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TWO NEW RECURSIVE METHODS FOR CONSTRUCTION OF WALSH MATRICES OF ORDER 2^n

F.M. Maalek Ghaini[1 §], R. Ghaffarnezhad Bidoki[2]

[1,2]Department of Mathematics
Yazd University
Yazd, IRAN
[1]e-mail: maalek@yazduni.ac.ir
[2]e-mail: ghaffar@sadi.ut.ac.ir

Abstract: In this paper we produce two new recursive methods for construction of Walsh matrices of orders 2^n. In the first method we construct Walsh matrices of orders 2^n, $\mathcal{W}_{2^n}$, from Walsh matrices of order 2^{n-2} using Kronecker products and a recursive method, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are known initially. In the second method we construct Walsh matrices of order 2^n, $\mathcal{W}_{2^n}$, from Hadamard matrices of order 2^n, $\mathcal{H}_{2^n}$, by rearranging the columns which achieved by multiplying an operational matrix P_{2^n} in it where the operational matrix P_{2^n} is achieved from a recursive method.

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1. Introduction

The Walsh system occupies a unique position among orthonormal systems on $[0, 1)$. It is the simplest complete, orthonormal system: each Walsh function takes only the values ± 1. It can be defined by using products of functions of mean zero (Rademacher functions) and up to measure preserving transformations, it is the only such system on a probability space whose functions have range $\{1, -1\}$ [9].

The Walsh system has influenced other areas of mathematics. Enflo [3] used it to construct a separable Banach space that has no basis. This solved

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§Correspondence author

a longstanding problem posed by Banach in the twenties. The fundamental theorem of martingale inequalities was first proved for the Walsh system [6]; and, the Walsh functions were used by Bethke [1] to analyze the behavior of genetic algorithms, a method to optimize nondifferentiable functions which works when the traditional methods either fail to converge or converge badly at suboptimal points.

Henderson [5] generated Walsh matrices by the modulo-2 product of two generating matrices: the natural binary code, and the transpose of the bit-reversed form of the first. As a result, the coefficients of the Walsh transform occur in bit-reversed order. By simply reordering the Walsh functions themselves to correspond to generation by the product of two such code matrices, neither or both in bit-reversed form, the Walsh coefficients occur in natural order.

In Section 2 we bring some preliminaries such as Walsh functions, Walsh matrices, Hadamard matrices and Kronecker products. In Section 3 we explain two new recursive methods for producing Walsh matrices. In Section 4 we compare the time consumption of two new recursive methods and the general method using *Matlab* package.

2. Some Preliminaries

In this section we define Walsh functions and Kronecker products.

2.1. Walsh Function

The Walsh functions (see [8]) have many properties similar to those of the trigonometric functions. For example they form a complete, total collection of orthogonal functions with respect to the space of square Lebesgue integrable functions. However, they are simpler in structure than the trigonometric functions because they take only the values 1 and -1 and they may be expressed as a linear combination of the Haar functions [4]. Therefore many proofs about the Haar functions carry over to the Walsh systems easily. Moreover, the Walsh functions are Haar wavelet packets; see [10] for a good account of the properties of the Haar wavelets and other wavelets. The Walsh (-Paley) system may be obtained, following Paley [6], from the Rademacher system [7]. The Rademacher system is an incomplete system of functions with respect to the space of Lebesgue integrable functions. We define Rademacher functions

$\{r_n\}_{n=1}^{\infty}$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases} \quad (1)$$

$$r_n(x) = r_0(2^n x - [2^n x]), \qquad n \in \mathbf{N}. \quad (2)$$

The Walsh (-Paley) system $\{W_n\}$ is then defined as follows:

$$W_0(x) = 1\,, \quad (3)$$

$$W_n(x) = \prod_{i=1}^{r} r_{n_i}(x)\,, \quad (4)$$

where $n = \sum_{i=1}^{r} 2^{n_i}$ is the unique decomposition of n into powers of two (the binary decomposition of n). The first 16 Walsh functions are shown in Figure 1.

2.2. Walsh Matrices

Definition 1. The Walsh matrix of order m, $\mathcal{W}_m$, has elements W_{ij}, where W_{ij} is the value of the i-th Walsh function in the j-th subinterval, i.e. $W_{ij} = W_i(\frac{j-1}{m})$.

Example 2. Suppose $m = 2^n$. The six $\mathcal{W}_m$ matrices, for $m = 2, 4, 8, 16, 32, 64$, are shown in Figure 2, where the white squares denote 1's in Walsh matrix and black squares denote the 1's in Walsh matrix.

2.3. Hadamard Matrices

A Hadamard matrix is a type of square $(-1, 1)$-matrix. A Hadamard matrix of order n is a solution to Hadamard's maximum determinant problem, i.e., it has the maximum possible determinant (in absolute value) of any complex matrix with elements in the unit circle [2], namely:

Definition 3. A square matrix of order m, $\mathcal{H}_m$, is a Hadamard matrix iff

$$\mathcal{H}_m \mathcal{H}_m^T = m I_m\,, \quad (5)$$

where I_m is the $m \times m$ identity matrix.

