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A COMMON FIXED POINT THEOREM FOR WEAK COMPATIBLE MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract: In this paper we prove common fixed point theorems for three mappings under the condition of weak compatible mappings, without taking any function continuous in intuitionistic fuzzy metric space. Our theorem is an extension of result of Fisher [8] to intuitionistic fuzzy metric spaces.

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1. Introduction

In 1965, the concept of fuzzy sets was introduced initially by Zadeh [23]. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. For example, Deng [4], Erceg [6], Fang [7], George and Veeramani [9], Kaleva and Seikkala [13], Kramosil and Michalek [14] have introduced the concept of fuzzy metric spaces in different ways. They also showed that every metric induces a fuzzy metric. Sharma and Tiwari [20] proved common fixed point theorems for three mappings under the condition of weak compatible mappings in fuzzy metric space and then extended this result to fuzzy 2-metric spaces and fuzzy 3-metric spaces.

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Alaca et al [1] using the idea of intuitionistic fuzzy sets [2], they defined the notion of intuitionistic fuzzy metric space as Park [16] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [14]. Further, they introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric spaces and proved the well-known fixed point theorems of Banach [3] and Edelstein [5] extended to intuitionistic fuzzy metric spaces with the help of Grabiec [10]. Many authors studied the concept of intuitionistic fuzzy metric space and its applications [11, 17, 18, 21, 22].

In this paper, we prove common fixed point theorems for three mappings under the condition of weak compatible mappings, without taking any function continuous in intuitionistic fuzzy metric space. Our theorem is an extension of result of Fisher [8], to intuitionistic fuzzy metric spaces.

2. Intuitionistic Fuzzy Metric Spaces

Definition 1. (see [19]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2. (see [19]) A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamondsuit$ is satisfies the following conditions:

(i) $\diamondsuit$ is commutative and associative;
(ii) $\diamondsuit$ is continuous;
(iii) $a \diamondsuit 0 = a$ for all $a \in [0,1]$;
(iv) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

The following definition of intuitionistic fuzzy metric space and the fundamental properties of intuitionistic fuzzy metric space due to Kramosil and Michalek [14] was given by Alaca et al [1].

Definition 3. (see [1]) A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(IFM-2) $M(x, y, 0) = 0$ for all $x, y \in X$;
(IFM-3) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(IFM-4) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
(IFM-6) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;
(IFM-7) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(IFM-8) $N(x, y, 0) = 1$ for all $x, y \in X$;
(IFM-9) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(IFM-10) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(IFM-11) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
(IFM-12) for all $x, y \in X$, $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;
(IFM-13) $\lim_{t\to\infty} N(x, y, t) = 0$ for all x, y in X.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 1. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \lozenge)$ such that t-norm $*$ and t-conorm $\lozenge$ are assosiated (see [15]), i.e., $x \lozenge y = 1 - ((1 - x) * (1 - y))$ for all $x, y \in X$.

Remark 2. In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Definition 4. (see [1]) Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. Then:

(i) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.$$

(ii) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \lim_{n\to\infty} N(x_n, x, t) = 0.$$

Remark 3. Since $*$ and $\lozenge$ are continuous, it follows from (IFM-5) and (IFM-11) that the limit of the sequence in intuitionistic fuzzy metric space is uniquely determined.

Definition 5. (see [1]) An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Lemma 1. *Let $\{y_n\}$ be a sequence in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. If there exists a number $k \in (0,1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)\,,$$
$$N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)\,, \tag{2.1}$$

for all $t > 0$ and $n =$ 1,2,... then $\{y_n\}$ is a Cauchy sequence in X.

Proof. For $t > 0$ and $k \in (0,1)$, we have $M(y_2, y_3, kt) \geq M(y_1, y_2, t) \geq M(y_0, y_1, \frac{t}{k})$ and $N(y_2, y_3, kt) \leq N(y_1, y_2, t) \leq N(y_0, y_1, \frac{t}{k})$ or $M(y_2, y_3, t) \geq M(y_0, y_1, \frac{t}{k^2})$ and $N(y_2, y_3, t) \leq N(y_0, y_1, \frac{t}{k^2})$. By simple induction with the condition (2.1), we have for all $t > 0$ and $n =$ 1, 2,...

