International Journal of Pure and Applied Mathematics

Volume 32 No. 4 2006, 1-4

STABLE AND SEMISTABLE COHERENT SYSTEMS ON SMOOTH PROJECTIVE CURVES

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Abstract: Fix a smooth genus $g \geq 2$ curve X. Here we prove an existence theoremfor coherent systems on X with certain parameters using a specialization argumentand a stronger existence theorem for general genus g curves recently provedby M. Teixidor i Bigas.

AMS Subject Classification: 14H60
Key Words: stable vector bundle, coherent system on a curve, semistable coherent system, stable coherent system

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Let X be a smooth and connected complex projective genus $g \geq 2$ curve. Fix integers $n > 0, k > 0, d$ and $\alpha \in \mathbb{R}$, $\alpha \geq 0$.For the theory of coherent systems, see [1], [2], and refences therein. We just recall that a coherent system of type (n, d, k)on X is a pair (E, V) with E a rank n vector bundle on X with degree d and V a k-dimensionallinear subspace of $H^0(X, E)$.Set $\mu_\alpha(E, V) = (d + \alpha \cdot k)/n$ and use the α-slope to define the notion of α-stabilityand α-semistability for coherent systems. In this paper we prove the following existence theoremfor stable coherent systems.

Theorem 1. *Let X be a smooth and connected curve with genus $g \geq 2$. Fix integers d, r, k with $k > r$ and $d \geq r$. AssumeMCD($(d, r, k) = 1$. Write*

Received: October 4, 2006

$$d = rd_1 + d_2, \quad k = rk_1 + k_2, \quad 0 \le d_2 < r, \quad 0 \le k_2 < r\,.$$

Assume

$$g - (k_1+1)(g - d_1 + k_1 - 1) \ge 1 \textit{ if } 0 \ne d_2 \ge k_2\,, \tag{1}$$

$$g - k_1(g - d_1 + k_1 - 1) > 1 \textit{ if } d_2 = k_2 = 0\,, \tag{2}$$

$$g - (k_1+1)(g - d_1 + k_1) \ge 1 \textit{ if } 0 \le d_2 < k_2\,. \tag{3}$$

Then there exists a coherent system (E, V) of type (r, d, k) on X which is α-stable for all real $\alpha > 0$and with E a semistable vector bundle.

A stronger form of Theorem 1 was proved in [4] when X is a general genus g curve (i.e. withoutthe very restrictive assumption MCD$((d, r, k) = 1)$,except that it was only claimed there that for every $\alpha > 0$ there is anα-stable coherent system (E_α, ϕ_α) of type (n, d, k). It is easy to see thatin that case we may find (E, V) which is α-stable for all $\alpha > 0$(see [4], middle of p. 8, in which she proved that (E, V) is α-stablefor all α). We will use the result provedin [4] to get by specialization the result for an arbitrary smooth genus g curve. Our proof were inspiredfrom a similar situation considered in [3].

Remark 1. Let (E, V) be a coherent system on a smooth projective curve and $\alpha > \beta > \gamma \ge 0$ real numbers. If $(E, V)$is α-stable and γ-stable, then it is β-stable (use [1], Lemma 6.1).

Remark 2. Let (E, V) be a coherent system on a smooth projective curve. If E is stable, then (E, V) is α-stablefor $0 < \alpha \ll 1$. If E is not semistable, then (E, V) is not α-stablefor $0 < \alpha \ll 1$.

Remark 3. Let (E, V) a coherent system of type (r, d, k) on a smooth projective curve. Assume that (E, V) is α-semistable,but not α-stable for all $0 < \alpha \ll 1$. Then E is semistable, but not stable (Remark 2)and there are an integer $t \ge 2$ dividing d, r, and k and a rank r/t subbundle F of E such that rank$(F) = r/t$,deg$(F) = d/k$ and $\dim(V \cap H^0(F)) = k/t$.

Remark 4. Fix integers $n > 0$, $k > 0$ and d. There are finitely many real numbers $0 < \alpha_1 < \cdots < \alpha_s$such that for every smooth and connected projective curve Y the following holds. Let (E, V) be a coherent system on Y not of the type described in Remark3. If (E, V) is α-semistable, but not α-stable, then $\alpha = \alpha_i$ for some$i$. Now take a coherent system (E, V) of type (n, d, k) on Y not of the type describedin Remark 3 and which is γ-semistable for some$\gamma \in \mathbb{R}$ such that $0 \le \gamma < \alpha_1$ and μ-semistable for some $\mu \in \mathbb{R}$such that $\alpha_s < \mu$. Then (E, V) is γ-stable and μ-stable. By Remark 1 $(E, V)$is α-stable for all α such that $\beta \le \alpha \le \gamma$; since$\mu > \alpha_s$, this is true only assuming that E is semistable.

