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A CONJECTURE OF CLAW-FREE HAMILTONIAN GRAPHS WITH NEIGHBORHOOD UNION

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Abstract: R.J. Faudree et al obtained that if G is a 3-connected claw-free graph of order n, and $|N(u) \cup N(v)| \geq (2n-3)/3$ for each pair of nonadjacent vertices u, v, then G is Hamiltonian. They conjectured that if G is a 3-connected claw-free graph of order n, and $|N(u) \cup N(v)| \geq (2n-6)/3$ for each pair of nonadjacent vertices u, v, then G is Hamiltonian. This paper we prove that if G is a 3-connected claw-free graph of order n, and $|N(u) \cup N(v)| \geq (2n-7)/3$ for each pair of nonadjacent vertices u, v, then G is Hamiltonian.

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1. Introduction

The graphs we consider in this paper are finite, undirected, without loops or multiple edges. For a graph G, we denote by δ (G) the minimum degree. The set of vertices of a graph G is denoted by V(G); the set of edges is denoted by E(G). If H and S are subsets of V(G) or subgraph of G, we denote by N_H (S) the set of vertices in H which are adjacent to some vertex in S and set $|$ N_H (S) $|$ $=d_H$ (S). In particular, when H=G and S={u}, then set N_G (S)=N(u) and set d_G (S)=d(u). We denote by G-H and G[S] the induced subgraphs of G on V(G)-V(H) and S, respectively. Denote a cycle of order m by $C_m = x_1 x_2 \ldots x_m x_1$. Set $N^+_{Cm}(u)=\{x_{i+1} : x_i \in N_{Cm}(u)\}$ and $N^-_{Cm}(u)=\{x_{i-1} : x_i \in N_{Cm}(u)\}$, $N^{\pm}_{Cm}(u)=N^+_{Cm}(u) \cup N^-_{Cm}(u)$. Neighborhood union NC=min$\{ \mid N(u) \cup N(v) \mid : u,v \in V(G), uv \notin E(G)\}$, where subscripts are taken modulo m. Notation

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will generally follow that in [1].

A graph G is called $K_{1,3}$-free (claw-free) if it has no induced subgraph isomorphic to $K_{1,3}$.

In 1988 Faudree, Gould and Lindquester [2] considered the following $K_{1,3}$ -free Hamiltonian graphs.

Theorem1. (see Faudree et al [2], [3]) *If G is a 3-connected $K_{1,3}$ -free graph of order n, and $NC \geq (2n-3)/3$, then G is Hamiltonian.*

They also showed the following conjecture.

Conjecture. (see Faudree et al [2], [3]) *If G is a 3-connected $K_{1,3}$ -free graph of order n such that $|N(x) \cup N(y)| \geq (2n-6)/3$ for each pair of nonadjacent vertices x, y, then G is Hamiltonian.*

In this paper we prove the further result.

Theorem 2. *If G is a 3-connected $K_{1,3}$ -free graph of order n such that $|N(x) \cup N(y)| \geq (2n-7)/3$ for each pair of nonadjacent vertices x, y, then G is Hamiltonian.*

Proof. Assume, to the contrary, that G is not Hamiltonian. Then we let C_m denote a longest cycle of G, G_1 denote a component of G-C_m. Since G is 3-connected, we let $x_i, x_j, x_k \in N_{Cm}(G_1)$ with $\{x_{i+1}, x_{i+2}, \ldots x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{k-1}\} \cap N_{Cm}(G_1) = \Phi$ and let $x \in V(G_1)$, $x_k \in N_{Cm}(x)$. Since C_m is a longest cycle, clearly none of $\{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}, x_{k-1}, x_{k+1}\}$ is adjacent to some vertex of G_1. Therefore G is $K_{1,3}$ -free graph.

Hence we have the following corollary.

