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WEAKLY STABLE HARMONIC MAPS BETWEEN COMPACT HOMOGENEOUS SPACES

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Abstract: Using Lichnerowicz's Theorem, we get weakly stable harmonic mappings $\overline{A_h}$ from compact homogeneous space $(G/T, g)$ into $(G/T, g')$ for arbitrary given G-invariant metrics g, g' a maximal torus T which is induced by inner automorphism $A_h : G \longrightarrow G$ $(h \in T)$, and calculate the nullity of Jacobi operator of such harmonic maps.

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1. Introduction

In this paper, we produce new examples of weakly stable harmonic maps. We consider the induced map $\overline{A_h} : (G/T, g) \to (G/T, g'), (h \in T)$, of inner automorphism $A_h : G \to G$. Here, g, g' are arbitrary given G-invariant Riemannian metrics on compact reducible homogeneous space G/T. Moreover, we treat the case G/T for a maximal torus T.

In Section 2 we introduce Guest's criterion (cf. Lemma 2.1) for the map between reductive homogeneous space G/H and G'/H' induced by a Lie group homomorphism from G into G'. In Section 3 using this criterion we show that every $\overline{A_h}$ is harmonic. In Section 4 we obtain a necessary and sufficient condition for $(G/T, g, J)$ to be Kähler. Using Lichnerowicz's Theorem (cf. Theorem 4.5), we get weakly stable harmonic mappings $\overline{A_h}$ $(h \in T)$, and calculate the nullity

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of Jacobi operator $J_{\overline{A_h}}$ of such harmonic mappings $\overline{A_h}$ (cf. Proposition 4.7, Theorem 4.8).

Finally, as example we give weakly stable harmonic mappings $\overline{A_h} : (SU(n)/T, g) \rightarrow (SU(n)$ (cf. Proposition 4.9).

2. Preliminaries

In this section, we review Guest's work which gives a nessary and sufficient condition for the map induced by a homomorphism $\theta : G \to G'$ between reductive homogeneous spaces $G/K, G'/K'$ with invariant Riemannian metrics to be harmonic (cf. [2]).

Let $\theta : G \rightarrow G'$ be a homomorphism of compact Lie groups G, G' such that $\theta(K) \subset K'$ for closed subgroups K, K'. We denote by $\mathfrak{g}$ (resp. $\mathfrak{k}, \mathfrak{g}'$ and $\mathfrak{k}'$) the Lie algebra of all left invariant vector fields on G (resp. K, G' and K'). Let f_θ be the map between reductive homogeneous spaces G/K, G'/K' induced by θ, that is, $f_\theta(xK) = \theta(x)K'$, $(x \in G)$. Let $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (direct sum of vector spaces) and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. Denote by $\mathfrak{X}(G/K)$ the space of all the left invariant vector fields on G/K. Recall the map

$$\mathfrak{g} \longrightarrow \mathfrak{X}(G/K)\,,$$

$$X \longmapsto X^\star_p(f) = (\frac{d}{dt}) f((\mathrm{expt})X \cdot p)|_{t=0} \quad (f \in C^\infty(G/K),\ p \in G/K),$$

under which we get the identification $\mathfrak{m} \simeq T_p(G/K)$.

The differential df_θ of the induced map f_θ is determined by its restriction to $O := \{K\} \in G/K$, which is given in terms of the Lie algebra homomorphism $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}'$ by

$$df_\theta(X) = \theta(X)_{\mathfrak{m}'}\,, \quad X \in \mathfrak{m}, \tag{2.1}$$

where $\theta(X)_{\mathfrak{m}'}$ denotes the $\mathfrak{m}'$-component of the element $\theta(X) \in \mathfrak{g}' = \mathfrak{m}' + \mathfrak{k}'$.

Let $< \, , \, >$ (resp. $< \, , \, >'$) be an inner product which is invariant with respect to $\mathrm{Ad}\,(K)$ (resp. $\mathrm{Ad}\,(K')$) on $\mathfrak{m}$ (resp. $\mathfrak{m}'$), where Ad denotes the adjoint representation of K (resp. K') in $\mathfrak{g}$ (resp. $\mathfrak{g}'$). This inner product $< \, , \, >$ (resp. $< \, , \, >'$) determines an invariant Riemannian metric $g_{< \, , \, >}$ (resp. $g'_{< \, , \, >'}$) on G/K (resp. G'/K'). Then, the Levi-Civita connection induced by $g_{< \, , \, >}$ is given as follows (cf. [2, 4])

$$\nabla_{X^\star} Y^\star|_0 = (-\frac{1}{2})[X, Y]_{\mathfrak{m}} + U(X, Y) \quad (X, Y \in \mathfrak{m}), \tag{2.2}$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > = < [Z,X]_{\mathfrak{m}}, Y > + < X, [Z,Y]_{\mathfrak{m}} > \qquad (X,Y,Z \in \mathfrak{m}), \quad (2.3)$$

and $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element $X \in \mathfrak{g} = \mathfrak{k} + \mathfrak{m}$.

