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ON LEFT NEGATIVELY ORDERED RPP SEMIGROUPS

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Abstract: A semigroup S equipped with a partial order " $\leq$ " is called a left negatively ordered semigroup if $ab \leq a$, for all $a, b \in S$. In this paper, we study the rpp semigroups equipped with the left negative Lawson partial order " $\leq_\ell$ ". In particular, we establish a structure theorem for the left negatively ordered rpp semigroups on which the Green relation $\mathcal{L}$ is a congruence. Some results recently obtained by Guo and Shum on rpp semigroups equipped with the Lawson negative partial order " $\leq_\ell$ " are extended and generalized.

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1. Introduction

A semigroup S is called a *right pp semigroup*, in short, a *rpp semigroup*, if for all

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$a \in S$, the right ideal aS^1, regarded as a right S^1-system, is projective. Dually, we can define the left pp semigroup (lpp semigroup). It was first pointed out by Fountain in [3] and [4] that a semigroup S is a rpp semigroup if and only if every $\mathcal{L}^*$-class of S contains at least one idempotent. Following Fountain [4], a semigroup S is called *abundant* if S is both rpp and lpp. It is easy to see that all regular semigroups are abundant.

In studying semigroups, Lawson [9] first considered several kinds of natural partial orders on an abundant semigroup S, say " $\leq_\ell$ ", " $\leq_r$ " and " $\leq$ ". These three kinds of natural partial orders are analogous to the natural partial order on regular semigroups, as introduced by K.S.S. Nambooripad in [10]. Indeed, we can always define the Lawson partial order " $\leq_\ell$ " on rpp semigroups. Several classes of rpp (lpp) semigroups equipped with the Lawson partial order " $\leq_\ell$ " have been recently studied by the authors in [5] and [7].

Let $\leq$ be a partial order on a semigroup S. Then we call S a *left negatively* (*right negatively*) *ordered semigroup* if $ab \leq a$ ($ab \leq b$), for all $a, b \in S$. Moreover, we simply call S a *negatively ordered semigroup* if S is both left negatively and right negatively ordered. The semigroups with negative partial order were first studied by Fidalgo Maia and Mitsch [2]. In this paper, we consider the rpp semigroups equipped with a negative Lawson partial order " $\leq_\ell$ ". The structure of the negatively Lawson ordered rpp semigroups will be investigated and the negatively Lawson adequate ordered semigroup equipped with the Lawson partial order " $\leq_\ell$ " will be characterized.

Throughout this paper, we shall use the terminologies and notations given by Fountain [3] and Howie [8]. For the sake of convenience, we let $E(S)$ to denote the set of idempotents of a semigroup S. Also, we denote by a^* the idempotents in $L^*_a \bigcap E(S)$ and by $a^\dagger$ the idempotents in $R^*_a \bigcap E(S)$.

2. Preliminaries

We first cite some well known definitions and results on rpp semigroups and abundant semigroups.

Definition 2.1. Let x, y be elements of an rpp semigroup S. Define the following Lawson partial order " $\leq_\ell$ " on S by

> $x \leq_\ell y$ if and only if $x \leq_{\mathcal{L}^*} y$ and there exists $f \in E(S) \bigcap L^*_x$ such that $x = yf$.

By using the same arguments as given by Lawson in [9] (see Lemma 2.1, Lemma 2.2, Proposition 2.5 and Proposition 2.7 in [9], also see [5]), we have the following lemma.

Lemma 2.2. (see [5]) *Let S be an rpp semigroup. Then, for $x, y \in S$ and $e \in E(S)$, the following statements holds:*

(i) " $\leq_\ell$ " is a partial order on S such that " $\leq_\ell$ " coincides with the usual idempotent order ω on $E(S)$, that is, $e\omega f$ if and only if $e = ef = fe$.

(ii) if $x \leq_\ell e$, then $x^2 = x$ in S.

(iii) if $x \leq_\ell y$ and y is a regular element in S, then x is also a regular element in S.