Corollary 1. *$x_{i-1} x_{i+1}, x_{j-1} x_{j+1}, x_{k-1} x_{k+1} \in E(G)$. Otherwise, if $x_{i-1} x_{i+1} \notin E(G)$. Together with $ux_{i-1}, ux_{i+1} \notin E(G)$, this contradicts that G is $K_{1,3}$ -free graph. Similarly, we have $x_{j-1} x_{j+1}, x_{k-1} x_{k+1} \in E(G)$.*

Corollary 2. *x_{k+1} is not adjacent to any vertex of $\{x_i, x_{i+1}, x_{i+2}, x_j, x_{j+1}, x_{j+2}\}$; x_{i+1} is not adjacent to any vertex of $\{x_k, x_{k+1}, x_{k+2}, x_j, x_{j+1}, x_{j+2}\}$; x_{j+1} is not adjacent to any vertex of $\{x_k, x_{k+1}, x_{k+2}, x_i, x_{i+1}, x_{i+2}\}$. Otherwise, if $x_{k+1} x_i \in E(G)$, we let P_1 (G_1) denote a path of G_1 with two end-vertices adjacent to x_k, x_i, respectively. Then cycle $x_k P(G_1) x_i x_{k+1} x_{k+2} \ldots x_{i-1} x_{i+1} x_{i+2} \ldots x_k$ is longer than C_m, a contradiction. Similarly, if $x_{k+1} x_{i+1} \in E(G)$, then a longer cycle $x_k P(G_1) x_i x_{i-1} x_{i-2} \ldots x_{k+1} x_{i+1} x_{i+2} \ldots x_k$ is formed, a contradiction. If $x_{k+1} x_{i+2} \in E(G)$, then a longer cycle $x_k P(G_1) x_i x_{i+1} x_{i-1} x_{i-2} \ldots x_{k+1} x_{i+2} x_{i+3} \ldots x_k$ is formed, a contradiction. By a similar argument, we can prove that x_{i+1} is not adjacent to any vertex of $\{x_k, x_{k+1}, x_{k+2}, x_j, x_{j+1}, x_{j+2}\}$, x_{j+1} is not adjacent to any vertex of $\{x_k, x_{k+1}, x_{k+2}, x_i, x_{i+1}, x_{i+2}\}$.*

We consider the following cases.

Case 1. There exists vertex x of G_1 such that $d(x) \leq (n-9)/3$.

In this case, if $x_k \in N(x)$, by $| N(x_{k+1}) \cup N(x) | \geq NC \geq (2n-7)/3$, we have $| N(x_{k+1})| \geq NC\text{-}d(x) + | \{x\} | \geq (n+5)/3$. If x_j, $x_k \in N_{Cm}$ (G_1) with $\{x_{j+1}, x_{j+2} \ldots, x_{k-1}\} \cap N_{Cm}$ (G_1), then we consider the following:

(1) When some vertex x_h of $\{x_{k+1}, x_{k+2}, \ldots, x_j\}$ is adjacent to x_{k+1}. Then we claim: x_{h-1} is not adjacent to x_{j+1} and x. Otherwise, if $x_{h-1}\, x_{j+1} \in E(G)$. Let P^* be a path of G_1 with two end-vertices adjacent to x_j, x_k, respectively. Then the cycle: $x_j\, P^* x_k\, x_{k-1} \ldots x_{j+1}\, x_{h-1}\, x_{h-2} \ldots x_{j+1}\, x_h\, x_{h+1} \ldots x_j$ is longer than C_m, a contradiction. If $x_{h-1}\, x \in E(G)$. Let P^* be a path of G_1 with two end-vertices adjacent to x_k, x_{h-1}, respectively. Then cycle: $x_k\, P^* x_{h-1}\, x_{h-2} \ldots x_{k+1}\, x_h\, x_{h-1} \ldots x_k$ is longer than C_m, a contradiction.

