i_j e^k_j c_{i+}(\overrightarrow{k}) c_{k-}(\overrightarrow{k}) = \delta^{ik} c_{i+}(\overrightarrow{k}) c_{k-}(\overrightarrow{k}) = c_{i+}(\overrightarrow{k}) c_{i-}(\overrightarrow{k}) \ ,$$

and (43) yields:

$$P_\alpha = -\int k_\alpha c_{i+}(\overrightarrow{k}) c_{i-}(\overrightarrow{k}) d\overrightarrow{k} \ . \quad (44)$$

5.3. The Energy

If we develop the relation (32b), by using the formula (30), we find for the Hamiltonian density of the field quantum the general expression:

$$H = \int \Big(M_{j,4}M_{j,4} - \frac{1}{2}M_{l,j}M_{l,j}\Big)d\overrightarrow{r} = \frac{1}{2}\int (M_{j,4}M_{j,4} - M_{j,\alpha}M_{j,\alpha})d\overrightarrow{r} \ . \quad (45)$$

Using the same procedure as for the momentum, to simplify the calculation we can write the operator (45) as a sum of four operatorial terms $H = H_{1+} + H_{1-} + H_{2+} + H_{2-}$, where:

$$H_{1\pm} = \frac{1}{2}\int (M_{j\pm,4}M_{j\pm,4} - M_{j\pm,\alpha}M_{j\pm,\alpha})d\overrightarrow{r} \ , \quad (46a)$$

$$H_{2\pm} = \frac{1}{2}\int (M_{j\pm,4}M_{j\mp,4} - M_{j\pm,\alpha}M_{j\mp,\alpha})d\overrightarrow{r} . \quad (46b)$$

Following the same method, it is easy to show that

$$H_{1+} = H_{1-} = 0 \ , \quad (47)$$

and

$$H_{2+} = H_{2-} = -\frac{1}{2}\int \omega M_{j+}(\overrightarrow{k})M_{j-}(\overrightarrow{k})d\overrightarrow{k} \ . \quad (48)$$

Adding these four terms, we find the momentum representation for the energy density:

$$H = -\int \omega M_{j+}(\overrightarrow{k})M_{j-}(\overrightarrow{k})d\overrightarrow{k} \ . \quad (49)$$

In the polarization frame, the Hamiltonian density is:

$$H = -\int \omega c_{j+}(\overrightarrow{k})c_{j-}(\overrightarrow{k})d\overrightarrow{k} \ . \quad (50)$$

5.4. The Spin

If we return to the definition (32c) of the spin tensor and consider a fixed moment, we obtain the space components of the spin:

$$I_{\alpha\beta} = \int \mathcal{S}_{\alpha\beta 4} d\overrightarrow{r} = i \int ((M_\beta M_{\alpha,4} - M_\alpha M_{\beta,4})) d\overrightarrow{r} \ . \tag{51}$$

A similar calculation gives:

$$I_{\alpha\beta} = i \int [M_{\beta+}(\overrightarrow{k}) M_{\alpha-}(\overrightarrow{k}) - M_{\alpha+}(\overrightarrow{k}) M_{\beta-}(\overrightarrow{k})] d\overrightarrow{k} \ . \tag{52}$$

Since the tensor (52) is an antisymmetric, it can be associated with an axial vector, having the significance of the spin moment:

$$I_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} \mathcal{S}_\gamma \ . \tag{53}$$

So, we can find the general expression for each of the spin components, by using the relations (52) and (53). For the component parallel to the direction of field propagation, we get:

$$\mathcal{S}_z \overset{\text{not.}}{=} \mathcal{S}_3 = i \int [M_{2+}(\overrightarrow{k}) M_{1-}(\overrightarrow{k}) - M_{1+}(\overrightarrow{k}) M_{2-}(\overrightarrow{k})] d\overrightarrow{k} \ . \tag{54}$$

Since in the polarization frame we have

$$M_{2+}(\overrightarrow{k}) M_{1-}(\overrightarrow{k}) = e_2^i c_{i+}(\overrightarrow{k}) e_1^k c_{k-}(\overrightarrow{k}) = \delta_2^i \delta_1^k c_{i+}(\overrightarrow{k}) c_{k-}(\overrightarrow{k})$$
$$= c_{2+}(\overrightarrow{k}) c_{1-}(\overrightarrow{k}) \ ,$$

this yelds

$$\mathcal{S}_z = i \int [c_{2+}(\overrightarrow{k}) c_{1-}(\overrightarrow{k}) - c_{1+}(\overrightarrow{k}) c_{2-}(\overrightarrow{k})] d\overrightarrow{k} \ . \tag{55}$$

A more significant form of the relation (55) can be displayed by defining a new group of operators:

$$c_{1\pm} = \frac{1}{\sqrt{2}}(b_{1\pm} + b_{2\pm}); \ c_{3\pm} = b_{3\pm}; \ c_{2\pm} = \frac{\pm i}{\sqrt{2}}(b_{1\pm} - b_{2\pm}); \ c_{4\pm} = b_{4\pm} \ , \tag{56a,b,c,d}$$

which leads to the third component of the spin:

$$\mathcal{S}_z = \int [b_{2+}(\overrightarrow{k}) b_{2-}(\overrightarrow{k}) - b_{1+}(\overrightarrow{k}) b_{1-}(\overrightarrow{k})] d\overrightarrow{k} \ . \tag{57}$$

6. Significance of the Operators

To find the meaning of the operators previously introduced, we must return to the quantization condition (29), which can be writen in a different form by means of (28):

$$[M_{i,0}(\overrightarrow{r}',t), M_j(\overrightarrow{r},t)] = -i\delta_{ij}\delta(\overrightarrow{r}' - \overrightarrow{r}) \ . \tag{58}$$