Recall that for Riemannian manifolds $(M,g), (N,h)$, a smooth map $f : M \to N$ is said to be *harmonic* if $\mathrm{tr}(\nabla df) = 0$, namely, the tension field $\tau(f)$ vanishes identically (cf. [1, 6]). For a harmonic map $\phi : (M,g) \to (N,h)$ (with M compact), ϕ is said to be *weakly stable* (resp. *stable*) (cf. [1, 6]) if the index of Jacobi operator J_ϕ is zero (resp. if index $(J_\phi) =$ nullity $(J_\phi) = 0$).

Guest [2, Lemma 2.1] obtained the following result.

Lemma 2.1. *The induced map $(G/K, g)$ into $(G'/K', g')$ is harmonic if and only if* $\sum_{i=1}^{m}\{[\theta(X_i)_{\mathfrak{k}'}, df_\theta(X_i)] + U'(df_\theta(X_i), df_\theta(X_i)) - df_\theta(U(X_i,X_i))\} = 0$, *where $\{X_i\}_{i=1}^{m}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $< \, , \, >$ and $m :=$ dim $(G/K) =$ dim $\mathfrak{m}$.*

3. Harmonic Maps between Compact Homogeneous Spaces

Let G be a compact semisimple Lie group and T be a maximal torus. We denote by $\mathfrak{g}$ (resp. $\mathfrak{t}$) the Lie algebra of G (resp. T). Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. We denote by Δ the set of all nonzero roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$, and by Δ^+ the set of all positive roots with respect to a fixed linear order in the dual space of $\{H \in \mathfrak{t}^c \mid \alpha(H) \in R \text{ for all } \alpha \in \Delta\}_R$. Let B be the Killing form of $\mathfrak{g}^c$. We define an inner product Q on $\mathfrak{g}$ by $Q(X,Y) := -B(X,Y)$, $(X,Y \in \mathfrak{g})$.

We choose an orthonormal basis of $\mathfrak{g}$ with respect to the inner product Q as follows : For $\alpha \in \Delta$, let E_α be a root vector such that $B(E_\alpha, E_{-\alpha}) = -1$ and $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ for $\alpha, \beta \in \Delta$ $(\alpha + \beta \neq 0)$, where $N_{\alpha,\beta}$ are real numbers defined by

$$\begin{cases} [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha, \beta, \alpha+\beta \in \Delta, \text{ and} \\ N_{\alpha,\beta} = 0 & \text{if } 0 \neq \alpha + \beta \notin \Delta. \end{cases} \quad (3.1)$$

Hence, $[E_\alpha, E_{-\alpha}] = -H_\alpha$, H_α being determined by $B(H, H_\alpha) = \alpha(H)$ for any $H \in \mathfrak{t}$. For $\alpha \in \Delta$, put $U_\alpha = E_\alpha + E_{-\alpha}$, $V_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ which belong to $\mathfrak{g}$. Then

$$\{U_\alpha/\sqrt{2},\ V_\alpha/\sqrt{2} \mid \alpha \in \Delta^+\} \quad (3.2)$$

is an orthonormal basis of $\mathfrak{g}$ with respect to Q.

For $\alpha \in \Delta^+$, let $\mathfrak{m}_\alpha$ be the real vector space generated by $\{U_\alpha, V_\alpha\}$. Then, $\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha$ is a Q-orthogonal $\mathrm{Ad}\,(K)$-invariant decomposition such that

$\mathrm{Ad}\,(T)|_{\mathfrak{m}_\alpha}$ is irreducible for every $\alpha \in \Delta^+$. Then, the space of all invariant Riemannian metrics on G/T is given by $\{<\ ,\ >= \bigoplus_{\alpha\in\Delta^+}(x_\alpha Q|_{\mathfrak{m}_\alpha}) \mid \text{each } x_\alpha > 0,\ (\alpha \in \Delta^+)\}$. Hence, for arbitrary given inner product $<\ ,\ >$ on $\mathfrak{m}$,

$$\{(U_\alpha/\sqrt{2x_\alpha}),\ (V_\alpha/\sqrt{2x_\alpha}\ |\alpha \in \Delta^+\} \tag{3.3}$$

is an orthonormal basis of $\mathfrak{m}$, where every x_α is positive constant determined by $<\ ,\ >$.