(iv) Let $y^* \in L_y^* \bigcap E(S)$ and $\omega(y^*) = \{f : f\omega y^*\}$. Then $x \leq_\ell y$ if and only if for all y^*, there exists an idempotent $f \in \omega(y^*)$ such that $x = yf$.

(v) If $x \leq_\ell y$ and $x\mathcal{L}^*y$, then $x = y$.

Let S be a semigroup and $a \in S$. Then we call S a *π-regular semigroup* if for any $a \in S$, there exists a positive integer m such that $a^m \in a^m S a^m$. The element a of S is called *completely regular* if there is an element x in S such that $a = axa$ and $ax = xa$, that is, $a\mathcal{H}e$ for some $e \in E(S)$. A semigroup S is called a *GV-semigroup* if S is π-regular and every regular element of S is completely regular. We call a semigroup S a *left group* if S is left simple semigroup satisfying the right cancellative law, i.e., it is isomorphic to the direct product of a left zero band and a group.

It is well known that a semigroup S with zero 0 is called a *nil-semigroup* if for all $x \in S$, there exists a positive integer n such that $x^n = 0$. A semigroup S is called a *nil-extension of the semigroup of type $\mathcal{P}$* if there exists a subsemigroup I of type $\mathcal{P}$ of S such that I is an ideal of S and S/I is a nil-semigroup.

The following result was due to Bogdanović [1].

Lemma 2.3. (see [1]) *Let S be a semigroup. Then S is a semilattice of nil-extensions of left groups and $E(S)$ is a subsemigroup of S if and only if S is a GV-semigroup and $E(S)$ is a left regular band.*

3. Characterizations of Negatively Lawson Ordered Semigroups

In this section, we give some characterization theorems for the (left; right) negatively ordered Lawson rpp semigroups, that is, S is an rpp semigroup equipped with a (left; right) negative Lawson partial order " $\leq_\ell$ ".

Proposition 3.1. *Let S be a left negatively ordered Lawson rpp semigroup. Then the following statements hold:*

(1) $E(S) = \{a^2 : a \in S\}$;

(2) $E(S)$ is a left regular band (a band satisfying the identity $xy = xyx$) and $E(S)$ is also a right ideal of S;

(3) $RegS$ (the set of regular elements of S)$= E(S)$;

(4) S is a GV-semigroup;

(5) S is a semilattice of nil-extension of a left zero band.

Proof. (1) Let $a \in S$. Then $a^2 \leq_\ell a$. Now, by Lemma 2.2, there exists $e \in E(L^*_{a^2})$ such that $a^2 = ae$. Hence $a^3 = a^2e = a^2$ so that $a^4 = a^2 \in E(S)$. Thus $\{a^2 : a \in S\} \subseteq E(S)$. The reverse inclusion is trivial. Hence, (1) is proved.

(2) Let $a \in S$ and $e \in E(S)$. Then, because S is left negatively ordered semigroup, $ea \leq_\ell e$ and by Lemma 2.2, we have $ea \in E(S)$. This shows that $E(S)$ is a right ideal of S. On the other hand, if $e, f \in E(S)$, then $ef \in E(S)$. Hence $ef = (ef)^2 = efef \leq_\ell efe \leq_\ell ef$ and clearly $efe = ef$. Thus $E(S)$ is indeed a left regular band.

(3) It is obvious that $E(S) \subseteq RegS$. If $a \in RegS$, then there exists $b \in S$ such that $aba = a$. Obverse that $ab \in E(S)$, by (2), we obtain $a = (ab)a \subseteq E(S)S \subseteq E(S)$. Hence $RegS \subseteq E(S)$. Thus $RegS = E(S)$.

(4) By (1), S is a π-regular semigroup. Since the idempotents of S are completely regular, by (3), we know that every regular element of S is completely regular and therefore, S must be a GV-semigroup.

