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NATURAL CONVECTION HEAT TRANSFER IN A DOUBLY-CONNECTED REGION, USING A SPECTRAL FINITE DIFFERENCE SCHEME

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Abstract: Axisymmetric steady-state Newtonian natural convection heat transfer from a near torus, which is heated at a uniform temperature and located concentrically in a vertical cylindrical cavity of finite height is analyzed. The cavity wall is kept at a constant uniform temperature. A spectral finite difference scheme is applied to get a steady-state solution as an initial and boundary-value problem.

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1. Introduction

Spectral finite difference scheme [1] posses a character of mathematical rigorousness in leading a system of simultaneous differential equations. Large effectiveness has been found in analyzing two-dimensional or axisymmetric laminar incompressible flow or forced/natural convection flow in a simply-connected region. However, for not a simply-connected region, in a mathematical point of

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view it is necessary to introduce multiply-connectedness in addition to usual boundary conditions, see [2]. Axisymmetric steady-state Newtonian natural convection heat transfer from a near torus, which is heated at a uniform temperature and located concentrically in a vertical cylindrical cavity of finite height is analyzed. The cavity wall is kept at a constant uniform temperature.

2. Analysis

2.1. Basic Equations

On temperature dependence, Boussinesq approximation is introduced and the viscous dissipation terms are neglected. Then the system of equations for energy transport, vorticity transport, and the vorticity - stream function relation is

$$J\frac{\partial T}{\partial t}+\frac{\partial(T,\psi/r)}{\partial(\alpha,\beta)}+\frac{\psi}{r^2}\frac{\partial(T,r)}{\partial(\alpha,\beta)}=\frac{1}{Pr\sqrt{Gr}}\left(\nabla\cdot\nabla+\nabla\ln r\cdot\nabla\right)T\,, \tag{1}$$

$$J\frac{\partial\zeta}{\partial t}+\frac{\partial(\zeta,\psi/r)}{\partial(\alpha,\beta)}+\frac{\partial(\zeta\psi/r^2,r)}{\partial(\alpha,\beta)}$$
$$=\frac{1}{\sqrt{Gr}}\left\{(\nabla\cdot\nabla)\,\zeta+(\nabla r)\cdot\nabla\,(\zeta/r)\right\}+\frac{\partial(T,z)}{\partial(\alpha,\beta)}\,, \tag{2}$$

$$J\zeta+(\nabla\cdot\nabla)\,(\psi/r)+(\nabla r)\cdot\nabla\left(\psi/r^2\right)=0,\quad J\equiv\frac{\partial(z,r)}{\partial(\alpha,\beta)}\,, \tag{3}$$

respectively, where ∇ is an operator in (α,β) which is an analytical plane in a conformal mapping from physical plane (z,r) such that $ia\mathrm{sn}(z+i\,r,k)=\{\exp(\alpha+i\beta)-1\}/\{\exp(\alpha+i\beta)+1\}$, $a>0$, $|z|\le K$, $0\le r\le K'$, $-\pi<\beta\le\pi$, sn : Jacobian elliptic function, k : modulus, $K(k)$: complete elliptic integral of the first kind, $K'\equiv K(\sqrt{1-k^2})$. The cavity wall is given by $\alpha=0$, i.e. $z=\pm K$ or $r=K'$, and the near torus by $\alpha=\alpha_0(<0)$ so that $\alpha_0\le\alpha\le 0$, z : vertically upward, $r=0$ being for the central axis, T : temperature, ζ : vorticity, ψ : axisymmetric stream function, Pr : Prandtl number, Gr : Grashof number. In these expressions all quantities are made dimensionless with respect to the reference length L such that (vertical height of cavity) / $L=2K$, reference velocity $U(\equiv\sqrt{Gr}\,\nu/L)$, and reference time L/U, where ν: kinematic viscosity of the fluid. Dimensionless temperature T is given by $T\equiv$ (local temperature $-T_w)/(T_c-T_w)$, T_c: uniform surface temperature of the near torus, T_w: uniform cavity wall temperature ($<T_c$). $Gr\equiv(T_c-T_w)L^3\beta^* g/\nu^2$, β^*: coefficient of thermal expansion of the fluid, g: acceleration constant of gravity. a is