Let $\overline{A_h}$ $(h \in T)$ be the map from $(G/T, g)$ into $(G/T, g')$ which is induced by a inner automorphism A_h of G. From now on in this paper, let g (resp. g') in G/T be the Riemannian metric determined by $\bigoplus_{\alpha\in\Delta^+}(x_\alpha Q|_{\mathfrak{m}_\alpha}) =: <\ ,\ >$ (resp. $\bigoplus_{\alpha\in\Delta^+}(y_\alpha Q|_{\mathfrak{m}_\alpha}) =:<\ ,\ >'$). The induced map $\overline{A_h} : (G/T, g) \to (G/T, g')$, $(h \in T)$, is harmonic if and only if

$$\begin{aligned}\sum_{\alpha\in\Delta^+}(x_\alpha^{-1}/2)\{U'(\mathrm{Ad}\,(h)U_\alpha, \mathrm{Ad}\,(h)U_\alpha) - \mathrm{Ad}\,(h)U(U_\alpha, U_\alpha)\}\\ + \sum_{\alpha\in\Delta^+}(x_\alpha^{-1}/2)\{U'(\mathrm{Ad}\,(h)V_\alpha, \mathrm{Ad}\,(h)V_\alpha) - \mathrm{Ad}\,(h)U(V_\alpha, V_\alpha)\} = 0.\end{aligned} \tag{3.4}$$

In fact, we get (3.4) from Lemma 2.1. of Section 2, using $[\mathrm{Ad}\,(h)X]_{\mathfrak{t}'} = 0$ $(X \in \mathfrak{m})$.

Now, we obtain the following lemma for later use.

Lemma 3.1. *For $\alpha, \beta \in \Delta^+$ with $\alpha < \beta$,*

$$\begin{aligned}
&U(U_\alpha, U_\alpha) = U(V_\alpha, V_\alpha) = U(U_\alpha, V_\alpha) = 0,\\
&U(U_\alpha, U_\beta) = (x_\alpha - x_\beta)\{(2x_{\alpha+\beta})^{-1}N_{\alpha+\beta,-\beta}U_{\alpha+\beta}\\
&\quad +(2x_{\beta-\alpha})^{-1}N_{\beta-\alpha,-\beta}U_{\beta-\alpha}\},\\
&U(U_\alpha, V_\beta) = (x_\alpha - x_\beta)\{(2x_{\alpha+\beta})^{-1}N_{\alpha+\beta,-\beta}V_{\alpha+\beta}\\
&\quad +(2x_{\beta-\alpha})^{-1}N_{\beta-\alpha,-\beta}V_{\beta-\alpha}\},\\
&U(V_\alpha, U_\beta) = (x_\alpha - x_\beta)\{(2x_{\alpha+\beta})^{-1}N_{\alpha+\beta,-\beta}V_{\alpha+\beta}\\
&\quad -(2x_{\beta-\alpha})^{-1}N_{\beta-\alpha,-\beta}V_{\beta-\alpha}\},\\
&U(V_\alpha, V_\beta) = (x_\beta - x_\alpha)\{(2x_{\alpha+\beta})^{-1}N_{\alpha+\beta,-\beta}U_{\alpha+\beta}\\
&\quad +(2x_{\beta-\alpha})^{-1}N_{\beta-\alpha,\alpha}U_{\beta-\alpha}\}.
\end{aligned}$$

Proof. Using (3.1), we obtain the following equations :

$$\left\{\begin{array}{l} [U_\alpha, V_\alpha]_{\mathfrak{m}} = 0,\ U_\alpha = U_{-\alpha},\ V_\alpha = -V_{-\alpha}, \\ [U_\beta, U_\alpha]_{\mathfrak{m}} = N_{\beta,\alpha}U_{\beta+\alpha} + N_{\beta,-\alpha}U_{\beta-\alpha}, \\ [U_\beta, V_\alpha]_{\mathfrak{m}} = N_{\beta,\alpha}V_{\beta+\alpha} - N_{\beta,-\alpha}V_{\beta-\alpha}, \\ [V_\beta, V_\alpha]_{\mathfrak{m}} = N_{\beta,-\alpha}U_{\beta-\alpha} - N_{\beta,\alpha}U_{\beta+\alpha}, \\ < U_\alpha, U_\alpha >= 2x_\alpha, \quad < V_\alpha, V_\alpha >= 2x_\alpha, \\ < U_\alpha, U_\alpha >'= 2y_\alpha, \quad < V_\alpha, V_\alpha >'= 2y_\alpha, \end{array}\right. \tag{3.5}$$

where $\alpha, \beta \in \Delta^+$. From (3.5), we get

$$[Z, U_\alpha] = \left\{\begin{array}{l} (1/\sqrt{2x_\beta})(N_{\beta,\alpha}U_{\beta+\alpha} + N_{\beta,-\alpha}U_{\beta-\alpha}) \text{ if } Z = U_\beta/\sqrt{2x_\beta}, \\ (1/\sqrt{2x_\beta})(N_{\beta,\alpha}V_{\beta+\alpha} + N_{\beta,-\alpha}V_{\beta-\alpha}) \text{ if } Z = V_\beta/\sqrt{2x_\beta} \end{array}\right. \tag{3.6}$$