Also we can obtain Hadamard matrices from a recursive method:

$$\mathcal{H}_{2^n} = \begin{bmatrix} \mathcal{H}_{2^{n-1}} & \mathcal{H}_{2^{n-1}} \\ -\mathcal{H}_{2^{n-1}} & \mathcal{H}_{2^{n-1}} \end{bmatrix}. \quad (6)$$

In Figure 3 we show the Hadamard matrices of order 2, 4, 8, 16, 32, 64.

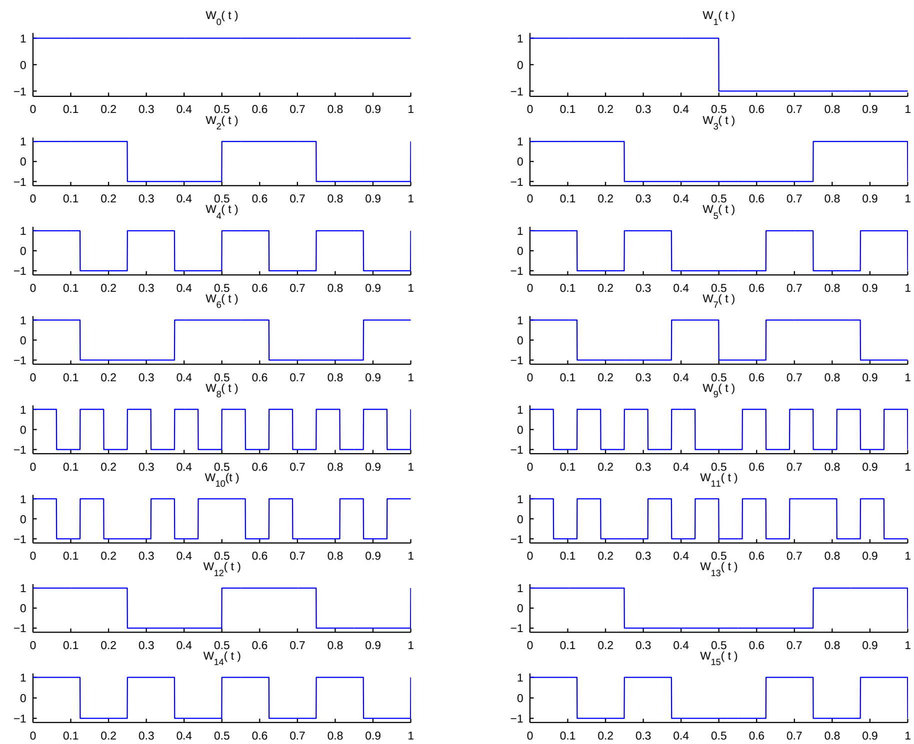

Figure 1: The first 16 Walsh functions

2.4. Kronecker Product

The Kronecker product is a binary matrix operator that maps two matrices of arbitrary dimensions into a larger matrix with a special block structure. Namely, given an $n \times m$ matrix A and a $p \times q$ matrix B

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}, \tag{7}$$

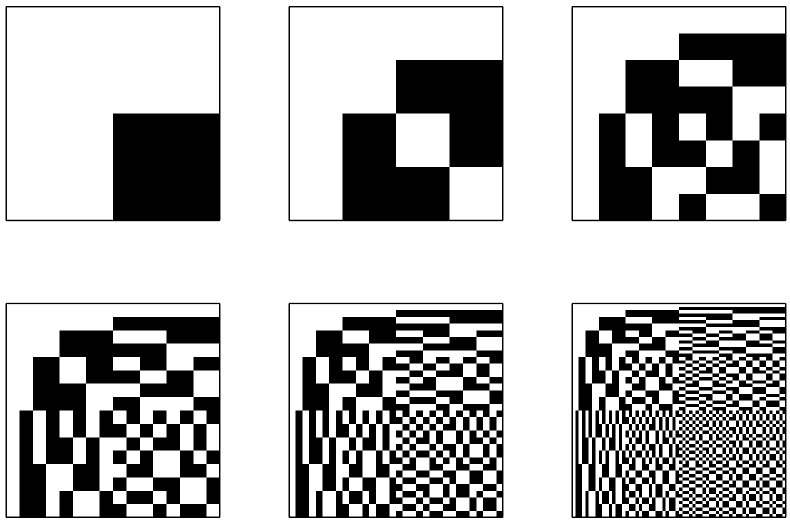

Figure 2: Walsh matrices of order 2, 4, 8, 16, 32, 64

their Kronecker product, denoted by $A \otimes B$, is the $(np) \times (mq)$ matrix with the block structure

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}. \quad (8)$$

3. Main Results

In this section we introduce two new methods for construction of Walsh matrices of order m in the case $m = 2^n$.

In the general method Walsh matrices are generated directly from definition of Walsh matrices (Definition 1).