a parameter such that $a = \mathrm{cn}(r_0, \sqrt{1-k^2})/\mathrm{sn}(r_0, \sqrt{1-k^2})$ where the point $(z = 0, r = r_0)$ corresponds to $\alpha = -\infty$ in the mapping, cn: Jacobian elliptic function. The physical z-component of velocity vector, v^z, and r-component of velocity vector, v^r, are given by $v^z = \frac{1}{r}\frac{\partial \psi}{\partial r}, v^r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$ respectively, and single non-zero component of vorticity, ζ, is related to $\zeta = \frac{\partial v^r}{\partial z} - \frac{\partial v^z}{\partial r}$. The perimeter of the section of the near torus (on the plane including the z-axis), L_0, is given by

$$L_0 \approx \frac{4\pi \exp \alpha_0}{\sqrt{1+a^2}\ \sqrt{1+k^2/a^2}} \left[1 + \left\{ \frac{k^2 - a^4}{(1+a^2)(a^2+k^2)} \right\}^2 \mathrm{e}^{2\ \alpha_0} \right] . \tag{4}$$

2.2. Physical Boundary Conditions

Dynamical boundary conditions are: No slip on the near torus and no slip on the cavity walls.

Thermal boundary conditions are:

$$T(\text{on the near torus}) = 1 \,, \tag{5}$$

$$T(\text{on the cavity wall}) = 0 \,. \tag{6}$$

In the (α, β) plane, the natural physical condition at $r = 0$ should be added:

$$v^r = 0 \ , \quad \zeta = 0 \ , \quad \frac{\partial T}{\partial r} = 0 \,, \tag{7}$$

which leads to

$$\frac{\partial \psi(0,\beta)}{\partial \beta} = 0 \ , \quad \zeta(0,\beta) = 0 \ , \quad \frac{\partial T(0,\beta)}{\partial \alpha} = 0 \quad [|\beta| \le \beta_0 \ (\equiv 2 \tan^{-1} a)] \,. \tag{8}$$

Finally the boundary conditions reduce without loss of generality to

$$\psi(0,\beta) = 0 \quad (|\beta| \le \pi) \,, \tag{9}$$

$$\frac{\partial \psi}{\partial \alpha}(0,\beta) = 0 \quad (\beta_0 < |\beta| \le \pi) \,, \tag{10}$$

$$\zeta(0,\beta) = 0 \quad (|\beta| \le \beta_0) \,, \tag{11}$$

$$\psi(\alpha_0,\beta) = \text{constant} \equiv c \quad (|\beta| \le \pi) \,. \tag{12}$$

Here c may be a function of time and other parameters.

$$\frac{\partial \psi}{\partial \alpha}(\alpha_0,\beta) = 0 \quad (|\beta| \le \pi) \,, \tag{13}$$

$$T(0,\beta)=0 \quad (\beta_0 \le |\beta| \le \pi)\,, \tag{14}$$

$$\frac{\partial T}{\partial \alpha}(0,\beta)=0 \quad (|\beta| < \beta_0)\,, \tag{15}$$

$$T(\alpha_0,\beta)=1 \quad (|\beta| \le \pi)\,. \tag{16}$$

2.3. Auxiliary Condition

Doubly-connectedness gives

$$\oint_{\alpha\,=\,\alpha_0} \frac{\partial p}{\partial \beta}\, d\beta = 0\,, \tag{17}$$

where p : pressure, and the component $\partial p/\partial\beta$ of ∇p can be obtained through the equation of motion for Cartesian expression (though not shown explicitly), using velocity vector $= \mathbf{0}$ at $\alpha = \alpha_0$ and $T(\alpha_0,\beta) = 1$. Thus equation (17) reduces to

$$\oint_{\alpha\,=\,\alpha_0} \left(\frac{\partial \zeta}{\partial \alpha} + \frac{\partial r}{\partial \alpha}\frac{\zeta}{r} \right) d\beta = 0\,. \tag{18}$$