(5) By (2) and (4), using Lemma 2.3, we can see immediately that S is a semilattice of nil-extensions of left groups and $E(S)$ is a subsemigroup of S. But since $RegS = E(S)$, each subsemigroup of S which is a left group must be a left zero band, and hence S is a semilattice of nil-extensions of left zero bands.This completes the proof. □

We now formulate the following theorem.

Theorem 3.2. *The following statements are equivalent on a rpp semigroup S:*

(1) S is a left negatively Lawson ordered semigroup;

(2) $E(S)$ is a right ideal of S and is also a left regular band;

(3) S is a semilattice of nil-extensions of left zero bands and $E(S)$ is a right ideal of S.

Proof. (1)⇒(2) and (1)⇒(3) follow directly from Proposition 3.1.

(2)⇒(1) Suppose that (2) holds. Then we let $a, b \in S$. Since S is a rpp semigroup, there exists $e \in E(S)$ such that $e\mathcal{L}^*a$. By noting that $\mathcal{L}^*$ is a right congruence on S, we have $eb\mathcal{L}^*ab$. But since $E(S)$ is a right ideal, $eb \in E(S)$. Note that $E(S)$ is a left regular band. We have $eb = e(eb) = e(eb)e = (eb)e$

and $eb \in \omega(e)$. On the other hand, since $a\mathcal{L}^*e$, $a = ae$ and $ab = a(eb)$. Thus $ab \leq_\ell a$. Therefore, S is a left negatively Lawson ordered semigroup.

(3)⇒(1) Suppose that (3) holds. Then, by using the same argument as above , we can easily show that (1) is equivalent to (2). Now, we need to verify that $E(S)$ is a left regular band. For this purpose, we simply let $S = \bigcup_{\alpha\in Y} S_\alpha$ be the semilattice decomposition of S satisfying the conditions in (3). It is now easy to see that $E(S) = \bigcup_{\alpha\in Y} E(S_\alpha)$. Hence $E(S)$ is a union of left zero bands. Thus, in proving $E(S)$ is a left regular band, we only need to show that $E(S)$ is a band,however, this follows from the fact that $E(S)$ is a right ideal of S. □

By a *right adequate semigroup*, we mean a rpp semigroup whose set of idempotents forms a semilattice. We have the following corollary.

Corollary 3.3. *The following statements are equivalent on a right adequate semigroup S:*

(1) S is a left negatively Lawson ordered semigroup;

(2) $E(S)$ is a right ideal of S;

(3) S is a semilattice of nil-semigroups and $E(S)$ is a right ideal of S.

Recall from Lawson [9] and also from Guo-Luo [6] that an abundant semigroup is idempotent-connected if and only if " $\leq_\ell$ " = " $\leq$ " = " $\leq_r$ ". Since the regular semigroups are idempotent-connected abundant semigroups, it is well known that " $\leq_\ell$ " = " $\leq$ " = " $\leq_r$ " on such semigroups. Now let S be an abundant semigroup. Then for all $a \in S$, there exists $e \in E(S)$ such that $e\mathcal{R}^*a$. Hence $a = ea$. Based on the above arguments, we can easily show that $S = E(S)$ if S is left negatively ordered semigroup under the Lawson partial order " $\leq_\ell$ ". Thus, by the above fact and Theorem 3.2, we obtain the following corollary.

Corollary 3.4. *The following statements are equivalent on an abundant semigroup S:*

(1) S is a left negatively Lawson ordered semigroup equipped with the partial order " $\leq_\ell$ ";

(2) S is a left regular band;

(3) S is a left negatively Lawson ordered semigroup under the Lawson partial order " $\leq$ ".

It is well known that a band is a semilattice if and only if it is both a left regular band and a right regular band. By Corollary 3.4, we immediately deduce the following theorem for abundant semigroups.

