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A REGULARIZED TRACE FORMULA FOR A DIFFERENTIAL OPERATOR OF SECOND ORDER WITH UNBOUNDED OPERATOR COEFFICIENTS GIVEN IN A FINITE INTERVAL

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Abstract: In this work, a formula for the regularized trace of second order differential operator given in a finite interval which has unbounded operator coefficients is found.

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1. Introduction

Let H be a separable Hilbert space and let $H_1 = L_2(H;[0,\pi])$ denotes the set of all measurable functions f with values in H and such that

$$\int_0^\pi \|f(x)\|_H^2 \, dx < \infty.$$

We consider the operators L_0 and L in H_1 which are formed by the differential expressions

$$l_0(y) = -y''(x) + Ay(x) \quad \text{and} \quad l(y) = -y''(x) + Ay(x) + Q(x)y(x)$$

and the same boundary conditions $y(0) = y'(\pi) = 0$ respectively. Suppose that A and $Q(x)$ in the above expressions satisfy the following conditions:

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(1) $A : D(A) \to H$ is a self adjoint operator. Moreover, $A \geq I$ and $A^{-1} \in \sigma_\infty(H)$, where I is an identity operator in H and $\sigma_\infty(H)$ is the set of all compact operators from H to H.

(2) For every $x \in [0,\pi]$, $Q(x) : H \to H$ is a self-adjoint compact operator. It is also a kernel operator $(Q(x) \in \sigma_1(H))$.

(3) The functions $\|Q^{(i)}(x)\|_{\sigma_1(H)}$ $(i = 0,1,2)$ are bounded and measurable in the interval $[0,\pi]$.

(4) For every $f \in H$, $\int_0^\pi (Q(x)f,f)_H \, dx = 0$.

We denote the norms in H and H_1 by $\|.\|_H$ or $\|.\|$ and $\|.\|_1$ respectively and denote the sum of eigenvalues of a kernel operator K by $\mathrm{tr}K = \mathrm{trace}K$. Moreover, $\sigma_1(H)$ denotes the space of kernel operators from H to H as in Cohberg and Krein [6].

The regularized trace formulas of scalar differential operators are studied by Gelfand and Levitan [9], Dikiy [7], Halberg and Kramer [12] and in many other works. In particular, the list of the works on this subject is given in Levitan Sargsyan [13] and Fulton and Prues [8].

The trace formulas for differential operators with operator coefficients are investigated by Adıgüzelov [1], Chalilova [5], Maksudov et al [14], Maksudov et al [15], Adıgüzelov et al [2], Albayrak et al [4], Gül [11], Adıgüzelov and Bakşi [3].

In this work we will firstly show how the concept of regularized trace for operator L is constructed and later will obtain a formula for this regularized trace.

2. Definition of Regularized Trace for Operator L

Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$ be the eigenvalues of the operator A and $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ be the orthonormal eigenvectors corresponding to these eigenvalues.

Moreover, D_0 denotes the set of the functions $y(x)$ in H_1 satisfying the conditions:

(1^o) $y(x)$ has continuous derivative of the second order with respect to the norm in the space H in the interval $[0,\pi]$.

(2^o) $Ay(x)$ is continuous with respect to the norm in the space H.

(3^o) $y(0) = y'(\pi) = 0$.

Here D_0 is dense in H_1 and the operator $L_0' : D_0 \to H_1$ defined by $L_0' = l_0(y)$ is symmetric. The eigenvalues of this operator are $(\frac{1}{2}+k)^2+\gamma_j$ $(k =$

$0, 1, 2, \cdots ; j = 1, 2, \cdots)$ and the orthonormal eigenvectors corresponding to these eigenvalues are $M_k \sin(k + \frac{1}{2})x.\varphi_j \quad (k = 0, 1, 2, \cdots ; j = 1, 2, \cdots)$, where $M_k = \sqrt{\frac{2}{\pi}}$ for $k = 0, 1, 2, \cdots$.

We can see that the orthonormal eigenvectors system of the symmetric operator L_0' is an orthonormal basis in the space H_1. Moreover, since this system is closed, the operator $L_0 = \overline{L_0'}$ is self-adjoint, Smirnov [17].

On the other hand, because of the fact that the operator $Q(x)$ satisfies condition(3), we can show that $Q(x)$ is a bounded, self-adjoint operator from H_1 to H_1. In this case, the operator $L = L_0 + Q$ will be a self-adjoint operator from $D(L) = D(L_0)$ to H_1.

Let R_λ^0 and R_λ be resolvents of the operators L_0 and L respectively

$$R_\lambda^0 = (L_0 - \lambda I)^{-1}, \qquad R_\lambda = (L - \lambda I)^{-1}$$

and let $\mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$ be the eigenvalues of operator L_0. Here every eigenvalue is repeated according to multiplicity number. Since the eigenvalues of operator L_0 are $(\frac{1}{2}+k)^2+\gamma_j \quad (k = 0, 1, 2, \cdots ; j = 1, 2, \cdots)$ and $\lim_{j\to\infty} \gamma_j = \infty$, we have $\lim_{n\to\infty} \mu_n = \infty$.