3. Spectral Finite Difference Formulation

Instead of ψ, ψ/r can be regarded as an unknown. The following Fourier decomposition is introduced:

$$\begin{bmatrix} \psi/r \equiv F \\ \zeta \\ T \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} F_{cn}(\alpha,t) \\ \zeta_{cn}(\alpha,t) \\ T_{cn}(\alpha,t) \end{bmatrix} \cos n\beta + \sum_{n=1}^{\infty} \begin{bmatrix} F_{sn}(\alpha,t) \\ \zeta_{sn}(\alpha,t) \\ T_{sn}(\alpha,t) \end{bmatrix} \sin n\beta\,. \tag{19}$$

Inserting equation (19) into equations (1)-(3) and decomposing them into each Fourier component of β produces a system of simultaneous differential equations in space α and time t in the interior of the domain. The resulting differential equations can be discretized in time and space by a finite difference approximation, using a non-uniform grid spacing such that for the n-th grid point α_n in α ($n \ge 0$)

$$\alpha_n = \alpha_0 + h \left\{ \frac{\sinh \gamma(n-1)}{\sinh \gamma} + 1 \right\} \quad (h > 0)\,, \tag{20}$$

γ: a suitable constant to be determined. $\gamma \to +0$ indicates a uniform grid spacing in α-coordinate. Actually $M+1$ grid points are allocated in $(\alpha_0, 0]$, that is, $\alpha_{M+1} = 0$. In this case equations (10) and (11) (a part of the boundary conditions) become

$$-\zeta(0,\beta) \approx \begin{cases} 0 & (|\beta| < \beta_0)\,, \\ \dfrac{2}{\alpha_M^2}\dfrac{1}{J}F(\alpha_M,\beta) & (\beta_0 < |\beta| \leq \pi)\,. \end{cases} \tag{21}$$

Equation (13) becomes

$$-\zeta(\alpha_0,\beta) \approx \frac{1}{rJ}\frac{2}{h^2}\left(r(\alpha_1,\beta)F(\alpha_1,\beta) - c\right) \quad (|\beta| \leq \pi)\,. \tag{22}$$

Likewise equations (14) and (15) become

$$T(0,\beta) \approx \begin{cases} T(\alpha_M,\beta) & (|\beta| < \beta_0)\,, \\ 0 & (\beta_0 \leq |\beta| \leq \pi)\,. \end{cases} \tag{23}$$

Treatment of the mixed boundary conditions is the same as in [1]. Thus the boundary conditions can be decomposed into Fourier components. Some examples are

$$F_{cn}(\alpha_0) = \frac{2c}{\pi}\int_0^\pi \frac{\cos n\beta}{r(\alpha_0,\beta)}d\beta \quad (n > 0)\,, \tag{24}$$

$$F_{c0}(\alpha_0) = \frac{c}{\pi}\int_0^\pi \frac{d\beta}{r(\alpha_0,\beta)}\,, \tag{25}$$

$$F_{sn}(\alpha_0) = 0 \quad (n > 0)\,, \tag{26}$$

$$F_{cn}(0) = 0 \quad (n \geq 0)\,, \tag{27}$$

$$F_{sn}(0) = 0 \quad (n \geq 1)\,. \tag{28}$$

If spectral decomposition equation (19) is applied to equation (18), then finite difference approximation yields

$$\begin{aligned} c\,D_0 + 3\sum_{n=0}^{\infty}\left\{\frac{-2F_{cn}(\alpha_1)}{h^2(h-q)} + \frac{F_{cn}(\alpha_2)}{h^2(2h-q)} + \frac{2F_{cn}(\alpha_3)}{q(h-q)(2h-q)}\right\}D_{1n} \\ + \sum_{n=0}^{\infty}\frac{-2F_{cn}(\alpha_1) + F_{cn}(\alpha_2)}{h^2}D_{2n} + \sum_{n=0}^{\infty}\frac{F_{cn}(\alpha_1)}{h}D_{3n} + \sum_{n=1}^{\infty}\frac{F_{cn}(\alpha_1)}{-h}D_{4n} \\ = 0\,, \end{aligned} \tag{29}$$