and

$$[Z, V_\alpha] = \left\{\begin{array}{l} (1/\sqrt{2x_\beta})(N_{\beta,\alpha}V_{\beta+\alpha} + N_{-\beta,\alpha}V_{-\beta+\alpha}) \text{ if } Z = U_\beta/\sqrt{2x_\beta}, \\ (1/\sqrt{2x_\beta})(N_{\beta,-\alpha}U_{\beta-\alpha} - N_{\beta,\alpha}U_{\beta+\alpha}) \text{if } Z = V_\beta/\sqrt{2x_\beta}. \end{array}\right. \tag{3.7}$$

Using the fact $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$ $(\alpha + \beta + \gamma = 0)$ (cf [3, p. 171]), $N_{\alpha,\beta} = -N_{\beta,\alpha}$, (2.3), and (3.5)–(3.7), we can obtain Lemma 3.1. □

Remark. In the cases of $\alpha > \beta$ in Lemma 3.1, α, β and $V_{\beta-\alpha}$ in each formular are changed into β, α and $-V_{\alpha-\beta}$. Interchanging x_α with y_α, we obtain $U'(U_\alpha, U_\beta)$, $U'(U_\alpha, V_\beta)$ and $U'(V_\alpha, V_\beta)$.

Moreover, we have

$$\left\{\begin{array}{l} \mathrm{Ad}\,(h)U_\alpha = \cos(\sqrt{-1}\alpha(H))U_\alpha - \sin(\sqrt{-1}\alpha(H))V_\alpha, \\ \mathrm{Ad}\,(h)V_\alpha = \sin(\sqrt{-1}\alpha(H))U_\alpha + \cos(\sqrt{-1}\alpha(H))V_\alpha, \end{array}\right. \tag{3.8}$$

where $\exp H = h$ $(H \in \mathfrak{t})$.

From (3.4), (3.8) and Lemma 3.1, we get the following theorem.

Theorem 3.2. *Let G be a compact connected semisimple Lie group, T a maximal torus. Let g and g' be arbitrary G-invariant metrics on G/T. Then, the map $\overline{A_h}$: $(G/T, g) \to (G/T, g'), (h \in T)$, which is induced from inner automorphism A_h of G is harmonic.*

4. Weakly Stable Harmonic Map $\overline{A_h}$

We define a field J of endomorphisms of $T(G/T)$ by $J(U_\alpha) = V_\alpha$ and $J(V_\alpha) = -U_\alpha$, $(\alpha \in \Delta^+)$. Then, $(G/T, g, J)$ and $(G/T, g', J)$ become almost Hermitian manifolds (cf. [1, p. 47]). From now on, we put $X_\alpha := (U_\alpha/\sqrt{2x_\alpha})$, $Y_\alpha := (V_\alpha/\sqrt{2x_\alpha})$, $X'_\alpha := (U_\alpha/\sqrt{2y_\alpha})$, and $Y'_\alpha := (V_\alpha/\sqrt{2y_\alpha})$ for each $\alpha \in \Delta^+$.

Lemma 4.1. *Almost complex structure J on G/T is integrable.*

Proof. For almost complex structure J to be integrable, it is necessary and sufficient that

$$S(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0 \tag{4.1}$$

holds for all X, Y $\in \mathfrak{X}(G/T)$. Here, S is a skew symmetric bilinear form from the product $\mathfrak{X}(G/T) \times \mathfrak{X}(G/T)$ of the $C^\infty(G/T)$-module $\mathfrak{X}(G/T)$ to $\mathfrak{X}(G/T)$.

Using $N_{\alpha,\beta} = -N_{\beta,\alpha}$, $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ and (3.5), we get

$$S(X_\beta, X_\alpha) = S(X_\beta, Y_\alpha) = S(Y_\beta, Y_\alpha) = 0. \tag{4.2}$$

Thus Lemma 4.1 is proved. □

The Kähler form ω (resp. ω') is defined on Hermitian manifold $(G/T, g, J)$ (resp. $(G/T, g', J)$) by $\omega(X,Y) = g(JX,Y)$ (resp. $\omega'(X,Y) = g'(JX,Y)$), X, $Y \in \mathfrak{X}(G/T)$.