This means that the limit of the sequence of eigenvalues $\left\{\frac{1}{\mu_n-\mu}\right\}_{n=1}^{\infty}$ of operator R_μ^0 is zero. That is,

$$\lim_{n\to\infty} \frac{1}{\mu_n - \mu} = 0 \qquad (\mu \ne \mu_n; \quad n = 1, 2, \cdots).$$

On the other hand, for every real μ which is not an eigenvalue of L_0, the operator R_μ^0 is self-sdjoint and the system of orthonormal eigenfunctions, $M_k \sin(k + \frac{1}{2})x.\varphi_j \quad (k = 0, 1, 2, \cdots ; j = 1, 2, \cdots)$ is complete. In this case, it is well known that R_μ^0 is a compact operator, Smirnov [17]. From the formula

$$R_\lambda^0 - R_\mu^0 = (\lambda - \mu) R_\lambda^0 R_\mu^0 .$$

It is obtained the compactness of operator R_λ^0 for every real number $\lambda \ne \mu_n \quad (n = 1, 2, \cdots)$. Therefore, the operator L_0 has pure discrete spectrum. Since the operator Q is a bounded self-adjoint operator, the spectrum of operator $L = L_0 + Q$ is also pure discrete, Smirnov [17].

Let $\lambda_1 \le \lambda_1 \le \cdots \le \lambda_1 \le \cdots$ be the eigenvalues of operator L. For every real μ which is not an eigenvalue of L, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \mu} = 0 .$$

Thus self-adjoint operator $R_\mu = (L-\mu I)^{-1}$ is a compact operator, Naimark [16]. From the relation

$$R_\lambda - R_\mu = (\lambda-\mu)R_\lambda R_\mu\,,$$

we obtain that for every $\lambda \neq \lambda_n\ (n=1,2,\cdots)$, R_λ is a compact operator.

Let $N(\lambda)$ be the number of eigenvalues of operator L_0 which is not greater than a positive number λ.

If $\gamma_j \sim aj^\alpha$ as $j\to\infty \quad (a>0,\ \alpha>2)$ that is, if

$$\lim_{j\to\infty}\frac{\gamma_j}{aj^\alpha}=1$$

then it can be found that $N(\lambda)\sim d\lambda^{\frac{2+\alpha}{2\alpha}}$, where

$$d=\frac{2}{\alpha a^{\frac{1}{\alpha}}}\int_0^\pi \cos^2 t \sin^{\frac{2}{\alpha}-1} t\, dt$$

and so

$$\mu_n \sim d_0 n^{\frac{2\alpha}{2+\alpha}} \quad \text{as} \quad j\to\infty \quad (d_0=d^{\frac{2\alpha}{2+\alpha}}) \tag{2.1}$$

is found, Gorbacuk and Gorbacuk [10].

Now, for the eigenvalues of the operator $L=L_0+Q$, an asymptotic formula can be found.

Since Q is a self-adjoint operator from H_1 to H_1 for every $y\in H_1$ we have

$$|(Qy,y)_1| \le \|Qy\|_1\|y\|_1 \le \|Q\|_1\|y\|_1^2\,,$$

or

$$(-\|Q\|y,y)_1 \le (Qy,y)_1 \le (\|Q\|y,y)_1.$$

This means that

$$-\|Q\|_1 I \le Q \le \|Q\|_1 I.$$

And so

$$L_0-\|Q\|_1 I \le L = L_0+Q \le L_0+\|Q\|_1 I.$$

In this case, it is well-known that

$$\mu_n-\|Q\|_1 \le \lambda_n \le \mu_n+\|Q\|_1,$$

Smirnov [17]. According to this, we can write

$$1-\frac{\|Q\|_1}{\mu_n} \le \frac{\lambda_n}{\mu_n} \le 1+\frac{\|Q\|_1}{\mu_n}.$$

By applying limit to each side of this inequality and by considering the equality

$$\lim_{n\to\infty}\frac{\mu_n}{d_0 n^{\frac{2\alpha}{2+\alpha}}}=1\,,\quad \text{we obtain}\quad \lim_{n\to\infty}\frac{\lambda_n}{\mu_n}=1\,.$$

Thus, we have

$$\lim_{n\to\infty}\frac{\lambda_n}{d_0 n^{\frac{2\alpha}{2+\alpha}}}=\lim_{n\to\infty}\left(\frac{\lambda_n}{\mu_n}\,\frac{\mu_n}{d_0 n^{\frac{2\alpha}{2+\alpha}}}\right)=\lim_{n\to\infty}\frac{\lambda_n}{\mu_n}\,\lim_{n\to\infty}\frac{\mu_n}{d_0 n^{\frac{2\alpha}{2+\alpha}}}=1\,,$$

or as $n\to\infty$, $\lambda_n\sim d_0 n^{\frac{2\alpha}{2+\alpha}}$.

Taking together with this last expression and relation (2.1) we write, if $\gamma_j\sim aj^{\alpha}$ $(a,\alpha>0)$ then as $n\to\infty$

$$\mu_n,\lambda_n\sim d_0 n^{\frac{2\alpha}{2+\alpha}}\,.$$

By using this formula, it can be showed that the sequence $\{\mu_n\}_{n=1}^{\infty}$ has a subsequence $\{\mu_{n_m}\}_{m=1}^{\infty}$ such that

$$\mu_k-\mu_{n_m}\geq d_1\left(k^{\frac{2\alpha}{2+\alpha}}-n_m^{\frac{2\alpha}{2+\alpha}}\right)\quad (k=n_m,n_m+1,n_m+2,\cdots)\,, \tag{2.2}$$

where d_1 is a positive constant.