$$M(y_{n+1}, y_{n+2}, t) \geq M(y_1, y_2, \frac{t}{k^n}),\; N(y_{n+1}, y_{n+2}, t) \leq N(y_1, y_2, \frac{t}{k^n}). \tag{2.2}$$

Thus by (2.2), (IFM-5), and (IFM-11), for any positive integer p and real number $t > 0$, we have

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, \frac{t}{p}) * \overset{p\text{-times}}{\cdots} * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\geq M(y_1, y_2, \frac{t}{pk^{n-1}}) * \overset{p\text{-times}}{\cdots} * M(y_1, y_2, \frac{t}{pk^{n+p-2}})\,, \end{aligned}$$

and

$$\begin{aligned} N(y_n, y_{n+p}, t) &\leq N(y_n, y_{n+1}, \frac{t}{p}) \Diamond \overset{p\text{-times}}{\cdots} \Diamond N(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\leq N(y_1, y_2, \frac{t}{pk^{n-1}}) \Diamond \overset{p\text{-times}}{\cdots} \Diamond N(y_1, y_2, \frac{t}{pk^{n+p-2}})\,. \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * \overset{p\text{-times}}{\cdots} * 1 \geq 1$$

and $\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 0 \Diamond \overset{p\text{-times}}{\cdots} \Diamond 0 \leq 0$ which implies that $\{y_n\}$ is a Cauchy sequence in X. □

Lemma 2. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$,*

$$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t)\,, \tag{2.3}$$

then $x = y$.

Proof. Since $t > 0$ and $k \in (0,1)$, we get $t > kt$. Using Remark 2 (in intuitionistic fuzzy metric space X, $M(x,y,.)$ is non-decreasing and $N(x,y,.)$ is non-increasing for all $x, y \in X$.), we have, $M(x,y,t) \geq M(x,y,kt)$ and $N(x,y,t) \leq N(x,y,kt)$. Using (2.3) and the definition of intuitionistic fuzzy metric, we have, $x = y$. □

Definition 6. (see [22]) Let A and B be maps from an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ into itself. The maps A and B are said to be compatible if, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 7. (see [12]) A pair of mappings A and S is said to be weakly compatible in fuzzy metric space if they commute at coincidence points.

Now, we give definition of weakly compatible in intuitionistic fuzzy metric spaces.

Definition 8. A pair of mappings A and S is said to be weakly compatible in intuitionistic fuzzy metric space if they commute at coincidence points, that is, the mappings A and S are weakly compatible if and only if $Ax = Sx$ implies $ASx = SAx$.

Example 1. Define A, $S : [0,4] \to [0,4]$ by

$$Ax = \begin{cases} x & \text{if} \quad x \in [0,1)\,, \\ 4 & \text{if} \quad x \in [1,4)\,, \end{cases} \quad \text{and } Sx = \begin{cases} 4-x & \text{if} \quad x \in [0,1)\,, \\ 4 & \text{if} \quad x \in [1,4)\,. \end{cases}$$

Then for any $x \in [1,4]$, $ASx = SAx$, showing that A, S are weakly compatible maps on $[0,4]$.

Example 2. Let $X = \mathbb{R}$ and define A, $S : \mathbb{R} \to \mathbb{R}$ by $Ax = \frac{x}{3}$, $x \in \mathbb{R}$ and $Sx = x^2$, $x \in \mathbb{R}$. Hence 0 and $\frac{1}{3}$ are two coincidence points for the maps A and S. Note that A and S commute at 0, i.e., $AS(0) = SA(0) = 0$, but $AS(\frac{1}{3}) = A(\frac{1}{9}) = \frac{1}{27}$ and $SA(\frac{1}{3}) = S(\frac{1}{9}) = \frac{1}{81}$ and so A and S are not weakly compatible maps on $\mathbb{R}$.

Example 3. Weakly compatible maps need not be compatible. Let $X = [2,20]$ and d be the usual metric on X. Define mappings A, $S : X \to X$ by $Ax = x$ if $x = 2$ or $x > 5$, $Ax = 6$ if $2 < x \leq 5$, $Sx = x$ if $x = 2$, $Sx = 12$ if

$2 < x \leq 5$, $Sx = x - 3$ if $x > 5$. The mappings A and S are non-compatible since sequence $\{x_n\}$ defined by $x_n = 5 + (\frac{1}{n})$, $n \geq 1$. Then

$$\lim_{n\to\infty} Sx_n = 2,\ \lim_{n\to\infty} SAx_n = 2, \text{ and } \lim_{n\to\infty} ASx_n = 6.$$

But they are weakly compatible since they commute at coincidence point at $x = 2$.