where

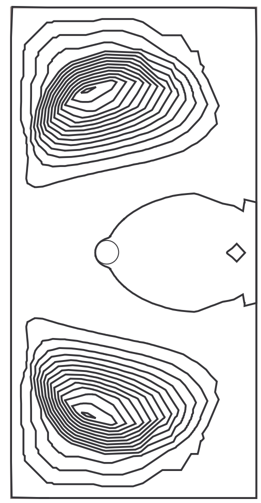

Figure 1: Streamlines. $\delta\psi = 5.0 \times 10^{-5}$, $C_L = 5.8 \times 10^{-4}$

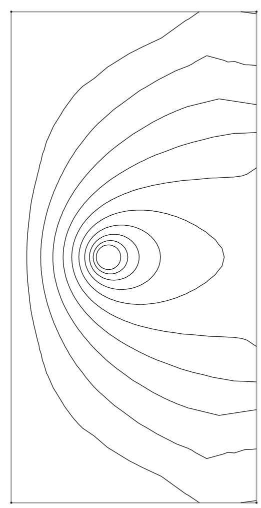

Figure 2: Isotherms. $\delta T = 0.1$, $Nu_m = 3.69$.

$$D_0 = \frac{-3}{h^2 q} \oint \frac{1}{rJ} d\beta + \frac{1}{h^2} \oint \left\{ \frac{1}{r^2} \frac{\partial}{\partial \alpha} \left(\frac{r}{J} \right) + \frac{1}{r^2} \frac{r_a}{J} \right\} d\beta$$
$$- \frac{1}{h} \oint \left\{ \frac{1}{rJ} \frac{\partial}{\partial \alpha} \left(\frac{r_a}{r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \alpha} \left(\frac{r_a}{J} \right) \right\} d\beta$$

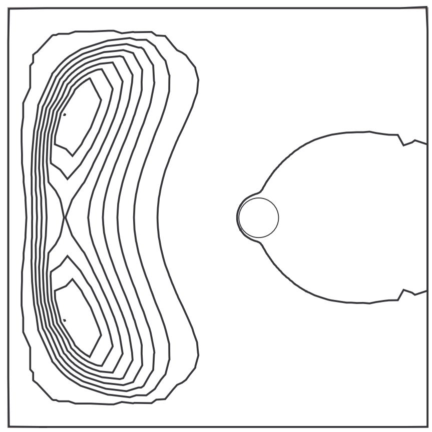

Figure 3: Streamlines. $\delta\psi = 5.0 \times 10^{-5}$, $C_L = 8.8 \times 10^{-4}$

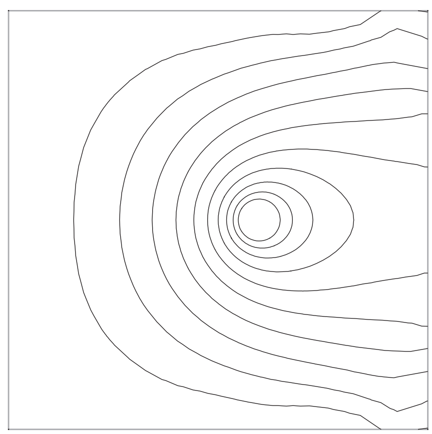

Figure 4: Isotherms. $\delta T = 0.1$, $Nu_m = 2.21$

$$+ \oint \left[\frac{1}{r} \left\{ \frac{\partial}{\partial \alpha} \left(\frac{r_a}{r^2} \right) + \frac{2J}{r^3} \right\} \frac{\partial}{\partial \alpha} \left(\frac{r}{J} \right) + \frac{r_a}{r^4} - \frac{r_b}{r^3} \frac{\partial}{\partial \alpha} \left(\frac{r_b}{J} \right) \right] d\beta \,, \quad (30)$$

$$D_{1n} = \oint \frac{\cos n\beta}{J} \, d\beta \,, \quad (31)$$

$$D_{2n} = \oint \cos n\beta \left\{ \frac{1}{r} \frac{\partial}{\partial \alpha} \left(\frac{r}{J} \right) + \frac{1}{r} \frac{r_a}{J} \right\} d\beta \,, \quad (32)$$

$$D_{3n} = \oint \left\{ \frac{-n^2}{J} - \frac{1}{r^2} + \frac{1}{r} \frac{\partial}{\partial \alpha} \left(\frac{r_a}{J} \right) \right\} d\beta \,, \quad (33)$$