(2) When some vertex x_h of $\{x_{j+1}, x_{j+2}, \ldots, x_k\} \setminus \{x_{k-1}, x_k\}$ is adjacent to x_{k+1}, then x_{h+1} is not adjacent to x_{j+1} and x. Otherwise, if they are adjacent, by a similar argument as above (1), clearly we have a cycle longer than C_m, a contradiction. x_{k+1} is not adjacent to x_{i+2}. Otherwise, if $x_{i+2}\, x_{k+1} \in E(G)$. Together with Corollary 1: $x_{i-1}\, x_{i+1} \in E(G)$, let P^* be a path of G_1 with two end-vertices adjacent to x_i, x_k, respectively. Then the cycle: $x_k\, P^* x_i\, x_{i+1}\, x_{i-1}\, x_{i-2} \ldots x_{k+1}\, x_{i+2}\, x_{i+3} \ldots x_k$ is longer than C_m, a contradiction. Since x_{i+1} is not adjacent to x_{j+1} and x.

(3) When some vertex y of $G\text{-}C_m$ is adjacent to x_{k+1}. Then y is not adjacent to x_{j+1} and x.

Now, we define a bijection f on $N(x_{k+1})$ as follows: Let $u \in N(x_{k+1})$,

$$f(u) = \begin{cases} v = u\,, & \text{for } \ u \notin V(C)\,, \\ v = x_{h-1}\,, & \text{for } \ u = x_h \in \{x_{k+1}\}\,, \\ v = x_{h+1}\,, & \text{for } \ u = x_h \in \{x_{j+1}, x_{j+2}, \ldots, x_k\} \setminus \{x_{k-1}\}\,. \end{cases}$$

From the previous arguments, for any $v \in f\ (N(x_{k+1}))$, we have vx_{j+1}, $vx \notin E(G)$. Note that for any $u \in \{x_{j+1}, x, x_{i+1}\}$, then $u \notin f\ (N(u_2) \cup N(x_2))$ and ux_{j+1}, $ux \notin E(G)$. Hence we have $| N(x_{j+1}) \cup N(x) | \leq | V(G) | - | N(x_{k+1}) \setminus \{x_k, x_{k-1}\} | - | \{x_{j+1}, x, x_{i+1}\} | \leq (2n-8)/3$, a contradiction.

Case 2. There exists vertex x in G_1 such that $d(x) \geq (n-2)/3$.

Subcase 2.1. There exists vertex x in G_1 such that $d(x) \geq (n-1)/3$.

In this case, clearly, by corollary: $x_{k-1}\, x_{k+1} \in E(G)$, so none of N^+_{Cm} (x) $\cup \{x_k, x_{k+2}, V(G_1)\}$ is adjacent to x_{i+1} or x_{j+1}. Otherwise, if there exists some vertex of N^+_{Cm} (x) $\cup \{x_k, x_{k+2}, V(G_1)\}$ is adjacent to x_{i+1} or x_{j+1}, by a similar argument of Case 1, we must obtain a longer cycle, a contradiction. For example, if x_{k+2} is adjacent to x_{i+1}. Let P^* be a path of G_1 with two end-vertices adjacent to x_i, x_k, respectively. Then cycle: $x_i\, P^* x_k\, x_{k+1}\, x_{k-1}\, x_{k-2} \ldots x_{i+1}\, x_{k+2}\, x_{k+3} \ldots x_i$ is longer than C_m, a contradiction.

Thus, we have $|\ N(x_{i+1}) \cup N(x_{j+1}) \leq |\ V(G)\ | - |\ N^{+}_{Cm}\ (x)\ | - |\ \{x_k,\ x_{k+2},\ V(G_1)\ \}\ | \leq (2n\text{-}8)/3$, a contradiction.

Subcase 2.2. There exists vertex x of G_1 such that d(x)=(n-2)/3.

Subcase 2.2.1. $|\ N_{Cm}\ (G_1)| \geq 5$.