Lemma 4.2. *The Kähler form ω (resp. ω') is closed if and only if*

$$x_\alpha + x_\beta = x_{\alpha+\beta} \text{ (resp. } y_\alpha + y_\beta = y_{\alpha+\beta}) \quad \text{holds for every } \alpha, \beta \in \Delta^+ \text{ with } \alpha+\beta \in \Delta^+. \tag{4.3}$$

Proof. From (3.5), we obtain for $\alpha < \beta < \gamma$ $(\alpha, \beta, \gamma \in \Delta^+)$

$$\begin{cases} dw(X_\gamma, X_\beta, X_\alpha) = 0, \\ dw(X_\gamma, X_\beta, Y_\alpha) = dw(X_\gamma, Y_\beta, X_\alpha) = -dw(Y_\gamma, X_\beta, X_\alpha) \\ = (-\sqrt{x_{\gamma-\beta}} N_{\gamma,-\beta}\delta_{\gamma-\beta,\alpha})/\sqrt{2x_\gamma x_\beta} \\ +(\sqrt{x_{\beta+\alpha}} N_{\beta,\alpha}\delta_{\beta+\alpha,\gamma})/\sqrt{2x_\beta x_\alpha} + (\sqrt{x_{\gamma-\alpha}} N_{\gamma,-\alpha}\delta_{\gamma-\alpha,\beta})/\sqrt{2x_\alpha x_\gamma}, \\ dw(X_\gamma, Y_\beta, Y_\alpha) = dw(Y_\gamma, X_\beta, Y_\alpha) = dw(Y_\gamma, Y_\beta, X_\alpha) = 0, \\ dw(Y_\gamma, Y_\beta, Y_\alpha) = dw(X_\gamma, X_\beta, Y_\alpha). \end{cases} \tag{4.4}$$

Here $\delta_{\beta,\alpha}(\beta, \alpha \in \Delta^+)$ is 1 if $\alpha = \beta$, and is 0 if $\alpha \neq \beta$. Moreover, using (3.5), we get for α, β $(\alpha, \beta \in \Delta^+)$

$$dw(X_\beta, X_\alpha, Y_\alpha) = dw(Y_\beta, X_\alpha, Y_\alpha) = 0. \tag{4.5}$$

Using $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$ $(\alpha+\beta+\gamma = 0)$, (4.4), and (4.5), we can get Lemma 4.2. □

Using (3.8), we get the following proposition.

$$\begin{cases} (J\circ d\overline{A_h})(X_\alpha) = (d\overline{A_h}\circ J)(X_\alpha) \\ = \sin(\sqrt{-1}\alpha(H))X_\alpha + \cos(\sqrt{-1}\alpha(H))Y_\alpha \quad (\alpha\in\Delta^+), \\ (J\circ d\overline{A_h})(Y_\alpha) = (d\overline{A_h}\circ J)(Y_\alpha) \\ = -\cos(\sqrt{-1}\alpha(H))X_\alpha + \sin(\sqrt{-1}\alpha(H))Y_\alpha \quad (\alpha\in\Delta^+)\,, \end{cases} \tag{4.6}$$

where $h = \exp H \quad (H \in \mathfrak{t})$. Hence, we have

$$J\circ d\overline{A_h} = d\overline{A_h}\circ J \qquad (h\in T). \tag{4.7}$$

Hence, we obtain the following result.

Lemma 4.3. $\overline{A_h} : (G/T, g, J) \to (G/T, g', J)$ *is holomorphic.*

Moreover, from Lemma 4.1 and Lemma 4.2, we get

Proposition 4.4. *If $x_\alpha + x_\beta = x_{\alpha+\beta}$ and $y_\alpha + y_\beta = y_{\alpha+\beta}$ hold for every $\alpha, \beta \in \Delta^+$ with $\alpha+\beta \in \Delta^+$, then $(G/T, g, J)$ and $(G/T, g', J)$ are Kähler manifolds.*

Now, in order to discuss stability of harmonic maps $\overline{A_h} \quad (h \in T)$ we introduce Lichnerowicz's Theorem.

Theorem 4.5. (cf. [1, 6]) *Let (M, g) and (N, h) be Kähler. Then:*

(i) *A holomorphic mapping $\phi : M \to N$ minimizes the energy in the homotopy class including ϕ,i.e., for a family $\phi_t \in C^\infty(M, N)$ with $\phi_0 = \phi$, depending smoothly on t, $E(\phi) \le E(\phi_t)$.*

(ii) *For holomorphic map $\phi : M \to N$,*

$$Ker(J_\phi) = \{S \in \Gamma(\phi^{-1}TN) \mid (DS)(X) := \widetilde{\nabla}_{JX}S - J\widetilde{\nabla}_X S = 0$$

for every $X \in \mathfrak{X}(M)\}$.

Here, $\widetilde{\nabla}$ is the connection on the induced bundle $\phi^{-1}TN$ given as $\widetilde{\nabla}_X S = {}^N\nabla_{\phi_\star X}S$, $(X \in T_xM,\ S \in \Gamma(\phi^{-1}TN))$. Let ∇ (resp. ∇') be the Riemannian connection corresponding to the metric g (resp. g') in G/T.

Using Lemma 3.1. and (2.2), we obtain the following lemma.