Theorem 3.5. *The following statements are equivalent on an abundant semigroup S:*

(1) S is a negatively Lawson ordered semigroup equipped with the Lawson partial order " $\leq_\ell$ ";

(2) S is a negatively Lawson ordered semigroup equipped with the Lawson partial order " $\leq_r$ ";

(3) S is a negatively Lawson ordered semigroup equipped with the Lawson partial order " $\leq$ ";

(4) S is a semilattice.

By the definition of the Lawson partial order " $\leq_\ell$ " on an rpp semigroup S, the partial order " $\leq_\ell$ " on S is not necessarily left-right symmetric. Hence, it is natural to ask when will a rpp semigroup admit a right negative Lawson partial order " $\leq_\ell$ " but " $\leq_\ell$ " itself is not left negative? (We notice that the "right negativity" and the "left negativity" are not dual concepts on rpp semigroups). We now devote to this kind of rpp semigroups.

Theorem 3.6. *Let S be a rpp semigroup. Then S is a right negatively Lawson ordered semigroup equipped with the partial order " $\leq_\ell$ " if and only if S itself is a right regular band.*

Proof. Suppose that S is a right negatively Lawson ordered semigroup equipped with " $\leq_\ell$ ". Now let $a \in S$. Then $a = aa^* \leq_\ell a^*$ and by Lemma 2.2, $a \in E(S)$. Hence S is a band. On the other hand, if $x, y \in S$, then $xy = (xy)^2 \leq_\ell yxy \leq_\ell xy$ and so $xy = yxy$. Thus, S is a right regular band. Conversely, if S is a right regular band, then $xy = y \circ xy = xy \circ y$, for all $x, y \in S$, and so $xy\omega y$. Obviously, $xy \leq_\ell y$. This shows that S is a right negatively Lawson ordered semigroup equipped with the partial order " $\leq_\ell$ ". □

4. A Structure Theorem

We finally consider a special rpp semigroup equipped with a left negative Lawson partial order " $\leq_\ell$ ".

Definition 4.1. Let S be a negatively Lawson ordered rpp semigroup equipped with " $\leq_\ell$ ". Then, S is called a *strictly left negatively Lawson ordered semigroup* equipped with " $\leq_\ell$ " if the relation $\mathcal{L}$ is also a left congruence on S.

Evidently, all left regular bands are left negatively Lawson ordered semigroups equipped with " $\leq_\ell$ ". In fact, there exist plenty of non-regular rpp semigroups equipped with the strictly left negative Lawson partial order $\leq_\ell$, for example, the right adequate semigroups equipped with the left negative

Lawson partial order " $\leq_\ell$ " are strictly left negatively Lawson ordered semigroup equipped with " $\leq_\ell$ ". In order to prove this claim, we let $S = \bigcup_{\alpha\in Y} M_\alpha$ be the semilattice decomposition of the right adequate semigroup S equipped with a left negative Lawson partial order " $\leq_\ell$ " into the nil-semigroups M_α with a zero element 0_α. Now let $a, b \in S$ and $a\mathcal{L}b$. Then $a, b \in M_\alpha$, for some $\alpha \in Y$. It is easy to see that either $a, b \in M_\alpha\backslash\{0_\alpha\}$ and $a \neq b$ or $a = b$. If $a, b \in M_\alpha\backslash\{0_\alpha\}$ and $a \neq b$, then there exist $x \in S_\beta$ and $y \in S_\gamma$ such that $a = xb$ and $b = ya$. These equalities imply that $\alpha \leq \beta$, and so $x^* \in M_\mu$, for some $\beta \leq \mu$. Note that $\alpha \leq \beta \leq \mu$. Thus $x^*b \in M_\mu M_\alpha \subseteq M_\alpha$ and $x^*b = 0_\alpha$. Thus $a = xb = xx^*b = x0_\alpha = 0_\alpha$, contrary to $a \neq 0_\alpha$. This shows that $\mathcal{L} = \Delta_S$, where Δ_S is the identity relation on S. Consequently, $\mathcal{L}$ is a congruence on S. Thus, S is indeed a strictly left negative Lawson semigroup equipped with " $\leq_\ell$ ".