Let the consecutive order $x_k,\ x_i,\ x_j,\ x_h,\ x_r \in N_{Cm}\ (G_1)$ with $\{x_{k+1},\ x_{k+2}\ \ldots,\ x_{i-1},\ x_{i+1},\ x_{i+2},\ \ldots,\ x_{j-1},\ x_{j+1},\ x_{j+2},\ \ldots,\ x_{h-1},\ x_{h+1},\ x_{h+2},\ \ldots,\ x_{r-1},\ x_{r+1}\ \} \cap N_{Cm}\ (G_1) = \Phi$.

Clearly, we have $|\ N(x_{j+1}) \cup N(x_{k+1})| \leq |\ V(G)\ | - |\ N^{+}_{Cm}\ (x)\ | - |\ V(G_1)| - |\ \{x_{k+2},\ x_{i+2},\ x_{r+2}\ \}\ | \leq (2n\text{-}8)/3$, a contradiction.

Subcase 2.2.2. $|\ N_{Cm}\ (G_1)| = 4$.

In this case, we claim: G_1 is a complete subgraph. Otherwise, if there exist two nonadjacent vertices u,v in G_1, then we can check $|\ N(x_{i+1}) \cup N(x_{j+1})| \leq |\ V(G)\ | - |\ N(u) \cup N(v)\ | - |\ \{u,v\}\ | \leq (2n\text{-}8)/3$, a contradiction.

Therefore, every component of G-C_m is complete subgraph.

Subcase 2.2.2.1. $|\ V(G_1)| \geq 2$.

Let the consecutive order $x_k,\ x_i,\ x_j,\ x_h \in N_{Cm}\ (G_1)$ with $\{x_{k+1},\ x_{k+2}\ \ldots,\ x_{i-1},\ x_{i+1},\ x_{i+2},\ \ldots,\ x_{j-1},\ x_{j+1},\ x_{j+2},\ \ldots, x_{h-1}\ \} \cap N_{Cm}\ (G_1) = \Phi$.

Then we have $|\ N(x_{j+1}) \cup N(x_{i+1})| \leq |\ V(G)\ | - |\ N^{+}_{Cm}\ (x)\ | - |\ V(G_1)| - |\ \{x_{k+2},\ x_{k+3},\ x_{r+2},\ x_{r+3}\ \}\ | \leq (2n\text{-}8)/3$, a contradiction.

Subcase 2.2.2.2. $|\ V(G_1)| = 1$.

In this case, we have d(x)=(n-2)/3=4, this implies n=14. By the corollary: $x_{i-1}\ x_{i+1},\ x_{j-1}\ x_{j+1},\ x_{k-1}\ x_{k+1} \in E(G)$, we can get a cycle longer than C_m, a contradiction.

Subcase 2.2.3. $|\ N_{Cm}\ (G_1)| = 3$.

Let the consecutive order $x_k,\ x_i,\ x_j \in N_{Cm}\ (G_1)$ with $\{x_{k+1},\ x_{k+2}\ \ldots,\ x_{i-1},\ x_{i+1},\ x_{i+2},\ \ldots,\ x_{j-1},\ x_{j+1}\ \} \cap N_{Cm}\ (G_1) = \Phi$.

By a similar to the proof of Subcase 2.2.2, we can obtain that every component of G-C_m is complete subgraph.

Subcase 2.2.3.1. $|\ V(G_1)| \geq 3$.

In this case, clearly, we have $|\ N(x_{j+1}) \cup N(x_{i+1})| \leq |\ V(G)\ | - |\ N^{+}_{Cm}\ (x)| - |\ V(G_1)| - |\{x_{k+2},\ x_{k+3},\ x_{k+4}\ \}\ | \leq |\ V(G)\ | - d(x) - |\ \{x\}\ | - 3 \leq (2n\text{-}8)/3$, a contradiction.

Subcase 2.2.3.2. $|\ V(G_1)| = 2$.

When $x_k,\ x_i,\ x_j \in N_{Cm}\ (x)$, then d(x)=(n-2)/3=4, this implies n=14. By Corollary (a): $x_{i-1}\ x_{i+1},\ x_{j-1}\ x_{j+1},\ x_{k-1}\ x_{k+1} \in E(G)$, easily we can obtain a cycle longer than C_m, a contradiction.