Lemma 4.6. *For every $\alpha, \beta \in \Delta^+$ with $\alpha < \beta$,*

(a) $\nabla'_{U^\star_\alpha} U^\star_\beta|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\beta} + y_{\alpha+\beta}N_{\beta,\alpha}\}U_{\alpha+\beta}/(2y_{\alpha+\beta})$
$+\{(y_\beta - y_\alpha)N_{\beta-\alpha,\alpha} + y_{\beta-\alpha}N_{\beta,-\alpha}\}U_{\beta-\alpha}/(2y_{\beta-\alpha})$,
$\nabla'_{U^\star_\beta} U^\star_\alpha|_0 = \{(y_\beta - y_\alpha)N_{\alpha+\beta,-\alpha} + y_{\alpha+\beta}N_{\alpha,\beta}\}U_{\alpha+\beta}/(2y_{\alpha+\beta})$
$+\{(y_\beta - y_\alpha)N_{\beta-\alpha,\alpha} + y_{\beta-\alpha}N_{\alpha,-\beta}\}U_{\beta-\alpha}/(2y_{\beta-\alpha})$,
(b) $\nabla'_{U^\star_\alpha} V^\star_\beta|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\beta} - y_{\alpha+\beta}N_{\alpha,\beta}\}V_{\alpha+\beta}/(2y_{\alpha+\beta})$
$+\{(y_\beta - y_\alpha)N_{\beta-\alpha,\alpha} - y_{\beta-\alpha}N_{\alpha,-\beta}\}V_{\beta-\alpha}/(2y_{\beta-\alpha})$,
$\nabla'_{U^\star_\beta} V^\star_\alpha|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\beta} - y_{\alpha+\beta}N_{\beta,\alpha}\}V_{\beta+\alpha}/(2y_{\alpha+\beta})$
$+\{(y_\alpha - y_\beta)N_{\beta-\alpha,\alpha} + y_{\beta-\alpha}N_{\beta,-\alpha}\}V_{\beta-\alpha}/(2y_{\beta-\alpha})$,
(c) $\nabla'_{V^\star_\alpha} U^\star_\beta|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\beta} + y_{\alpha+\beta}N_{\beta,\alpha}\}V_{\beta+\alpha}/(2y_{\alpha+\beta})$
$+\{(y_\alpha - y_\beta)N_{\beta-\alpha,\alpha} - y_{\beta-\alpha}N_{\beta,-\alpha}\}V_{\beta-\alpha}/(2y_{\beta-\alpha})$,
$\nabla'_{V^\star_\beta} U^\star_\alpha|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\beta} + y_{\alpha+\beta}N_{\alpha,\beta}\}V_{\beta+\alpha}/(2y_{\alpha+\beta})$
$+\{(y_\beta - y_\alpha)N_{\beta-\alpha,\alpha} + y_{\beta-\alpha}N_{\alpha,-\beta}\}V_{\beta-\alpha}/(2y_{\beta-\alpha})$,
(d) $\nabla'_{V^\star_\alpha} V^\star_\beta|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\alpha} + y_{\alpha+\beta}N_{\alpha,\beta}\}U_{\beta+\alpha}/(2y_{\alpha+\beta})$
$+\{(y_\beta - y_\alpha)N_{\beta-\alpha,\alpha} + y_{\beta-\alpha}N_{\beta,-\alpha}\}U_{\beta-\alpha}/(2y_{\beta-\alpha})$,
$\nabla'_{V^\star_\beta} V^\star_\alpha|_0 = \{(y_\alpha - y_\beta)N_{\alpha+\beta,-\alpha} - y_{\alpha+\beta}N_{\alpha,\beta}\}U_{\beta+\alpha}/(2y_{\alpha+\beta})$
$+\{(y_\beta - y_\alpha)N_{\beta-\alpha,\alpha} - y_{\beta-\alpha}N_{\beta,-\alpha}\}U_{\beta-\alpha}/(2y_{\beta-\alpha})$,
(e) $\nabla'_{U^\star_\alpha} U^\star_\alpha|_0 = \nabla'_{U^\star_\alpha} V^\star_\alpha|_0 = \nabla'_{V^\star_\alpha} U^\star_\alpha|_0 = \nabla'_{V^\star_\alpha} V^\star_\alpha|_0 = 0.$

Proposition 4.7. *Let g and g' be G-invariant Riemannian metrics on G/T which satisfy the following condition:*

$$x_\alpha + x_\beta = x_{\alpha+\beta} \text{ and } y_\alpha + y_\beta = y_{\alpha+\beta} \tag{4.8}$$

hold for every $\alpha, \beta \in \Delta^+$ with $\alpha + \beta \in \Delta^+$. The nullity of Jacobi operator $J_{\overline{A_h}}$ of each harmonic map $\overline{A_h} : (G/T, g) \to (G/T, g'), (h \in T)$, is $\dim G/T$.