The limit given in the form

$$\lim_{m\to\infty}\sum_{k=1}^{n_m}(\lambda_k-\mu_k)$$

is called the regularized trace of operator L such that the sequence $\{\mu_n\}_{n=1}^{\infty}$ has a subsequence $\{\mu_{n_m}\}_{m=1}^{\infty}$ satisfying the inequality (2.2).

If $\gamma_j\sim aj^{\alpha}$ as $j\to\infty$ $(a>0,\alpha>2)$ then by using the inequality (2.2) we obtain that

$$\lambda_{n_m}<\frac{1}{2}\left(\mu_{n_m+1}+\mu_{n_m}\right)<\lambda_{n_m}+1$$

for large values of m.

Since $\lambda_n,\mu_n\sim dn^{\frac{2\alpha}{2+\alpha}}$, if $\alpha>2$ and $\lambda\neq\lambda_k$ $(k=1,2,\cdots)$ then the series $\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda_k-\lambda}\right)$ and $\sum_{k=1}^{\infty}\left(\frac{\lambda}{\mu_k-\lambda}\right)$ are convergent on the circle $|\lambda|=b_m=2^{-1}(\mu_{n_m}+\mu_{n_m+1})$ for large values of m. Moreover, since

$$\lambda_k,\mu_k\sim d_0 k^{\frac{2\alpha}{2+\alpha}}\quad\text{as}\quad k\to\infty$$

if $\alpha > 2$ and $\lambda \neq \lambda_k, \mu_k$ $(k = 1, 2, \cdots)$ then the series $\sum_{k=1}^{\infty} |\mu_k - \lambda|^{-1}$ and $\sum_{k=1}^{\infty} |\lambda_k - \lambda|^{-1}$ are convergent. Therefore R_λ^0 and R_λ are the kernel operators and we find that

$$\operatorname{tr}(R_\lambda - R_\lambda^0) = \operatorname{tr} R_\lambda - \operatorname{tr} R_\lambda^0 = \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right),$$

see Cohberg and Krein [6].

If we multiply this equation with $\frac{\lambda}{2\pi i}$ and integrate on the circle $|\lambda| = b_m = 2^{-1}(\mu_{n_m} + \mu_{n_m+1})$ then we obtain that

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \lambda \operatorname{tr}(R_\lambda - R_\lambda^0) d\lambda$$
$$= \frac{1}{2\pi i} \left(\int\limits_{|\lambda|=b_m} \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda_k - \lambda} \right) d\lambda - \int\limits_{|\lambda|=b_m} \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu_k - \lambda} \right) d\lambda \right),$$

or

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \lambda \operatorname{tr}(R_\lambda - R_\lambda^0) d\lambda$$
$$= -\sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{\lambda}{\lambda - \lambda_k} d\lambda - \frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{\lambda}{\lambda - \mu_k} d\lambda \right\}. \tag{2.3}$$

Since $\mu_{n_m} < b_m < \mu_{n_m+1}$ it can be shown that for large values of m

$$\{\lambda_k, \mu_k\}_1^{n_m} \subset K(0, b_m) = \{\lambda : |\lambda| < b_m\} \tag{2.4}$$

and

$$\lambda_k, \mu_k \in \overline{K(0, b_m)} = \{\lambda : |\lambda| < b_m\} \quad (k \geq n_m + 1)\,.$$

Therefore,

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{\lambda\, d\lambda}{\lambda - \mu_k} = \begin{cases} \mu_k & \text{if } k < n_m\,, \\ 0 & \text{if } k \geq n_m + 1\,, \end{cases}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{\lambda\,d\lambda}{\lambda-\lambda_k}=\begin{cases}\lambda_k & \text{if } k<n_m\,,\\ 0 & \text{if } k\geq n_m+1\,.\end{cases}$$

Thus, the equation (2.3) comes to the form

$$\sum_{k=1}^{n_m}(\lambda_k-\mu_k)=-\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\lambda\,\mathrm{tr}(R_\lambda-R_\lambda^0)\,d\lambda\,. \tag{2.5}$$

It is known that $R_\lambda=R_\lambda^0-R_\lambda QR_\lambda^0$ for $\lambda\in\rho(L)\bigcap\rho(L_0)$. From here, for any natural number $p\geq 2$ we obtain the equality

$$R_\lambda-R_\lambda^0=\sum_{j=1}^{p}(-1)^j R_\lambda^0(QR_\lambda^0)^j+(-1)^{p+1}R_\lambda(QR_\lambda^0)^{p+1}\,.$$

If we substitute this expression in (2.5) we find that

$$\sum_{k=1}^{n_m}(\lambda_k-\mu_k)=\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\lambda\,\mathrm{tr}\left[\sum_{j=1}^{p}(-1)^{j+1}R_\lambda^0(QR_\lambda^0)^j+(-1)^pR_\lambda(QR_\lambda^0)^{p+1}\right]d\lambda\,,$$

or

$$\sum_{k=1}^{n_m}(\lambda_k-\mu_k)=\sum_{j=1}^{p}D_{mj}+D_m^{(p)}\,, \tag{2.6}$$

where

$$D_{mj}=\frac{(-1)^{j+1}}{2\pi i}\int\limits_{|\lambda|=b_m}\lambda\,\mathrm{tr}[R_\lambda^0(QR_\lambda^0)^j]\,d\lambda\,, \tag{2.7}$$

$$D_m^{(p)}=\frac{(-1)^p}{2\pi i}\int\limits_{|\lambda|=b_m}\lambda\,\mathrm{tr}[R_\lambda(QR_\lambda^0)^{p+1}]\,d\lambda\,. \tag{2.8}$$