Subcase 2.2.3.3. $|\ V(G_1)| = 1$.

By a similar argument of Subcase 2.2.3.2, we can obtain a cycle longer than C_m, a contradiction.

Case 3. (n-8)/3 $\leq$ d(x) $\leq$ (n-3)/3 for any vertex x of G-C_m.

Subcase 3.1. There exists a component G_1 of G-C_m such that $|N_{Cm}(G_1)| \geq 4$.

Let the consecutive order $x_k, x_i, x_j, x_h \in N_{Cm}(G_1)$ with $\{x_{k+1}, x_{k+2} \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{h-1}\} \cap N_{Cm}(G_1) = \Phi$.

By $|N(x_{k+1}) \cup N(x)| \geq NC \geq (2n\text{-}7)/3$, this implies that $|N(x_{k+1})| \geq NC\text{-}d(x) + |\{x_k\}| \geq (n\text{-}1)/3$, and $|N(x_{i+1})| \geq (n\text{-}1)/3$, $|N(x_{j+1})| \geq (n\text{-}1)/3$, $|N(x_{h+1})| \geq (n\text{-}1)/3$.

Subcase 3.1.1. There exists a component G_1 of G-C_m such that $|N_{Cm}(G_1)| \geq 5$.

Let the consecutive order $x_k, x_i, x_j, x_h, x_r \in N_{Cm}(G_1)$. By $|N(x_{k+1})| \geq NC\text{-}d(x) + |\{x_k\}| \geq (n\text{-}1)/3$ and since C_m is a longest cycle of G, we have $|N(x_{i+1}) \cup N(x)| \leq |V(G)| - |V(G_1)| - |(N(x_{j+1}) \setminus \{x_{j-1}, x_j\}| - |\{x, x_{i+1}, x_{k+1}, x_{h+1}, x_{r+1}\}| \leq (2n\text{-}8)/3$, a contradiction.

Subcase 3.1.2. There exists a component G_1 of G-C_m such that $|N_{Cm}(G_1)| = 4$.

Subcase 3.1.2.1. There exists a vertex x of G-C_m with (n-8)/3 $\leq$ d(x) $\leq$ (n-6)/3. In this case, we have $|N(x_{i+1})| \geq NC\text{-}d(x) + |\{x_i\}| \geq (n\text{+}2)/3$. Since C_m is a longest cycle of G, clearly we have $|N(x_{k+1}) \cup N(x)| \leq |V(G)| - |V(G_1)| - |(N(x_{i+1}) \setminus \{x_{i-1}, x_i\}| - |\{x, x_{k+1}, x_{j+1}, x_{h+1}\}| \leq (2n\text{-}8)/3$, a contradiction.

Subcase 3.1.2.2. (n-5)/3 $\leq$ d(x) $\leq$ (n-3)/3 for every vertex x of G-C_m.

Subcase 3.1.2.2.1. $|V(G_1)| \geq 2$.

In this case, we have $|N(x_{i+1})| \geq NC\text{-}d(x) + |\{x_i\}| \geq (n\text{-}1)/3$. Since C_m is a longest cycle of G, clearly we have $|N(x_{k+1}) \cup N(x)| \leq |V(G)| - |V(G_1)| - |(N(x_{i+1}) \setminus \{x_{i-1}, x_i\}| - |\{x, x_{k+1}, x_{j+1}, x_{j+2}, x_{h+1}, x_{h+2}\}| \leq (2n\text{-}8)/3$, a contradiction.

Subcase 3.1.2.2.2. $|V(G_1)| = 1$.

Since G is $K_{1,3}$ -free graph, thus, $x_{k-1} x_{k+1}, x_{i-1} x_{i+1}, x_{j-1} x_{j+1}, x_{h-1} x_{h+1} \in E(G)$. By (n-5)/3 $\leq$ d(x)=4 $\leq$ (n-3)/3. This implies 15 $\leq$ n $\leq$ 17. Therefore, we have d(x)=(n-5)/3, this implies n=17. Then:

If $d(x_{k+1}) \geq 3$ or $d(x_{i+1}) \geq 3$, then we can get a longer cycle, a contradiction.