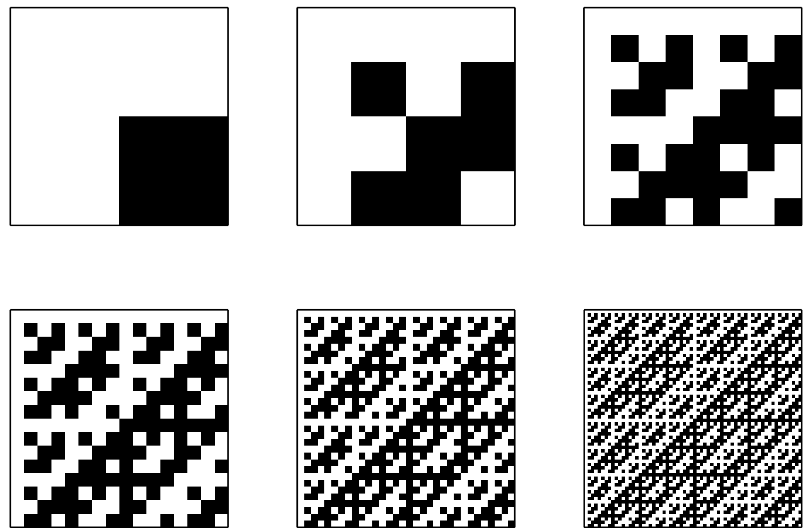

Figure 3: Hadamard matrices of order 2, 4, 8, 16, 32, 64

3.1. The First Method

If we separate Walsh matrix of order 4, $\mathcal{W}_4$, into 4 block-2×2 matrices, then we obtain

$$\mathcal{W}_4 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \hline 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right], \tag{9}$$

or

$$\mathcal{W}_4 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \tag{10}$$

in which

$$W_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad W_{12} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$
$$W_{21} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad W_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{11}$$

We can construct $\mathcal{W}_m$ for $m = 2^n$ and $n \geq 2$ by the following recursive algorithm in which

$$\mathcal{W}_1 = 1 \qquad \text{and} \qquad \mathcal{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$\begin{aligned} \mathcal{W}_{2^n} &= \left[\begin{array}{c|c} \mathcal{W}_{2^{n-2}} \otimes W_{11} & \mathcal{W}_{2^{n-2}} \otimes W_{12} \\ \hline \mathcal{W}_{2^{n-2}} \otimes W_{21} & \mathcal{W}_{2^{n-2}} \otimes W_{22} \end{array}\right] \qquad (12) \\ &= \begin{bmatrix} \mathcal{W}_{2^{n-2}} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \mathcal{W}_{2^{n-2}} \otimes \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ \mathcal{W}_{2^{n-2}} \otimes \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} & \mathcal{W}_{2^{n-2}} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Example 4. By using the above algorithm we construct $\mathcal{W}_8$.
We have

$$\begin{aligned} \mathcal{W}_8 &= \left[\begin{array}{c|c} \mathcal{W}_2 \otimes W_{11} & \mathcal{W}_2 \otimes W_{12} \\ \hline \mathcal{W}_2 \otimes W_{21} & \mathcal{W}_2 \otimes W_{22} \end{array}\right] \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}. \end{aligned}$$

3.2. The Second Method

In this section we construct a Walsh matrix from rearranging columns of a Hadamard matrix.

Suppose we want to construct an m-Walsh matrix in the case $m = 2^n$. Putting

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{13}$$

$$P_{2^n} = \begin{bmatrix} P_{2^{n-1}} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} & P_{2^{n-1}} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}, \qquad n \geq 2, \tag{14}$$

then we have

$$\mathcal{W}_{2^n} = P_{2^n} \times \mathcal{H}_{2^n} \tag{15}$$

Example 5. By using the above algorithm we construct $\mathcal{W}_8$.
We have

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow P_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

now from (15) and (6) we have

$$\begin{aligned} \mathcal{W}_8 &= P_8 \times \mathcal{H}_8 \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}. \end{aligned}$$

Methods	128	256	512	1024	2048	4096
General method	2.90	13.71	63.73	291.06	1311.37	5900.0
Rec. M. ($\mathcal{W}_4$)	0.00	0.00	0.04	0.29	1.93	15.48
Rec. M. (Hadamard)	0.00	0.01	0.02	0.03	0.19	0.76

Table 1: Three methods and their time consuming by *Matlab* package

4. Comparing Methods

In this section we compare the ordinary method with the two new recursive methods.

Using *Matlab* package and for $m = 128, 256, 512, 1024, 2048, 4096,$ we obtain the Table 1. In this table we see that for example for $m = 512$ the first recursive method is approximately 1360 times and the second recursive method is 4085 times faster than the ordinary method. For $m = 4056$ the first recursive method is approximately 381 times and the second recursive method is approximately 7706 times faster than the ordinary method.

Comment 1. We have not any proof for the above results, but we have obtained them experimentally.

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