For every natural number j, it can be shown that the operator function $(QR_\lambda^0)^j$ is analytic according to the norm in $\sigma_1(H_1)$ in the resolvent region $\rho(L_0)$ of the operator L_0. Moreover,

$$\mathrm{tr}[R_\lambda^0(QR_\lambda^0)^j]=\mathrm{tr}[(QR_\lambda^0)^{j-1}(Q(R_\lambda^0)^2)]=\mathrm{tr}[(QR_\lambda^0)^{j-1}\frac{d}{d\lambda}(QR_\lambda^0)]\,,$$

$$\mathrm{tr}\{\frac{d}{d\lambda}[(QR_\lambda^0)^j]\}=j\mathrm{tr}[(QR_\lambda^0)^{j-1}\frac{d}{d\lambda}(QR_\lambda^0)]\,.$$

From the last two relations one obtains

$$\mathrm{tr}[R_\lambda^0(QR_\lambda^0)^j] = \frac{1}{j}\mathrm{tr}\{\frac{d}{d\lambda}[(QR_\lambda^0)^j]\}\,.$$

If this expression is substituted in (2.7), then we find that

$$\begin{aligned} D_{mj} = \frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_m} \lambda\,\mathrm{tr}\{\frac{d}{d\lambda}[(QR_\lambda^0)^j]\}\,d\lambda \\ = \frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_m} \mathrm{tr}\{\frac{d}{d\lambda}[\lambda(QR_\lambda^0)^j - (QR_\lambda^0)^j]\}\,d\lambda \\ = \frac{(-1)^{j}}{2\pi ij}\int\limits_{|\lambda|=b_m} \mathrm{tr}[(QR_\lambda^0)^j]\,d\lambda + \frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_m} \frac{d}{d\lambda}\mathrm{tr}\{[\lambda(QR_\lambda^0)^j]\}\,d\lambda\,. \end{aligned}$$

By using (2.4), we can show that

$$\int\limits_{|\lambda|=b_m} \frac{d}{d\lambda}\mathrm{tr}\{[\lambda(QR_\lambda^0)^j]\}\,d\lambda = 0\,.$$

Because of this, we obtain that

$$D_{mj} = \frac{(-1)^{j}}{2\pi ij}\int\limits_{|\lambda|=b_m} \mathrm{tr}[(QR_\lambda^0)^j]\,d\lambda\,. \tag{2.9}$$

Let $\{\psi_q(x)\}_{q=1}^{\infty}$ be the system of orthonormal eigenvectors corresponding to the eigenvalues $\{\mu_q(x)\}_{q=1}^{\infty}$ of operator L_0 respectively. Since for $k = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots$

$$M_k \sin\left(\frac{1}{2}+k\right)x\,\varphi_j$$

is the system of orthonormal eigenvectors corresponding to the eigenvalues $\left(\frac{1}{2}+k\right)^2 + \gamma_j$ of operator L_0 respectively, we have

$$\psi_q(x) = M_{k_q}\sin\left(\frac{1}{2}+k_q\right)x\,\varphi_{j_q}\ \ (q = 1, 2, ...) \tag{2.10}$$

Theorem 2.1. *If the operator function $Q(x)$ satisfies the conditions (2), (3) and (4) and if as $j \to \infty$ $\gamma_j \sim a_j^{\alpha}$ $(0 < a < \infty,\ 2 < \alpha < \infty)$ then*

$$\lim_{m\to\infty} D_{m1} = \frac{1}{4}[\mathrm{tr}Q(\pi) - \mathrm{tr}Q(0)]\,.$$

Proof. From equation (2.9) we have

$$D_{m1} = \frac{-1}{2\pi i} \int\limits_{|\lambda|=b_m} \operatorname{tr}(QR_\lambda^0)\, d\lambda\,. \tag{2.11}$$

Since, for every $\lambda \in \rho(L_0)$, QR_λ^0 is a kernel operator and $\{\psi_q(x)\}_{q=1}^\infty$ is an orthonormal basis of the space H_1 then the equality

$$\operatorname{tr}(QR_\lambda^0) = \sum_{q=1}^{\infty} (QR_\lambda^0 \psi_q, \psi_q)_1$$

holds.

If we substitute this expression in (2.11) and consider the relation

$$R_\lambda^0 \psi_q = (L_0 - \lambda I)^{-1} \psi_q = (\mu_q - \lambda)^{-1} \psi_q\,,$$

we obtain that

$$D_{m1} = \frac{-1}{2\pi i} \int\limits_{|\lambda|=b_m} \Big[\sum_{q=1}^{\infty} (QR_\lambda^0 \psi_q, \psi_q)_1\Big]\, d\lambda$$

$$= \frac{-1}{2\pi i} \int\limits_{|\lambda|=b_m} \Big[\sum_{q=1}^{\infty} \frac{1}{\mu_q - \lambda} (Q\psi_q, \psi_q)_1\Big]\, d\lambda = \sum_{q=1}^{\infty} (Q\psi_q, \psi_q)_1 \, \frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q}\,.$$