Proof. We use the notation in (ii) of Theorem 4.5. We put

$$S = \mathrm{Ad}\,(h) \sum_{\alpha \in \Delta^+} (f^\beta U^\star_\beta + g^\beta V^\star_\beta),\ (f^\beta, g^\beta \in C^\infty(G/K)).$$

In this proof, we identify U_α, V_α with $U^\star_\alpha, V^\star_\alpha$ $(\alpha \in \Delta^+)$. Using Lemma 4.6, for $\alpha, \beta \in \Delta^+$ with $\alpha > \beta$

$$\begin{aligned}
&\{\mathrm{Ad}\,(h)U_\beta \text{ (resp. } \mathrm{Ad}\,(h)V_\beta) - \text{component of } (DS)(U_\alpha)\} \\
&\quad = \{\mathrm{Ad}\,(h)U_\beta \text{ (resp. } \mathrm{Ad}\,(h)V_\beta) - \text{component} \\
&\quad\quad \text{of } D(\mathrm{Ad}\,(h)(f^{\alpha-\beta}U_{\alpha-\beta} + g^{\alpha-\beta}V_{\alpha-\beta} + f^\beta U_\beta + g^\beta V_\beta))(U_\alpha)\},
\end{aligned}$$

and for $\alpha, \beta \in \Delta^+$ with $\alpha < \beta$

$$\begin{aligned}\{\mathrm{Ad}\,(h)U_\beta \text{ (resp. } \mathrm{Ad}\,(h)V_\beta) - \text{component of } (DS)(U_\alpha)\}\\ = \{\mathrm{Ad}\,(h)U_\beta \text{ (resp. } \mathrm{Ad}\,(h)V_\beta) - \text{component}\\ \text{of } D(\mathrm{Ad}\,(h)(f^\beta U_\beta + g^\beta V_\beta + f^{\beta+\alpha}U_{\beta+\alpha} + g^{\beta+\alpha}V_{\beta+\alpha}))(U_\alpha)\}.\end{aligned}$$

Using $N_{-\alpha,-\beta} = N_{\alpha,\beta} = -N_{\beta,\alpha}$, $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$ $(\alpha + \beta + \gamma = 0)$, $2\alpha \notin \Delta$ $(\alpha \in \Delta^+)$, (4.7), (4.8), and Lemma 4.6, we get

$$\begin{aligned}(DS)(U_\alpha) = \sum_{(\alpha>)\beta} \{(V_\alpha f^\beta + U_\alpha g^\beta - 2g^{\alpha-\beta}N_{\alpha-\beta,\beta})\mathrm{Ad}\,(h)U_\beta\\ + (V_\alpha g^\beta - U_\alpha f^\beta - 2f^{\alpha-\beta}N_{\alpha-\beta,\beta})\mathrm{Ad}\,(h)V_\beta\}\\ + (V_\alpha f^\alpha + U_\alpha g^\alpha)\mathrm{Ad}\,(h)U_\alpha + (V_\alpha g^\alpha - U_\alpha f^\alpha)\mathrm{Ad}\,(h)V_\alpha\\ + \sum_{(\alpha<)\beta} \{(V_\alpha f^\beta + U_\alpha g^\beta)\mathrm{Ad}\,(h)U_\beta + (V_\alpha g^\beta - U_\alpha f^\beta)\mathrm{Ad}\,(h)V_\beta\}. \quad (4.9)\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}(DS)(V_\alpha) = \sum_{(\alpha>)\beta} \{(-U_\alpha f^\beta + V_\alpha g^\beta + 2f^{\alpha-\beta}N_{\alpha-\beta,\beta})\mathrm{Ad}\,(h)U_\beta\\ - (U_\alpha g^\beta + V_\alpha f^\beta + 2g^{\alpha-\beta}N_{\alpha-\beta,\beta})\mathrm{Ad}\,(h)V_\beta\}\\ + (V_\alpha g^\alpha - U_\alpha f^\alpha)\mathrm{Ad}\,(h)U_\alpha - (V_\alpha f^\alpha + U_\alpha g^\alpha)\mathrm{Ad}\,(h)V_\alpha\\ + \sum_{(\alpha<)\beta} \{(-U_\alpha f^\beta + V_\alpha g^\beta)\mathrm{Ad}\,(h)U_\beta - (U_\alpha g^\beta + V_\alpha f^\beta)\mathrm{Ad}\,(h)V_\beta\}. \quad (4.10)\end{aligned}$$

Hence, $(DS)(U_\alpha) = (DS)(V_\alpha) = 0$ if and only if

$$U_\alpha g^\beta = -V_\alpha f^\beta, \quad U_\alpha f^\beta = V_\alpha g^\beta \quad (\alpha, \beta \in \Delta^+). \quad (4.11)$$

Now, if $DS = 0$, then f^β and g^β $(\beta \in \Delta^+)$ are harmonic functions. In fact, these follow from (e) of Lemma 4.6 and (4.11). Moreover, since G/T is compact, f^β and g^β $(\beta \in \Delta^+)$ are constant functions. Hence (the nullity of Jacobi operator $J_{\overline{A_h}}$ of harmonic map $\overline{A_h}$) = dim G/T. □

From Lemma 4.3, Proposition 4.4, Theorem 4.5 and Proposition 4.7, we have the following theorem.

