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A CONJECTURE ON BICONNECTED GRAPHS AND REGULAR CELLULATIONS OF THE 3-SPHERE

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Abstract: We conjecture that every biconnected graph with at least two cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere. The conjecture is true for planar and hamiltonian graphs. An equivalent formulation of the conjecture was given in De Agostino [2], involving the concept of spatiality degree of a graph.

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1. Introduction

Let X be a CW-complex (Lundell et al [4]) on the 3-sphere $S^3 = \{x \in R^4 : |x| = 1\}$ with its standard topology. X is also called a *cellulation* of the 3-sphere. The ascending sequence $X^0 \subset X^1 \subset X^2 \subset X^3 = X$ of closed subspaces of X satisfies the following conditions:

(1) X^0 is a discrete set of points (0-cells).

(2) For $0 < k \leq 3$, $X^k - X^{k-1}$ is the disjoint union of open subspaces, called k-cells, each of which homeomorphic to the open k - dimensional ball $U^k (= \{x \in R^k : |x| < 1\})$.

X^k is the k-dimensional skeleton of X and is a k-dimensional CW-complex for $1 \leq k \leq 3$ on a subspace of the 3-sphere. X is a *regular* CW-complex if the boundary of every k-cell is homeomorphic to the $k-1$ dimensional sphere S^{k-1}, for $1 \leq k \leq 3$. Then, X is called a regular cellulation of S^3.

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We conjecture that every biconnected graph with at least two cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere. The conjecture is true for planar and hamiltonian graphs. We want to point out that we assume that a graph is simple, that is, it has no loops on one vertex and no multiple edges between two vertices.

An equivalent formulation of the conjecture was given in De Agostino [2], involving the concept of spatiality degree of a graph. The *spatiality degree* of a graph G with at least four cycles is the maximum number of 3-cells that the cellulation of a 3-sphere can have with G as 1-dimensional skeleton, assuming that two distinct 2-cells of the complex cannot share the same boundary and the two-dimensional skeleton is regular (De Agostino [2] and Luccio et al [3]). Let us rename with V, E, F and C the sets of 0-, 1-, 2- and 3-cells of the cellulation of the 3-sphere. It is a well-known result that the Euler characteristic $|V| - |E| + |F| - |C|$ is a topological invariant and it is equal to zero on S^3 with its standard topology. From such result it was derived that $2(|E| - |V|)$ is an upper bound to the spatiality degree of a biconnected graph (Crescenzi et al [1]). Observe that when the upper bound is reached the cellulation is regular since each 3-cell is delimited by exactly three 2-cells.

It was proved in Crescenzi et al [1] that the spatiality degree of biconnected planar graphs and hamiltonian graphs is equal to $2(|E| - |V|)$. Based on these results, we formulated the conjecture that the spatiality degree of every biconnected graph $G = (V, E)$ with at least four cycles is $2(|E| - |V|)$ in De Agostino [2]. In Section 2 we show that every biconnected planar or Hamiltonian graph with at least two cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere. We also give the conjecture that this is true for every biconnected graph and show its equivalence with the conjecture presented in De Agostino [2].

2. The Conjecture

We show the following facts.

Theorem 1. *Every biconnected planar graph $G = (V, E)$ with at least two cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere.*

Proof. The embedding of G into the 2-sphere provides a regular cellulation of the 3-sphere with two 3-cells. □

Theorem 2. *Every hamiltonian graph $G = (V, E)$ with at least two cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere.*

Proof. We embed V into the 3-sphere. Let $v_1, v_2, ...v_n, v_1$ be the sequence of vertices (0-cells) ordered by a hamiltonian cycle h of G, where $|V| = n$. We embed the edges on h (1-cells) into the 3-sphere so that we have a one-dimesional complex X. Then, we add to X a 2-cell with boundary h. Let us consider any edge, say (v_i, v_j), which does not belong to h, with $i < j$. We add to X the edge (v_i, v_j) as a 1-cell and two 2-cells with the cycles $v_1, ..., v_i, v_j, v_{j-1}, ..., v_1$ and $v_i, ..., v_n, v_{n-1}, ..., v_j, v_i$ as boundaries, respectively. At this point, we could add to X two 3-cells to obtain a regular cellulation of the 3-sphere. Instead, we add only one 3-cell so that we can embed similarly the remaining edges of G. After all the edges have been embedded, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as one-dimensional skeleton. □

From Theorem 2, it follows that a biconnected graph with n vertices is the one-dimensional skeleton of a regular cellulation of the 3-sphere if it is a complete graph. Over all the graphs with n vertices, the complete graph is an obvious case where the genus is maximized. On the other hand, Theorem 1 shows that such property holds when the genus of the graph is 0. This consideration suggests this property might hold when the graph lies, as far as embeddability into surfaces is concerned, in between a planar one and a complete one. Therefore, we formulate the following conjecture.

Conjecture. *Every biconnected graph $G = (V, E)$ with at least two cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere.*

Now, we show the following result.

Theorem 3. *A biconnected graph $G = (V, E)$ with at least four cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere if and only if its spatiality degree is $2(|E| - |V|)$.*

Proof. Let X be a regular cellulation of S^3 with G as one-dimensional skeleton. Without loss of generality, we can assume no cells of X share the same boundary. In fact, if any pair of 2- or 3-cells shared the same boundary we could remove one with no effect on the one-dimensional skeleton. On the other hand, the one-dimensional skeleton observes such property since the graph is simple. As mentioned in the introduction, the spatiality degree of a graph $G = (V, E)$ is $2(|E| - |V|)$ if and only if G is the one-dimensional skeleton of a regular cellulation of the 3-sphere where each 3-cell is delimited by exactly three 2-cells. Therefore, to prove the theorem we need to show that if a biconnected graph $G = (V, E)$ with at least four cycles is the one-dimensional skeleton of a regular cellulation of the 3-sphere then it is the one-dimensional skeleton of a regular cellulation of the 3-sphere where each 3-cell is delimited by exactly three 2-cells. Let c be a 3-cell of X and let f be a 2-cell of X on the boundary of c. Since X is regular and no cells of X share the same boundary, there must be two 2-cells f_1 and f_2 on the boundary of c such that the intersection between the boundary of f and the boundary of $f_1(f_2)$ is a simple path of at least one edge. Let b and b_1 be the boundaries of f and f_1 respectively. The *sum* of b and b_1, denoted as $b + b_1$, is a cycle determined by the symmetric 1-cell difference between b and b_1. It suffices to show that if the boundary of c has more than three 2-cells, it is possible to "split" the cell by adding a new 2-cell. In fact, if c is delimited by more than three faces neither $b + b_1$ nor $b + b_2$ may be a boundary of a 2-cell on the boundary of c, where b_2 is the boundary of f_2. It follows that they cannot be both boundaries of a 2-cell, otherwise they would intersect. Thus, we can add a new 2-cell with one of these boundaries and then split c. Then, by iterating this operation the statement of the theorem follows. □

As a note, we want to point out that a biconnected graph with two cycles has always a third one (the symmetric 1-cell difference). If a biconnected graph $G = (V, E)$ has only three cycles, it is the one-dimensional skeleton of a regular cellulation of the 3-sphere with only two 3-cells and these 3-cells have the same boundary. Then, it follows from Theorem 3 that the conjecture formulated in this paper is equivalent to the one in De Agostino [2].

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