From (2.10) and by using the equality

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q} = \begin{cases} 1\,, & \text{if} \quad q \le n_m\,, \\ 0\,, & \text{if} \quad q > n_m\,, \end{cases}$$

we find that

$$\begin{aligned} D_{m1} &= \sum_{q=1}^{n_m} (Q\psi_q, \psi_q)_1 = \sum_{q=1}^{n_m} \int\limits_0^\pi (Q(x)\psi_q(x), \psi_q(x))\, dx \\ &= \sum_{q=1}^{n_m} \int\limits_0^\pi \left(Q(x) M_{k_q} \sin\left(\frac{1}{2} + k\right) x\, \varphi_{j_q}, M_{k_q} \sin\left(\frac{1}{2} + k\right) x\, \varphi_{j_q}\right) dx \\ &= \sum_{q=1}^{n_m} M_{k_q}^2 \int\limits_0^\pi \sin^2\left(\frac{1}{2} + k_q\right) x (Q(x)\varphi_{j_q}, \varphi_{j_q})\, dx \end{aligned}$$

$$= \frac{1}{2}\sum_{q=1}^{n_m} M_{k_q}^2 \int\limits_0^\pi (1-\cos(2k_q+1)x)(Q(x)\varphi_{j_q},\varphi_{j_q})\,dx.$$

From the condition (4) for $Q(x)$ and since

$$M_k = \sqrt{2\pi^{-1}} \qquad (k=0,1,2,\cdots),$$

it follows that

$$D_{m1} = \frac{-1}{\pi}\sum_{q=1}^{n_m}\int\limits_0^\pi \cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})\,dx\,, \tag{2.12}$$

it can be proved that the series

$$\sum_{k=1}^\infty\sum_{j=1}^\infty\int\limits_0^\pi (Q(x)\varphi_j,\varphi_j)\cos(2k+1)x\,dx\,,$$

absolutely converges. It is known that

$$\lim_{m\to\infty}\sum_{q=1}^{n_m}\int\limits_0^\pi \Big(\cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})\Big)\,dx$$
$$= \sum_{k=1}^\infty\sum_{j=1}^\infty\int\limits_0^\pi (Q(x)\varphi_j,\varphi_j)\cos(2k+1)x\,dx\,.$$

By applying limit, as $m\to\infty$, to the equation (2.12) and by considering the last relation above

$$\lim_{m\to\infty} D_{m1} = \frac{-1}{\pi}\sum_{k=1}^\infty\sum_{j=1}^\infty\int\limits_0^\pi (Q(x)\varphi_j,\varphi_j)\cos(2k+1)x\,dx$$

is found. By subtracting and adding the expression

$$(Q(x)\varphi_j,\varphi_j)\cos 2kx\,,$$

into the integral one obtains

$$\lim_{m\to\infty} D_{m1} =$$

$$\frac{-1}{\pi}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int\limits_{0}^{\pi}\Big[(Q(x)\varphi_j,\varphi_j)(\cos(2k+1)x+\cos 2kx)-(Q(x)\varphi_j,\varphi_j)\cos 2kx\Big]\,dx.$$

It can be written the expression

$$\frac{-1}{\pi}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos rx\,dx\,,$$

instead of first term in the right side of this equality. Thus, we have

$$\lim_{m\to\infty} D_{m1} = \frac{-1}{\pi}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos rx\,dx + \frac{1}{\pi}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos 2kx\,dx.$$

We can write this equation in the form

$$\lim_{m\to\infty} D_{m1} = \frac{-1}{\pi}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos rx\,dx + \frac{1}{2\pi}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos kx\,dx + (-1)^k\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos kx\,dx\right],$$

and so we have

$$\begin{aligned}\lim_{m\to\infty} D_{m1} &= -\frac{1}{2}\sum_{j=1}^{\infty}\left\{\sum_{r=1}^{\infty}\left[\frac{2}{\pi}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos rx\,dx\right]\cos k0\right\}\\ &+ \frac{1}{4}\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\infty}\Big[\frac{2}{\pi}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos kx\,dx\Big]\cos k0\right.\\ &+ \left.\sum_{k=1}^{\infty}\Big[\frac{2}{\pi}\int\limits_{0}^{\pi}(Q(x)\varphi_j,\varphi_j)\cos kx\,dx\Big]\cos k\pi\right\}.\end{aligned}$$

Since $Q(x)$ satisfies the condition (4), the sum with respect to r in the first term on the right side of this expression is the value at 0 of Fourier series according to functions $\{\cos rx\}_{r=0}^{\infty}$ in the interval $[0,\pi]$ of the function

$(Q(x)\varphi_j, \varphi_j)_H$ having the derivative of second order. Similarly, the sums in the second term with respect to k are the values at the points 0 and π respectively of Fourier series with respect to the functions $\{\cos mx\}_{m=0}^{\infty}$ in the same interval of that function.

For this reason we obtain that

$$\begin{aligned}\lim_{m\to\infty} D_{m1} &= -\frac{1}{2}\sum_{j=1}^{\infty}[(Q(0)\varphi_j, \varphi_j)] + \frac{1}{4}\sum_{j=1}^{\infty}[(Q(0)\varphi_j, \varphi_j) + (Q(\pi)\varphi_j, \varphi_j)] \\ &= \frac{1}{4}[\operatorname{tr} Q(\pi) - \operatorname{tr} Q(0)].\end{aligned}$$

This proves the theorem. □

3. A Formula for the Regularized Trace of the Operator L

In this section, we obtain a formula for the limit $\{\lim_{m\to\infty} \sum_{k=1}^{n_m}(\lambda_k - \mu_k)\}$ that we called the regularized trace of the operator L in the previous section.