Proposition 4.2. *Let S be a strictly left negatively Lawson ordered rpp semigroup equipped with the Lawson partial order " $\leq_\ell$ ". Then the following statements hold:*

(1) If $a, b \notin E(S)$ such that $a\mathcal{L}b$, then $a = b$;

(2) $\mathcal{L}$ preserves the $\mathcal{L}^*$-classes;

(3) $S/\mathcal{L}$ is a right adequate semigroup equipped with a left negative Lawson order " $\leq_\ell$ ".

Proof. (1) Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be the semilattice decomposition of S into the nil-extensions S_α of left zero bands E_α. Without loss of generality, we let $a \in S_\alpha$ and $b \in S_\beta$. Suppose on the contrary that $a \neq b$. Then there exist $x \in S_\gamma$ and $y \in S_\mu$ such that $a = xb$ and $b = ya$. These equalities imply that $\alpha \leq \beta$, $\alpha \leq \gamma$, $\beta \leq \alpha$ and $\beta \leq \mu$. Hence $\alpha = \beta$. By the above proof before Proposition 4.2, $x^*b \in E_\beta$. Since $\gamma \geq \alpha$, we have $xb = x(x^*b) = x(x^*b)(x^*b) \in S_\beta E_\beta \subseteq E_\beta$ and so $a = xb \in E_\alpha$. This is a contradiction and hence, $a = b$.

(2) It suffices to verify that $(a\mathcal{L})\mathcal{L}^*(a^*\mathcal{L})$, for all $a \in S$. Assume that $(a\mathcal{L})(x\mathcal{L}) = (a\mathcal{L})(y\mathcal{L})$ for $x, y \in S^1$, that is, $ax\mathcal{L}ay$. Then by (1), $ax = ay$ or $ax, ay \in E(S)$.

— If $ax = ay$ holds, then this equality leads to $a^*x = a^*y$ since $a\mathcal{L}a^*$, and hence $(a^*\mathcal{L})(x\mathcal{L}) = (a^*\mathcal{L})(y\mathcal{L})$.

— Suppose that $ax, ay \in E(S)$ holds. Since $\mathcal{L}$ is a right congruence on S, giving that $ax\mathcal{L}a^*x$ and $ay\mathcal{L}a^*y$, we have $a^*x\mathcal{L}^*a^*y$ and $a^*x\mathcal{L}a^*y$. Hence $a^*x\mathcal{L}a^*y$. Thus $(a^*\mathcal{L})(x\mathcal{L}) = (a^*\mathcal{L})(y\mathcal{L})$.

We have now proved that $(a\mathcal{L})(x\mathcal{L}) = (a\mathcal{L})(y\mathcal{L})$ which implies that $(a^*\mathcal{L})(x\mathcal{L}) = (a^*\mathcal{L})(y\mathcal{L})$. By this result together with the fact that $(a\mathcal{L})(a^*\mathcal{L}) = (aa^*)\mathcal{L} = a\mathcal{L}$, we can easily deduce that $(a\mathcal{L})\mathcal{L}^*(a^*\mathcal{L})$.

(3) By (1), $\mathcal{L}$ is an idempotent-pure and $\mathcal{L} = \Delta_{S\backslash E(S)} \bigcup \mathcal{L}^{E(S)}$, where $\Delta_{S\backslash E(S)}$ is the identity relation on $S\backslash E(S)$. Hence, again by (2), $S/\mathcal{L}$ is a right adequate semigroup. Let $a, b \in S$. Then $ab \leq_\ell a$ since S is left negative Lawson semigroup equipped with $\leq_\ell$. Hence there exists $f \in E(S)$ such that $ab = af$, and thereby, $(a\mathcal{L})(b\mathcal{L}) = (a\mathcal{L})(f\mathcal{L})$. This leads to $(a\mathcal{L})(b\mathcal{L}) \leq_\ell (a\mathcal{L})$. Thus, $S/\mathcal{L}$ is a right adequate semigroup equipped with the Lawson partial order " $\leq_\ell$ ". □

We next give a construction theorem for the right adequate semigroups equipped with a left negative partial order " $\leq_\ell$ ".