Theorem 4.8. *Let g and g' G-invariant Riemannian metrics on G/T which satisfy the following : $x_\alpha + x_\beta = x_{\alpha+\beta}$ and $y_\alpha + y_\beta = y_{\alpha+\beta}$ hold for every $\alpha, \beta \in \Delta^+$ with $\alpha + \beta \in \Delta^+$. Then, every mapping $\overline{A_h} : (G/T, g) \to$*

$(G/T, g'), (h \in T)$, *is weakly stable harmonic map with nullity* $(J_{\overline{A_h}}) = \dim G/T$.

Example. The Lie algebra $\mathfrak{sl}_n(C)$ of $SL_n(C)$ is complexification of the Lie algebra $\mathfrak{su}(n)$ of $G := SU(n)$. Let E_{ij} denote a square matrix with the (i,j)-entry being 1, and all the other entries being 0. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_n(C)$ which consists of the diagonal matrices of trace 0. If $e_i : \mathfrak{h} \to C$ is a linear map defined by $e_i(\sum_{j=1}^n \alpha_j E_{jj}) = \alpha_i$, then

$$[H, E_{ij}] = (e_i - e_j)(H) \cdot E_{ij} \quad (H \in \mathfrak{h}). \tag{4.12}$$

Here, the non-zero roots of $\mathfrak{sl}_n(C)$ with respect to $\mathfrak{h}$ are

$$e_i - e_j \quad (1 \le i,\ j \le n,\ \ i \ne j). \tag{4.13}$$

Let B be the Killing form of $\mathfrak{sl}_n(C)$, i.e., $B(X,Y) = 2n\ \mathrm{Trace}(XY)\ \ (X, Y \in \mathfrak{sl}_n(C))$. We choose an orthonormal basis of $\mathfrak{su}(n)$ with respet to $Q := -B$ as follows : For i, j such that $1 \le i < j \le n$, let $E_{e_i - e_j}$ (resp. $E_{e_j - e_i}$) denote the root vectors with (i,j)-entry being $1/\sqrt{2n}$ (resp. the (j,i)-entry being $-1/\sqrt{2n}$) and all the other entries being 0. Then $B(E_{e_i-e_j}, E_{e_j-e_i}) = -1$. We put $U_{ij} := E_{e_i-e_j} + E_{e_j-e_i}$ and $V_{ij} := \sqrt{-1}(E_{e_i-e_j} - E_{e_j-e_i})$, where $1 \le i, j \le n$ and $i \ne j$. Then,

$$\{U_{ij}/\sqrt{2},\ V_{ij}/\sqrt{2} \mid 1 \le i \le n-1,\quad 1 \le i < j \le n\} \tag{4.14}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to Q, where $\mathfrak{su}(n) = \mathfrak{m} \oplus \mathfrak{t}$. Here, $\mathfrak{t}$ is the Lie alebra of the Lie group $T := \{A \in SU(n) \mid A \text{ is diagonal}\}$. Riemannian metric g (resp. g') on $SU(n)/T$ is determined by $<\ ,\ > = \underset{i<j}{\oplus} x_{ij} Q|_{\mathfrak{m}_{ij}}$ (resp. $<\ ,\ >' = \underset{i<j}{\oplus} y_{ij} Q|_{\mathfrak{m}_{ij}}$). Here $\mathfrak{m}_{ij}$ is the real vector space generated by $\{U_{ij}, V_{ij}\}$. Then, from Proposition 3.7 and Theorem 3.8 we obtain

Proposition 4.9. *Let* g, g' *be* $SU(n)$*-invariant Riemannian metrics on* $SU(n)/T$ *which satisfy the following condition*:

$$x_{ij} + x_{jk} = x_{ik}, \quad y_{ij} + y_{jk} = y_{ik} \tag{4.15}$$

hold for i, j, k *with* $1 \le i < j < k \le n$. *Then every map* $\overline{A}_h : (SU(n)/T, g) \to (SU(n)/T, g')$, $(h \in T)$ *is a weakly stable harmonic map, and the nullity of Jacobi operator* $J_{\overline{A}_h}$ *of* $\overline{A}_h$ $(h \in T)$ *is* $(n^2 - n)$.

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