Since $\{\psi_q(x)\}_1^{\infty}$ is an orthonormal basis of the space H_1, for every $y \in H_1$ we have

$$y = \sum_{k=1}^{\infty}(y, \psi_k)_1\psi_k \quad \text{and} \quad R_\lambda^0 y = \sum_{k=1}^{\infty}(y, \psi_k)_1 R_\lambda^0\psi_k.$$

According to this, we obtain that

$$QR_\lambda^0\psi_{k_1} = \frac{Q\psi_{k_1}}{\mu_{k_1} - \lambda},$$

$$\begin{aligned}(QR_\lambda^0)^2\psi_{k_1} &= QR_\lambda^0\left(\frac{Q\psi_{k_1}}{\mu_{k_1} - \lambda}\right) = \frac{1}{\mu_{k_1} - \lambda}(Q\psi_{k_1}) \\ &= \frac{1}{\mu_{k_1} - \lambda}Q\left[\sum_{k_2=1}^{\infty}\frac{(Q\psi_{k_1}, \psi_{k_2})_1}{\mu_{k_1} - \lambda}\psi_{k_2}\right] = \frac{1}{\mu_{k_1} - \lambda}\sum_{k_2=1}^{\infty}\frac{(Q\psi_{k_1}, \psi_{k_2})_1}{\mu_{k_2} - \lambda}Q\psi_{k_2}.\end{aligned}$$

Similarly, it follows that

$$(QR_\lambda^0)^3\psi_{k_1} = \frac{1}{\mu_{k_1} - \lambda}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\frac{(Q\psi_{k_1}, \psi_{k_2})_1}{\mu_{k_2} - \lambda}\frac{(Q\psi_{k_2}, \psi_{k_3})_1}{\mu_{k_3} - \lambda}Q\psi_{k_3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(QR_\lambda^0)^n\psi_{k_1} = \frac{1}{\mu_{k_1}-\lambda}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[\prod_{j=1}^{n-1}\frac{(Q\psi_{k_j},\psi_{k_{j+1}})_1}{\mu_{k_{j+1}}-\lambda}\right]Q\psi_{k_n}\,. \tag{3.1}$$

This shows that

$$\mathrm{tr}\,(QR_\lambda^0)^n = \sum_{k_1=1}^{\infty}\Big((QR_\lambda^0)^n\psi_{k_1},\psi_{k_1}\Big)_1$$

$$= \sum_{k_1=1}^{\infty}\left(\frac{1}{\mu_{k_1}-\lambda}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[\prod_{j=1}^{n-1}\frac{(Q\psi_{k_j},\psi_{k_{j+1}})_1}{\mu_{k_{j+1}}-\lambda}\right]Q\psi_{k_n},\psi_{k_n}\right) = \sum_{k_2=1}^{\infty}$$

$$\sum_{k_3=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[\frac{(Q\psi_{k_1},\psi_{k_2})_1}{\mu_{k_2}-\lambda}\,\frac{(Q\psi_{k_2},\psi_{k_3})_1}{\mu_{k_3}-\lambda}\cdots\frac{(Q\psi_{k_{n-1}},\psi_{k_n})_1}{\mu_{k_n}-\lambda}\,\frac{(Q\psi_{k_n},\psi_{k_1})_1}{\mu_{k_1}-\lambda}\right]$$

$$= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\prod_{j=1}^{n}(\mu_{k_j}-\lambda)^{-1}(Q\psi_{k_j},\psi_{k_{\rho(j)+1}})_1\,, \tag{3.2}$$

where

$$\rho(j) = \begin{cases} j & \text{if} \quad j<n\,,\\ 0 & \text{if} \quad j=n\,,\end{cases}$$

By using this last equation (3.2), equation (2.9) comes to the form

$$D_{mj} = \frac{(-1)^j}{2\pi ij}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots$$

$$\sum_{k_j=1}^{\infty}\left[\left(\int\limits_{|\lambda|=b_m}\prod_{q=1}^{j}(\mu_{k_q}-\lambda)^{-1}d\lambda\right)\prod_{q=1}^{j}(Q\psi_{k_q},\psi_{k_{\rho(q)+1}})_1\right], \tag{3.3}$$

or

$$D_{mj} = \frac{(-1)^j}{2\pi ij}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots$$

$$\sum_{k_j=1}^{\infty}{}^{*}\left[\left(\int\limits_{|\lambda|=b_m}\prod_{q=1}^{j}(\mu_{k_q}-\lambda)^{-1}d\lambda\right)\prod_{q=1}^{j}(Q\psi_{k_q},\psi_{k_{\rho(q)+1}})_1\right], \tag{3.4}$$

where the symbol "$*$" denotes that these are numbers among $\mu_{k_1},\mu_{k_2},\cdots,\mu_{k_j}$ less than or greater than b_m.