Theorem 4.3. *Let S be a right adequate semigroup. Then S is a left negatively Lawson ordered semigroup equipped with the Lawson partial order " $\leq_\ell$ " if and only if S is a semilattice of nil-semigroups M_α with zero element 0_α with $\alpha \in Y$ and $0_\alpha M_\beta = \{0_{\alpha\beta}\}$, for all $\alpha,\ \beta \in Y$.*

Proof. In view of Corollary 3.3, the proof is only a matter of routine verification. We omit the details. □

Let Y be a semilattice and $T = \bigcup_{\alpha \in Y} M_\alpha$ the semilattice decompsoition of the right adequate semigroup T equipped with left negative Lawson partial order " $\leq_\ell$ " into the nil-semigroups M_α with a zero element 0_α for $\alpha \in Y$. Let $E = \bigcup_{\alpha \in Y} E_\alpha$ be the semilattice decomposition of the left regular band E into left zero bands E_α with $\alpha \in Y$. Putting $S_\alpha = (M_\alpha \backslash \{0_\alpha\}) \bigcup E_\alpha$ and write $S = (T\backslash\{0_\alpha : \alpha \in Y\}) \bigcup E$. Obviously, $S = \bigcup_{\alpha \in Y} S_\alpha$. We now use $\mathcal{T}(S)$ to denote the semigroup of mappings (on the left) from S into itself. Suppose that the mapping

$$\Psi : \ S \to \mathcal{T}(S); \quad s \mapsto \psi_s,$$

where $\psi_s : S \to S; \quad x \mapsto {}^s x$, satisfies the following conditions:

(S1) if $s \in S_\alpha$ and $x \in S_\beta$, then ${}^s x \in S_{\alpha\beta}$.

(S2) if $s \in M_\alpha \backslash \{0_\alpha\}$ and $x \in M_\beta \backslash \{0_\beta\}$ with $sx \notin E(T)$, then ${}^s x = sx$.

(S3) if $s \in T\backslash\{0_\alpha : \alpha \in Y\}$ and ${}^s x = t \notin E$ for some $x \in E_\beta$, then ${}^s y = t$ for all $y \in S_\beta$.

(S4) if $s \in T\backslash\{0_\alpha : \alpha \in Y\}$ and ${}^s x = t \in E(T)$ for some $x \in E_\beta$, then ${}^s y \in E$ for all $y \in S_\beta$.

(S5) if $e \in E$ and $x \in S$, then ${}^e x \in E$. In addition, if $x \in E$, then ${}^e x = ex$.

(S6) for all $s, t \in S$, $\psi_s \psi_t = \psi_{{}^s t}$.

(S7) for all $s \in T\backslash\{0_\alpha : \alpha \in Y\}$, if $s\mathcal{L}^* 0_\alpha$, then

(S71) ${}^s x = s$ for all $x \in E_\alpha$.

(S72) for all $x, y \in E$ and $e \in E_\alpha$, ${}^s x = {}^s y$ implies that ${}^e x = {}^e y$.

Define a multiplication "$\star$" on S by

$$s \star t = {}^{s}t \qquad (s, t \in S).$$

Now, it is clear that "$\star$" is well defined.

Lemma 4.4. *$(S, \star)$ is a semigroup.*

Proof. Let $a, b, c \in S$. Then

$$(a \star b) \star c = {}^{a}b \star c = \psi_{{}^{a}b}(c) = \psi_a \psi_b(c) = \psi_a(b \star c) = a \star (b \star c).$$

This shows that "$\star$" satisfies the associative law. □

Definition 4.5. We call the above semigroup $(S, \star)$ the union product of the right adequate semigroup T equipped with a left negative Lawson partial order "$\leq_\ell$" and the left regular band E with respect to the structure mapping Ψ. We shall denote this semigroup by $LNLO(T, E; \Psi)$.