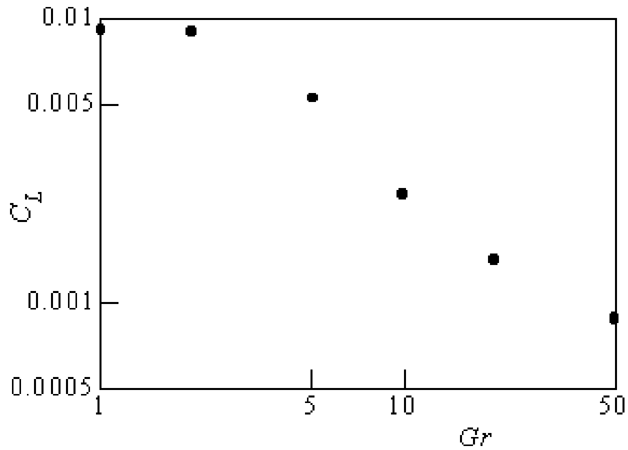

Figure 5: Lift coeficient C_L vs. Gr

$$D_{4n} = \oint n \sin n\beta \, \frac{r_b}{rJ} \, d\beta \,, \tag{34}$$

$$q \equiv \alpha_3 - \alpha_0 = h \left(2 \cosh \gamma + 1 \right) \,. \tag{35}$$

In the preceding expressions $\oint$ stands for integration along $\alpha = \alpha_0$, and

$$r_a \equiv \frac{\partial r}{\partial \alpha} \,, \tag{36}$$

$$r_b \equiv \frac{\partial r}{\partial \beta} \,. \tag{37}$$

4. Time Integration Scheme

Given a stationary and uniform (zero) thermal field, the discretized equations can be integrated with respect to time semi-implicitly to produce a steady-state solution, see [1], [2].

5. Characteristics

Mean Nusselt number (averaged over the near torus surface, based on the reference length L), Nu_m, is given by

$$Nu_m = -\oint r\frac{\partial T}{\partial \alpha}d\beta \Big/ \oint r\sqrt{J}d\beta\,. \tag{38}$$

Lift coefficient C_L is given by

$$\begin{aligned} C_L &\equiv \frac{\text{Lift on the near torus} - \text{(stationary buoyancy)}}{\rho U^2 L^2} \\ &= \pi \oint \left(r^2\frac{\partial \zeta}{\partial \alpha} - r\zeta\frac{\partial r}{\partial \alpha}\right) d\beta \Big/ \sqrt{Gr}\,, \end{aligned} \tag{39}$$

where ρ is the density of the fluid.

6. Numerical Results

The following numerical results correspond to the configuration

$$\frac{2\ \mathrm{e}^{\alpha_0}}{\sqrt{1+a^2}\ \sqrt{1+k^2/a^2}} = \frac{1}{20}K'\,, \tag{40}$$

which means horizontal diameter of the near torus $\approx (1/10)$ of the radius of the cavity. Figures 1 and 2 are streamlines and isotherms in the left half section at $Pr = 0.7$ (air), $Gr = 50, K'/K = 1, a = 0.6$, where $\gamma \approx 0.020888, (h \approx 0.07522), M = 30$.

Figures 3 and 4 are streamlines and isotherms in the left half section at $Pr = 0.7$ (air), $Gr = 50, K'/K = 2, a = 0.6$, where $\gamma \approx 0.020888, (h \approx 0.07101), M = 30$.

Figure 5 shows dependence of the lift coefficient on Gr at $Pr = 0.7$, $K'/K = 2$, $a = 0.6$.

7. Conclusions

One concrete form for multiply-connectedness is proposed. Application of a spectral finite difference scheme to the current multiply-connected region in natural convection is found to be effective.

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