From equation (3.4) it can be seen that

$$
\begin{aligned}
D_{m2}
&= \frac{1}{4\pi i}\sum_{k_1=1}^{n_m}\sum_{k_2=n_m+1}^{\infty}\left[\int\limits_{|\lambda|=b_m}\frac{d\lambda}{(\lambda-\mu_{k_1})(\lambda-\mu_{k_2})}\right](Q\psi_{k_1},\psi_{k_2})_1(Q\psi_{k_2},\psi_{k_1})_1 \\
&+ \frac{1}{4\pi i}\sum_{k_1=n_m+1}^{\infty}\sum_{k_2=1}^{n_m}\left[\int\limits_{|\lambda|=b_m}\frac{d\lambda}{(\lambda-\mu_{k_1})(\lambda-\mu_{k_2})}\right](Q\psi_{k_1},\psi_{k_2})_1(Q\psi_{k_2},\psi_{k_1})_1 \\
&= \frac{1}{2\pi i}\sum_{k=1}^{n_m}\sum_{j=n_m+1}^{\infty}\left[\int\limits_{|\lambda|=b_m}\frac{d\lambda}{(\lambda-\mu_k)(\lambda-\mu_j)}\right](Q\psi_k,\psi_j)_1(Q\psi_j,\psi_k)_1\,. \qquad (3.5)
\end{aligned}
$$

Since $k \le n_m$ and $j \ge n_m+1$ we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{d\lambda}{(\lambda-\mu_k)(\lambda-\mu_j)} \\
&\qquad= \frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{1}{(\mu_k-\mu_j)}\left[\frac{1}{(\lambda-\mu_k)}-\frac{1}{(\lambda-\mu_j)}\right]d\lambda \\
&\qquad= \frac{1}{\mu_k-\mu_j}\left[\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{d\lambda}{(\lambda-\mu_k)}-\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{d\lambda}{(\lambda-\mu_j)}\right]=\frac{1}{\mu_k-\mu_j}\,.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|D_{m2}| &= \left|\sum_{k=1}^{n_m}\sum_{j=n_m+1}^{\infty}(\mu_k-\mu_j)^{-1}(\psi_k,Q\psi_j)_1\overline{(\psi_k,Q\psi_j)_1}\right| \\
&= \sum_{k=1}^{n_m}\sum_{j=n_m+1}^{\infty}(\mu_j-\mu_k)^{-1}|(\psi_k,Q\psi_j)_1|^2 \le \sum_{k=1}^{n_m}\sum_{j=n_m+1}^{\infty}(\mu_j-\mu_{n_m})^{-1}|(\psi_k,Q\psi_j)_1|^2 \\
&\le \sum_{j=n_m+1}^{\infty}(\mu_j-\mu_{n_m})^{-1}\sum_{k=1}^{\infty}|(\psi_j,Q\psi_k)_1|^2 = \sum_{j=n_m+1}^{\infty}(\mu_j-\mu_{n_m})^{-1}\|Q\psi_j\|_1^2 \\
&\le \sum_{j=n_m+1}^{\infty}(\mu_j-\mu_{n_m})^{-1}\|Q\|_1^2\,,
\end{aligned}
$$

and so we have

$$
|D_{m2}| \le \|Q\|_1^2\Omega_m\,, \qquad (3.6)
$$

where

$$\Omega_m = \sum_{j=n_m+1}^{\infty} (\mu_j - \mu_{n_m})^{-1} \qquad (m = 1, 2, \cdots)\,.$$

In a similar form, by using (3.4) and considering the inequalities (2.2) and

$$x^{1+\delta} - (x-1)^{1+\delta} > x^{\delta} \qquad (x > 1, \quad \delta > 0)\,,$$

it can be shown that

$$|D_{m3}| \le \|Q\|_1^3 \Omega_m (\Omega_m + 4 d_1^{-1} n_m^{1-\delta}), \quad d_1 = \frac{d_0}{4},\ \delta = \frac{\alpha - 2}{\alpha + 2}\,. \tag{3.7}$$

Moreover, if $\gamma_j \sim a j^{\alpha}$ as $j \to \infty$ $(0 < a < \infty,\ 2 < \alpha < \infty)$ then it is satisfied that

$$\|R_\lambda^0\|_{\sigma_1(H_1)} < \text{const.} n_m^{1-\delta} \qquad (\delta = \frac{\alpha - 2}{\alpha + 2})\,, \tag{3.8}$$

on the circle $|\lambda| = b_m$.

On the other hand, since $Q(x)$ is a bounded self-adjoint operator from H_1 to H_1 and

$$\mu_n - \|Q\|_1 \le \lambda_n \le \mu_n + \|Q\|_1\,,$$

if $\gamma_j \sim a j^{\alpha}$ as $j \to \infty$ $(a > 0,\ \alpha > 2)$ then for the large values of m, the inequality

$$\|R_\lambda\|_1 < \frac{d_1}{4} n_m^{-\delta} \qquad (\delta = \frac{\alpha - 2}{\alpha + 2})\,, \tag{3.9}$$

is also satisfied on the circle $|\lambda| = b_m$.

Now we are ready to prove the following theorem.

Theorem 3.1. *Suppose that* $\gamma_j \sim a j^{\alpha}$ *as* $j \to \infty$ $(0 < a < \infty, 2 < \alpha < \infty)$. *If the operator function* $Q(x)$ *satisfies the conditions (2) and (3) then for* $j \ge 2$

$$\lim_{m \to \infty} D_{mj} = 0\,.$$

Proof. We can give a restriction to for the magnitude of the expression D_{m1} : From (2.9) we have

$$|D_{mj}| \le \frac{1}{2\pi j} \int\limits_{|\lambda| = b_m} |\text{tr}\,(Q R_\lambda^0)^j|\,|d\lambda| \le \frac{1}{2\pi j} \int\limits_{|\lambda| = b_m} \|Q R_\lambda^0\|_{\sigma_1(H_1)}^j |d\lambda|$$

$$\le \frac{1}{2\pi j} \int\limits_{|\lambda| = b_m} \|Q R_\lambda^0\|_{\sigma_1(H_1)} \|Q R_\lambda^0\|_{\sigma_1(H_1)}^{j-1} |d\lambda|$$

$$\leq \frac{1}{2\pi j} \int\limits_{|\lambda|=b_m} \|Q\|_1 \|R_\lambda^0\|_{\sigma_1(H_1)} \|QR_\lambda^0\|_{\sigma_1(H_1)}^{j-1} |d\lambda|$$

$$\leq \frac{1}{2\pi j} \int\limits_{|\lambda|=b_m} \|Q\|_1^j \|R_\lambda^0\|_{\sigma_1(H_1)} \|R_\lambda^0\|_{\sigma_1(H_1)}^{j-1} |d\lambda| . \quad (3.10)$$