If $d(x_{k+1}) = d(x_{i+1}) = 2$, then we have $|N(x_{i+1}) \cup N(x_{k+1})| \leq 4 \leq (2n\text{-}8)/3$, a contradiction.

Subcase 3.2. $|N_{Cm}(V(G_1))| = 3$ for any component G_1 of G-C_m.

Let the consecutive order $x_k, x_i, x_j \in N_{Cm}(G_1)$ with $\{x_{k+1}, x_{k+2} \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{j-1}, x_{j+1}\} \cap N_{Cm}(G_1) = \Phi$.

In this case, if there exist two nonadjacent vertices u,v of G_1, then we have $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N(u) \cup N(v)| - |\{u,v\}| \leq (2n\text{-}8)/3$, a contradiction.

Therefore, this shows that every component of G-C_m is complete subgraph. Since $(n-8)/3 \leq d(x)=3 \leq (n-2)/3$ for every vertex x of G-C_m. Then we consider the following subcases:

Subcase 3.2.1. $|V(G_1)| = 1$.

In this case by $(n-8)/3 \leq d(x) \leq (n-3)/3$ and d(x)=3, this implies $12 \leq n \leq 17$.

When $(n-6)/3 \leq d(x)=3 \leq (n-2)/3$, we have $11 \leq d(x) \leq 15$. Thus, $\min\{|j-i|, |k-j|, |i-k|\} \leq 4$. Without loss of generality, let $|j-i| \leq 4$. (a). When $|j-i| \leq 3$, clearly we can get a longer cycle. (b). When $|j-i| = 4$, clearly, x_{i+2} is adjacent to at most one vertex of $\{x_{i+1}, x_{k-1}\}$. Hence $|N(x) \cup N(x_{i+2})| \leq |\{x_i, x_j, x_k, x_{i+1}, x_{k-1}\}| = 5 \leq (2n-8)/3$, a contradiction.

When $(n-8)/3 \leq d(x)=3 \leq (n-7)/3$, this implies $16 \leq n \leq 17$, we have $\min\{|j-i|, |k-i|, |i-k|\} \leq 5$. Without loss of generality, let $|j-i| \leq 5$. Since C_m is a longest cycle of G, thus, x_{i+2} is adjacent to x_{i+1}, x_{i+3}, x_{i+4} of C_m, then we have $|N(x_{i+2}) \cup N(x)| \leq |N(x)| + |\{x_{i+1}, x_{i+3}, x_{i+4}\}| = 6 \leq (2n-8)/3$, a contradiction.

Subcase 3.2.2. $|V(G_1)| \geq 2$.

If there exists vertex x of G-C_m with $(n-8)/3 \leq d(x) \leq (n-6)/3$. Then by $|N(x_{i+1})| \geq NC - d(x) + |\{x_i\}| \geq (n+2)/3$, we have $|N(x_{k+1}) \cup N(x)| \leq |V(G)| - |N(x_{i+1}) \setminus \{x_{i-1}, x_i\}| - |\{x, x_{k+1}, x_{j+1}, x_{j+2}\}| \leq (2n-8)/3$, a contradiction.

If $(n-5)/3 \leq d(x) \leq (n-2)/3$ for any vertex x of G-C_m.

Then, we claim: every vertex of G_1 is adjacent to x_i, x_j and x_k. Otherwise, if there exists some vertex u of G_1 is not adjacent to x_i or x_j or x_k. Without loss of generality, say u is not adjacent to x_i. Then u must be adjacent to x_j and x_k. Otherwise, if u is not adjacent to x_k, then we have $|N(x_{j+1}) \cup N(x_{k+1})| \leq |V(G)| - |N^+_{Cm}(u)| - |V(G_1)| - |\{x_{k+1}, x_i, x_{i+1}, x_{i+2}\}| \leq (2n-8)/3$, a contradiction. Thus, $|N(x_{j+1}) \cup N(x_{k+1})| \leq |V(G)| - |N^+_{Cm}(u)| - |V(G_1)| - |\{x_i, x_{i+1}, x_{i+2}\}| = n - d(u) - 4$.