Decomposing the commutator (58) in the sum

$$\begin{aligned}
&[M_{i,0}(\overrightarrow{r}',t), M_j(\overrightarrow{r},t)] \\
&\quad = [M_{i+,0}(\overrightarrow{r}',t), M_{j+}(\overrightarrow{r},t)] + [M_{i+,0}(\overrightarrow{r}',t), M_{j-}(\overrightarrow{r},t)] \\
&\qquad + [M_{i-,0}(\overrightarrow{r}',t), M_{j+}, (\overrightarrow{r},t)] + [M_{i-,0}(\overrightarrow{r}',t), M_{-j,0}(\overrightarrow{r},t)] \ ,
\end{aligned}$$

and performing the transformation to the momentum representation, we find:

$$\begin{aligned}
&[M_{i\pm,0}(\overrightarrow{k},t), M_{j\pm}(\overrightarrow{k}',t)] = 0; \\
&\qquad [M_{i\pm,0}(\overrightarrow{k},t), M_{j\mp}(\overrightarrow{k}',t)] = \pm\delta_{ij}\delta(\overrightarrow{k} - \overrightarrow{k}') \ .
\end{aligned} \tag{59a,b}$$

Introducing (36) in (59a,b), we get to the following commutator relations:

$$[c_{i\pm}(\overrightarrow{k}), c_{j\pm}(\overrightarrow{k}')] = 0; \ [c_{i\pm}(\overrightarrow{k}), (-c_{j\mp}(\overrightarrow{k}'))] = \pm\delta_{ij}\delta(\overrightarrow{k} - \overrightarrow{k}') \ , \tag{60a,b}$$

where the "-" sign, in the second relation, is just a consequence of the Lagrangean choice.

An analogy to the quantum field theory, on the basis of relations (60a,b), shows that $c_{i+}(\overrightarrow{k})$ represents *the emission operator* of a combined field quantum, in the impuls state $\overrightarrow{k}$, and $(-c_{i-}(\overrightarrow{k}))$ is *the anihilation operator*. In this case, the density of the field quanta, in the momentum state $\overrightarrow{k}$, is given by $N_i(\overrightarrow{k}) = -c_{i+}(\overrightarrow{k})c_{i-}(\overrightarrow{k})$, while the momentum and the energy can be writen as:

$$P_\alpha = \int k_\alpha \sum_i N_i(\overrightarrow{k}) d\overrightarrow{k}; \ H = \int \omega \sum_i N_i(\overrightarrow{k}) d\overrightarrow{k} \ . \tag{61a,b}$$

Similar considerations can be developed for the second group of operators. So, if we express $b_{\{1,2\}\pm}$ in (56a,b,c,d), we are left with:

$$[b_{\{1,2\}+}(\overrightarrow{k}), b_{\{1,2\}-}(\overrightarrow{k}')] = \frac{1}{2}\Big[[c_{1+}(\overrightarrow{k}), c_{1-}(\overrightarrow{k}')] \mp i \underbrace{[c_{2+}(\overrightarrow{k}), c_{1-}(\overrightarrow{k}')]}_{=0}$$

$$\pm\, i \underbrace{[c_{1+}(\overrightarrow{k}), c_{2-}(\overrightarrow{k}')]}_{=0} + [c_{2+}(\overrightarrow{k}), c_{2-}(\overrightarrow{k}')]\Big] = \delta(\overrightarrow{k} - \overrightarrow{k}') \; .$$

Denoting $N_2'(\overrightarrow{k}) \stackrel{\text{not.}}{=} b_{2+}(\overrightarrow{k})b_{2-}(\overrightarrow{k})$; $N_1'(\overrightarrow{k}) \stackrel{\text{not.}}{=} b_{1+}(\overrightarrow{k})b_{1-}(\overrightarrow{k})$, the projection of spin moment on the propagation axis becomes:

$$\mathcal{S}_z = \int [N_2'(\overrightarrow{k}) - N_1'(\overrightarrow{k})] d\overrightarrow{k} \; . \tag{62}$$

The combined field quantum is a boson, so we may say that $N_2'(\overrightarrow{k})$ represents the density of the field quanta with the spin $+1$, while $N_1'(\overrightarrow{k})$ signifies the density of the combined bosons with the spin -1.

7. Final Remarks

To complete the model, we should also find the *microscopic condition of calibration* for the combined field. It is easy to show that, by introducing *the metric operator*, we can find a *Lorentz-Fermi type condition* ($M_{j-,j} \mid \psi >= 0$, or $< \psi \mid M_{j+,j} = 0$) which, by averaging, leads to the macroscopic Lorentz-type condition.

If conditions (23a,b) are imposed, we arrive at a complete characterization of the combined field as a free field, in quantum approach, even in the presence of a material magnetofluid medium. This means that *a perfect conducting magnetofluid, that satisfies (23a,b) and for which the dissipative mechanical effects are neglijable* represents a *free space* for the combined field. Using the well-known method of quantization, we found analytic expressions for the momentum, the energy and the spin momentum of the field quanta. The general formalism shows that this quantum is a boson, with zero rest-mass, like the photon, but, in addition, it intrisecally contains the interaction between the photon and the real sources of the magnetofluid model previously described.

So, we may conclude that the photon interacting with such a medium is equivalent to a *free combined field quantum.* Thus, the interacting photons density is equal to the free combined bosons density and the energy of the combined quanta can be decomposed in the sum between the interaction energy and the free photon energy. Even if this approach does not offer any direct information about the diffusion probability of the photons, we may consider that the model allows us, for this peculiar case, a coherent treatment of the electromagnetic field in interaction with the real sources on a microscopic level.

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