Proposition 4.6. *The following conditions hold in the semigroup $LNLO(T, E; \Psi) = (S, \star)$:*

(1) $E(S) = E$ is a left regular band and also a right ideal of S.

(2) if $s \in T \backslash \{0_\alpha : \alpha \in Y\}$ and $s\mathcal{L}^* 0_\beta$ in the semigroup T, then $s\mathcal{L}^* e$, for all $e \in E_\beta$.

(3) S is a left negatively ordered semigroup under the Lawson partial order "$\leq_\ell$".

(4) $\mathcal{L}$ is a congruence on S.

Proof. (1) By (S5), $E \subseteq E(S)$. Now let $s \in E(S) \backslash E$. Then ${}^{s}s = s$ and by (S2), $s^2 = s$. It follows that $s = 0_\alpha$ for some $\alpha \in Y$. This leads to $s \notin S$, a contradiction. Thus $E(S) = E$ as sets. By a routine calculation, $E(S) \cong E$ is a left regular band. On the other hand, we can show that $E(S)$ is a right ideal of S by (S5).

(2) Suppose that $s \in S$ satisfy the conditions in (2), and $e \in E_\beta$. Then, by (S71), $s \star e = s$. If $x, y \in S$ and $s \star x = s \star y$, then $s \star (e \star x) = s \star (e \star y)$ and by (S72), $(e \star x =)\ e \star (e \star x) = e \star (e \star y)\ (= e \star y)$. If $s \star x = s$, then by using the arguments in the above proof, $s \star x = s \star e$ and so $e \star x = e \star e = e$. Thus, $s\mathcal{L}^* e$ and (2) is proved.

(3) By (2), S is a rpp semigroup and hence by (1) and Theorem 3.2, S is left negatively Lawson ordered semigroup equipped with "$\leq_\ell$".

(4) If $a \in RegS$, then there exists $b \in S$ such that $a \star b \star a = a$. It follows that $a \star b \in E(S) = E$. By (S4), $a = a \star b \star a \in E$. Thus $RegS = E$. Now let

$a, b, x \in S$ and $a\mathcal{L}b$. Then $a, b \in E$ or $a, b \notin E$. We now consider the following two cases:

(a) $a, b \notin E$. By (S5), $a\mathcal{L}^T b$ and by Proposition 4.2, $a = b$. Obviously, $(x \star a)\mathcal{L}(x \star b)$.

(b) $a, b \in E$. Then $a\mathcal{L}^E b$ and $ab = a, ba = b$ in E.

— If $x \star a \in E_\alpha$ for $\alpha \in Y$, then by (S4) and (S5), $x \star b, a \star x \in E_\alpha$. Hence

$$\begin{aligned}(x \star a) \star (x \star b) &= x \star (a \star x) \star b = x \star a \star (a \star x) \star b \\ &= x \star a \star (a \star x) \star a \star b = x \star a \star (a \star x) \star a \\ &= (x \star a) \star (x \star a) = x \star a\end{aligned}$$

and similarly, we have $(x \star b) \star (x \star a) = (x \star b)$. It follows that $(x \star a)\mathcal{L}(x \star b)$.

— If $x \star a \notin E$ and $x \star a \in M_\alpha$ for $\alpha \in Y$, then by (S4), $x \star b \notin E$ and by (S1), $x \star b \in M_\alpha$. By (S3), $x \star a = x \star b$. That is, $(x \star a)\mathcal{L}(x \star b)$.