If we choose $Q(x) \equiv 0$ identically then we have

$$R_\lambda = R_\lambda^0 .$$

It means that

$$\|R_\lambda\|_1 < \frac{d_1}{4} n_m^{-\delta} \qquad (\delta = \frac{\alpha - 2}{\alpha + 2}) . \quad (3.11)$$

From (3.8), (3.10) and (3.11), it follows that

$$|D_{mj}| \leq \text{const.} \int\limits_{|\lambda|=b_m} n_m^{1-\delta} n_m^{-\delta(j-1)} |d\lambda| \leq \text{const.}\ \mu_{n_m} n_m^{1-\delta j} .$$

Since $\mu_{n_m} \leq \text{const.}\, n_m^{1+\delta}$ we have

$$|D_{mj}| \leq \text{const.}\, n_m^{2-\delta(j-1)} .$$

Clearly, if $j > 1 + 2\delta^{-1}$ then

$$\lim_{m\to\infty} D_{mj} = 0 .$$

For $j = 2$ since $\lim\limits_{m\to\infty} \Omega_m = 0$, from (3.6) we obtain that

$$\lim_{m\to\infty} D_{m2} = 0 .$$

Similarly, from (3.7) we see that

$$\lim_{m\to\infty} D_{m3} = 0 .$$

It follows that for $j = 2, 3, \cdots , |2\delta^{-1}| + 1$

$$\lim_{m\to\infty} D_{mj} = 0 \qquad \square$$

Our main result in this paper is given by the following theorem.

Theorem 3.2. *Suppose that* $\gamma_j \sim aj^{\alpha}$ *as* $j \to \infty (0 < a < \infty, 2 < \alpha < \infty)$. *If the operator function* $Q(x)$ *satisfies the conditions (2), (3) and (4) then the formula*

$$\lim_{m\to\infty}\sum_{k=1}^{n_m}(\lambda_k-\mu_k)=\frac{1}{4}[tr\,Q(\pi)-tr\,Q(0)]$$

is satisfied.

Proof. By using Theorem 2.1 and Theorem 3.1, from equation (2.6) we write that

$$\lim_{m\to\infty}\sum_{k=1}^{n_m}(\lambda_k-\mu_k)=\frac{1}{4}[\mathrm{tr}\,Q(\pi)-\mathrm{tr}\,Q(0)]+\lim_{m\to\infty}D_m^{(p)}\,. \qquad (3.12)$$

Here, by (2.8) we have

$$D_m^{(p)}=\frac{(-1)^p}{2\pi i}\int\limits_{|\lambda|=b_m}\lambda\mathrm{tr}\,[R_\lambda(QR_\lambda^0)^{p+1}]\,d\lambda\,.$$

We can give a restriction to for the magnitude of this expression as in the following

$$\begin{aligned}
|D_m^{(p)}| &\le \frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}|\lambda||\mathrm{tr}\,[R_\lambda(QR_\lambda^0)^{p+1}]|\,|d\lambda|\\
&\le b_m\int\limits_{|\lambda|=b_m}\|R_\lambda(QR_\lambda^0)^{p+1}\|_{\sigma_1(H_1)}\,|d\lambda|\\
&\le b_m\int\limits_{|\lambda|=b_m}\|R_\lambda\|_1\|(QR_\lambda^0)^{p+1}\|_{\sigma_1(H_1)}\,|d\lambda|\\
&\le b_m\int\limits_{|\lambda|=b_m}\|R_\lambda\|_1\|(QR_\lambda^0)^{p}\|_1\|(QR_\lambda^0)\|_{\sigma_1(H_1)}\,|d\lambda|\\
&\le b_m\int\limits_{|\lambda|=b_m}\|R_\lambda\|_1\|Q\|_1^p\|R_\lambda^0)\|_1^p\|Q\|_1\|R_\lambda^0)\|_{\sigma_1(H_1)}\,|d\lambda|\,.
\end{aligned}$$

From (3.8) and (3.9) we obtain that

$$|D_m^{(p)}|\le \mathrm{const.}\, b_m^2 n_m^{-(p+1)\delta}n_m^{1-\delta}\,.$$

Since $b_m \leq \text{const.}\, n_m^{1+\delta}$ then we have

$$|D_m^{(p)}| \leq \text{const.}\, n_m^{-(p+2)\delta+1} n_m^{2(1+\delta)} = \text{const.}\, n_m^{3-p\delta}\,.$$

It follows that for $p > 3\delta^{-1}$

$$\lim_{m\to\infty} D_m^{(p)} = 0\,.$$

If we substitute this result in equation (3.12) we obtain the regularized trace formula of operator L as

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4}[\operatorname{tr} Q(\pi) - \operatorname{tr} Q(0)].$$

The proof is done.

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