When $d(u) \geq (n-4)/3$. We have $|N(x_{j+1}) \cup N(x_{k+1})| \leq |V(G)| - |N^+_{Cm}(u)| - |V(G_1)| - |\{x_i, x_{i+1}, x_{i+2}\}| = n - d(u) - 4 \leq (2n-8)/3$, a contradiction.

When $d(u)=(n-5)/3$, if $|V(G_1)| = 2$, we have $d(u)=(n-5)/3=3$, this implies n=14. Since G is $K_{1,3}$ -free graph, thus, $x_{k-1} x_{k+1}$, $x_{i-1} x_{i+1}$, $x_{j-1} x_{j+1} \in E(G)$, easily we can get a longer cycle, a contradiction.

If $|V(G_1)| \geq 3$. Since $(n-7)/3 \leq d(x) \leq (n-2)/3$ for every vertex x of G-C_m. Hence $|N(x_{k+1}) \cup N(x_{j+1})| \leq |V(G)| - |N^+_{Cm}(u)| - |V(G_1)| - |\{x_{i+3}, x_{i+2}, x_{i+1}, x_i\}| \leq (2n-8)/3$, a contradiction.

Therefore, the claim that every vertex of G_1 is adjacent to x_i, x_j and x_k is true.

We consider the following.

By the above claim: Since C_m is a longest cycle of G, clearly we have that x_{k+3} is not adjacent to x_{i+1} and x_{j+1}.

When $d(x) \geq (n-4)/3$, since x is not adjacent to x_{k-1}, x_{k+1} and x_{k+2}, and every vertex of $\{x_k, x_{k+2}, x_{k+3}\}$ is not adjacent to x_{i+1} and x_{j+1}. Hence $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N^+_{Cm}(x)| - |V(G_1)| - |\{x_{k+2}, x_{k+3}, x_k\}| \leq (2n-8)/3$, a contradiction.

When $d(x) \leq (n-6)/3$. By $|N(x_{i+1})| \geq NC- |N(x)| + |\{x_i\}| \geq (n+2)/3$, we have $|N(x_{k+1}) \cup N(x)| \leq |V(G)| - |N(x_{i+1}) \setminus \{x_{i-1}, x_i\}| - |\{x, x_{k+1}, x_{j+1}, x_{j+2}\}| \leq (2n-8)/3$, a contradiction.

When $d(x)=(n-5)/3$, if $|V(G_1)| = 2$, easily we have $d(x)=(n-5)/3=4$, this implies $n=17$. When x_{i+2} is adjacent to some vertex of $C_m \setminus \{x_{i+1}, x_{i+3}\}$, we can get a longer cycle, a contradiction.

When x_{i+2} is not adjacent to some vertex of $C_m \setminus \{x_{i+1}, x_{i+3}\}$, and x_{j+2} is not adjacent to some vertex of $C_m \setminus x_{j+1}, x_{j+3}$, then we can check that $|N(x_{i+1}) \cup N(x_{j+1})| \leq 4 \leq (2n-8)/3$, a contradiction.

If $|V(G_1)| \geq 3$, by $|N(x_{i+1})| \geq NC- |N(x)| + |\{x_i\}| \geq (n+1)/3$, we have $|N(x_{k+1}) \cup N(x)| \leq |V(G)| - |N(x_{i+1}) \setminus \{x_{i-1}, x_i\}| - |\{x, x_{k+1}, x_{j+1}, x_{j+2}, x_{j+3}\}| \leq (2n-8)/3$, a contradiction.

Therefore, the proof of theorem is complete. □

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