As in [1], Section 6, we alsoget that if E is stable, then (E, V) is α-stable for all $\alpha > \alpha_s$ and for all $0 < \alpha < \alpha_1$. Hence(E, V) is α-stable for all $\alpha > 0$ (although, in general it is only 0-semistable, i.e.E may be properly semistable).

Proof of Theorem 1. Let$0 < \alpha_1 < \cdots < \alpha_s$ be the critical values for the type (n, d, k).Fix $\alpha > 0$. Fix a smooth and connected genus g projective curve X. Take a smooth and connected affine curve T, $o \in T$,and a proper smooth morphism $\pi : Y \to T$ such that $Y_o := \pi^{-1}(o) \cong X$ andfor each curve $Y_t := \pi^{-1}(t)$, $t \in T\backslash\{o\}$, there is an α-stable coherent system $(F_t, V_t)$of type (n, d, k) on Y_t (use [4] and take as Y_t, $t \neq o$, a general genus g curve). Furthermore,by the proof in [4] we may also assume that each F_t, $t \neq o$, is stable. Up to a finite covering of T, we may also assumethe existence of a relative coherent system $(\mathcal{F}, \mathcal{V})$ on $Y\backslash Y_o$ such that$\mathcal{F}|Y_t \cong F_t$ and $\mathcal{V}|Y_t = V_t$for all $t \neq o$. By a closedness property of semistability and α-semistability we may also find (up toanother finite covering of T) a flat family of morphisms $\psi : \mathcal{O}_Y^k \to \mathcal{E}$ on Y such that$\mathcal{E}|Y\backslash Y_o = \mathcal{F}$, $\psi|Y_t$ induces the morphism $\psi_t : \mathcal{O}_{Y_t}^k \to F_t$defining the coherent system (F_t, V_t),$F_o := \mathcal{E}|Y_o$ is semistable and the pair $(F_o, \phi|Y_o)$ satisfies all the properties of anα-semistabe coherent system,except that we do not claim that the map $(\phi|Y_o)_* : H^0(Y_o, \mathcal{O}_{Y_o}^k) \to H^0(Y_o, E_o)$ is injective(see [1], p. 688).The map $(\phi|Y_o)_*$ is injective because $(F_o, \phi|Y_o)$ is α-semistable [1], p. 688. Thus (E_o, V_o) is an α-semistable coherent system on X. This coherent systemis equivalent to a unique α-polystable coherent system (E', V') obtained as the graded object associatedto a Jordan-Hölder filtration of (E_o, V_o) ([1], p. 689).The coherent system (E', V') may depend from $\alpha$and hence we will write $(E(\alpha), V(\alpha)) := (E', V')$.

Claim. *We may find (E', V') such that $(E(\alpha), V(\alpha)) = (E(\beta), V(\beta))$ for all positive α, β.*

Proof of Claim. The limit linear series constructed in [4] does not dependof the real number α. Hence we may find $\pi : Y \to T$ such that the stable vector bundle $F_t$does not depends from the choice of $\alpha \in \mathbb{R}$, $\alpha \geq 0$, for all $t \neq o$. We may take as V_ta general k-dimensional linear subspace of $H^0(Y_t, F_t)$ (even if $h^0(Y_t, F_t) > k$) for the following reason.Let $G(k, H^0(Y_t, F_t))$ denote the Grassmannian of all k-dimensional linear subspaces of $H^0(Y_t, F_t)$.Hence $G(k, H^0(Y_t, F_t))$ is irreducible. Hence a finite intersection of non-zero Zariski open subsets of $G(k, H^0(Y_t, F_t))$is non-empty. Since E_t is stable, we need to check the α-stability condition only forthe real numbers α_i. Hence in the construction we may take E' independent from α. We take as V'a general k-dimensional linear subspace of $H^0(X, E')$ and conclude the proof of the Claim as above. □

Since (E', V') is α-stable for some $0 < \alpha < \alpha_1$ and for all $\alpha \gg 0$, (E', V') is

α-stablefor all $\alpha > 0$ and it is 0-semistable, i.e. E is semistable. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] S. Bradlow, O. Garcìa-Prada, V. Munoz, P. Newstead, Coherent systems and Brill-Noether theory, *Internat. J. Math.*, **14**, No. 7 (2003), 683-733.

[2] S. Bradlow, O. Garcìa-Prada, V. Mercat, V. Munoz, P. Newstead, On the geometry of moduli spaces of coherent systems on algebraic curves, *ArXiv*: math.AG/0407523.

[3] M. Teixidor-i-Bigas, On Lange's conjecture, *J. Reine Angew. Math.*, **528** (2000), 81-99.

[4] M. Teixidor-i-Bigas, Existence of coherent systems, *ArXiv*: math.AG/0606348.