All the above show that $(x \star a)\mathcal{L}(x \star b)$. Hence, $\mathcal{L}$ is a left congruence on S. This proves (4). □

By Proposition 4.6, $LNLO(T, E; \Psi)$ is a strictly left negatively Lawson ordered semigroup equipped with " $\leq_\ell$ ". We still need to prove that any rpp semigroup equipped with a strictly left negative Lawson partial order " $\leq_\ell$ " is isomorphic to some $LNLO(T, E; \Psi)$. For this purpose, we let $S = \bigcup_{\alpha \in Y} S_\alpha$ be the semilattice decomposition of S into nil-extensions S_α of left zero bands E_α. Then $E = \bigcup_{\alpha \in Y} E_\alpha$ is the semilattice decomposition of the left regular band E of idempotents of S into left zero bands E_α. Hence, $S/\mathcal{L} = \bigcup_{\alpha \in Y} S_\alpha/\mathcal{L}^{S_\alpha}$ is a right adequate semigroup with the left negative Lawson partial order $\leq_\ell$. We now use 0_α to denote the zero element of $S_\alpha/\mathcal{L}^{S_\alpha}$. Then by Proposition 4.2, $S_\alpha/\mathcal{L}^{S_\alpha} = (S_\alpha \backslash E_\alpha) \bigcup \{0_\alpha\}$. Obviously, $S = (S/\mathcal{L} \backslash \{0_\alpha : \alpha \in Y\} \bigcup E$.

Define

$$\phi_s : S \to S; \qquad x \mapsto {}^s x = sx \qquad (s, x \in S),$$

and $\Phi : S \to \mathcal{T}(S); \quad s \mapsto \phi_s$.

Lemma 4.7. *The above triple $(S/\mathcal{L}, E; \Phi)$ satisfies conditions (S1-S7).*

Proof. The conditions (S1), (S2) and (S6) hold trivially. Also, condition (S5) follows from the fact that E is a right ideal of S.

Now let $x, y \in E_\beta$ and $s \in S$. Then $x\mathcal{L}y$. Since $\mathcal{L}$ is a congruence on S, $sx\mathcal{L}sy$.

— If ${}^s x = sx \notin E$, then $sy \notin E$. Otherwise, $sy \in E$ and $sx \in RegS$. By Proposition 3.1 (3), $sx \in E$. This is a contradiction. By Proposition 4.2, $sx = sy$. That is, ${}^s x = {}^s y$. This shows that condition (S3) holds.

— If ${}^{s}x = sx \in E$, then by the above proof ${}^{s}y = sy \in E$. Hence condition (S4) also holds.

Because S is a rpp semigroup, there exists $\beta \in Y$ such that $s\mathcal{L}^*e$ for all $e \in E_\beta$. By Proposition 4.2, $s\mathcal{L}^*0_\beta$ in the semigroup $S/\mathcal{L}$. On the other hand, $s\mathcal{L}^*e$ implies that $se = s$, that is, ${}^{s}e = s$. Now let $x, y \in E$ and ${}^{s}x = {}^{s}y$, i.e., $sx = sy$. Then $ex = ey$, i.e. ${}^{e}x = {}^{e}y$, since $x\mathcal{L}^*e$. If $s\mathcal{L}^*0_\alpha$, then $0_\alpha = 0_\beta$ since $S/\mathcal{L}$ is a right adequate semigroup, and thereby, $\alpha = \beta$. Thus condition (S7) holds. □

Theorem 4.8. $S \cong LNLO(S/\mathcal{L}, E; \Phi)$.

Proof. By a routine computation, we can show that the identical mapping is an isomorphism of S onto the semigroup $LNLO(S/\mathcal{L}, E; \Phi)$. □

We now arrive at the main result of this section.

Theorem 4.9. *Any union product $LNLO(T, E; \Psi)$ of the right adequate semigroup T equipped with the left negative Lawson partial order " $\leq_\ell$ " together with a left regular band E with respect to the structure mapping Ψ is a rpp semigroup equipped with a strictly left negative Lawson partial order " $\leq_\ell$ ".*

Conversely, any rpp semigroup equipped with a strictly left negative Lawson partial order " $\leq_\ell$ " can be constructed in the above manner.

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