3. A Common Fixed Point Theorem for Weakly Compatible Mappings

Fisher [8] proved the following theorem for three mappings in complete metric space.

Theorem 1. (see [8]) *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and satisfy:*

$$d(Ax, Ay) \leq \alpha d(Sx, Ty)$$

for all $x, y \in X$ and $0 < \alpha < 1$. Indeed S, T and A have a unique common fixed point.

Now, we extend the above Fisher's Theorem to intuitionistic fuzzy metric space. We prove the following result.

Theorem 2. *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0, 1]$. Let A, B and S be mappings from X into itself such that:*

$$A(X) \cup B(X) \subset S(X), \tag{3.1}$$

the pairs $\{S, B\}$ and $\{S, A\}$ are weakly compatible, (3.2)

there exists a number $k \in (0, 1)$ such that: (3.3)

$$M(Ax, By, kt) \geq M(Sx, Sy, t) * M(Ax, Sx, t) * M(Sy, By, t) * M(Sy, Ax, t) * M(Sx, By, (2-\alpha)t)$$

and

$$N(Ax, By, kt) \leq N(Sx, Sy, t) \Diamond N(Ax, Sx, t) \Diamond N(Sy, By, t) \Diamond N(Sy, Ax, t) \Diamond N(Sx, By, (2-\alpha)t)$$

for all x, $y \in X$, $\alpha \in (0,2)$, $t > 0$.

Then A, B *and* S *have a unique common fixed point.*

Proof. We define a sequence $\{x_n\}$ and $\{y_n\}$ such that

$$y_{2n} = Sx_{2n} = Bx_{2n-1} \text{ and } y_{2n+1} = Sx_{2n+1} = Ax_{2n},\ n = 1,\ 2,...$$

We shall prove that $\{y_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.3), for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$, we write

$$\begin{aligned} M(Ax_{2n}, Bx_{2n+1}, kt) \geq\ & M(Sx_{2n}, Sx_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) \\ & * M(Sx_{2n+1}, Bx_{2n+1}, t) * M(Sx_{2n+1}, Ax_{2n}, t) \\ & * M(Sx_{2n}, Bx_{2n+1}, (2-\alpha)t) \end{aligned}$$

and

$$\begin{aligned} N(Ax_{2n}, Bx_{2n+1}, kt) \leq\ & N(Sx_{2n}, Sx_{2n+1}, t) \Diamond N(Ax_{2n}, Sx_{2n}, t) \\ & \Diamond N(Sx_{2n+1}, Bx_{2n+1}, t) \Diamond N(Sx_{2n+1}, Ax_{2n}, t) \\ & \Diamond N(Sx_{2n}, Bx_{2n+1}, (2-\alpha)t), \end{aligned}$$

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq\ & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) \\ & * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+1}, t) \\ & * M(y_{2n}, y_{2n+2}, (1+q)t) \end{aligned}$$

and

$$\begin{aligned} N(y_{2n+1}, y_{2n+2}, kt) \leq\ & N(y_{2n}, y_{2n+1}, t) \Diamond N(y_{2n+1}, y_{2n}, t) \\ & \Diamond N(y_{2n+1}, y_{2n+2}, t) \Diamond N(y_{2n+1}, y_{2n+1}, t) \\ & \Diamond N(y_{2n}, y_{2n+2}, (1+q)t)\,. \end{aligned}$$

Therefore, using (IFM-5) and (IFM-11), we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq\ & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, qt) \end{aligned}$$

and

$$N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \Diamond N(y_{2n+1}, y_{2n+2}, t)$$

$$\Diamond N(y_{2n}, y_{2n+1}, t) \Diamond N(y_{2n+1}, y_{2n+2}, qt) .$$

Then we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, qt)$$

and

$$N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \Diamond N(y_{2n+1}, y_{2n+2}, t) \Diamond N(y_{2n+1}, y_{2n+2}, qt).$$

Since the t-norm $*$ is continuous, the t-conorm $\Diamond$ is continuous, $M(x, y, .)$ and $N(x, y, .)$ are left continuous and right continuous, respectively, letting $q \to 1$, we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) &\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t), \\ N(y_{2n+1}, y_{2n+2}, kt) &\leq N(y_{2n}, y_{2n+1}, t) \Diamond N(y_{2n+1}, y_{2n+2}, t). \end{aligned} \tag{3.4}$$

Similarly, we have also

$$\begin{aligned} M(y_{2n+2}, y_{2n+3}, kt) \geq & \quad M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t), \\ N(y_{2n+2}, y_{2n+3}, kt) \leq & \quad N(y_{2n+1}, y_{2n+2}, t) \Diamond N(y_{2n+2}, y_{2n+3}, t). \end{aligned} \tag{3.5}$$

Thus by (3.4) and (3.5), it follows that

$$\begin{aligned} M(y_{n+1}, y_{n+2}, kt) &\geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t), \\ N(y_{n+1}, y_{n+2}, kt) &\leq N(y_n, y_{n+1}, t) \Diamond N(y_{n+1}, y_{n+2}, t) , \end{aligned}$$

for all $n = 1, 2, ...$ and so for positive integers n, p

$$\begin{aligned} M(y_{n+1}, y_{n+2}, kt) &\geq M(y_n, y_{n+1}, t) * M\left(y_{n+1}, y_{n+2}, \frac{t}{k^p}\right), \\ N(y_{n+1}, y_{n+2}, kt) &\leq N(y_n, y_{n+1}, t) \Diamond N\left(y_{n+1}, y_{n+2}, \frac{t}{k^p}\right). \end{aligned}$$

Thus since $M\left(y_{n+1}, y_{n+2}, \frac{t}{k^p}\right) \to 1$ and $N\left(y_{n+1}, y_{n+2}, \frac{t}{k^p}\right) \to 0$ as $p \to \infty$ we have

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) \text{ and } N(y_{n+1}, y_{n+2}, kt) \leq N(y_n, y_{n+1}, t).$$

By Lemma 1, $\{y_n\}$ is a Cauchy sequence in X. Since the space X is complete, the sequence converges to some point z in X. Also the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n-1}\}$ also converge to z.

Since $A(X) \subset S(X)$, there exists a point $u \in X$ such that $Au = z$. Then using (3.3), with $\alpha = 1$, we write

$$\begin{aligned} M(Au, Bx_{2n+1}, kt) &\geq M(Su, Sx_{2n+1}, t) * M(Au, Su, t) \\ &* M(Sx_{2n+1}, Bx_{2n+1}, t) * M(Sx_{2n+1}, Au, t) * M(Su, Bx_{2n+1}, t) \\ &\geq M(z, Sx_{2n+1}, t) * M(Au, z, t) * M(Sx_{2n+1}, Bx_{2n+1}, t) \\ &\qquad * M(Sx_{2n+1}, Au, t) * M(z, Bx_{2n+1}, t) \end{aligned}$$

and

$$\begin{aligned} N(Au, Bx_{2n+1}, kt) &\leq N(Su, Sx_{2n+1}, t) \Diamond N(Au, Su, t) \\ &\Diamond N(Sx_{2n+1}, Bx_{2n+1}, t) \Diamond N(Sx_{2n+1}, Au, t) \Diamond N(Su, Bx_{2n+1}, t) \\ &\leq N(z, Sx_{2n+1}, t) \Diamond N(Au, z, t) \Diamond N(Sx_{2n+1}, Bx_{2n+1}, t) \\ &\qquad \Diamond N(Sx_{2n+1}, Au, t) \Diamond N(z, Bx_{2n+1}, t). \end{aligned}$$

Taking the limit $n \to \infty$, we have

$$M(Au, z, kt) \geq M(Au, z, t) \text{ and } N(Au, z, kt) \leq N(Au, z, t).$$

Therefore by Lemma 2, we have $Au = z$. Since $Su = z$. Therefore $Au = z = Su$. Similarly since $B(X) \subset S(X)$, there exists a point $v \in X$ such that $Sv = z$. Then using (3.3), with $\alpha = 1$, we write

$$M(z, Bv, kt) \geq M(z, Sv, t) * 1 * M(Sv, Bv, t) * M(Sv, z, t) * M(z, Bv, t)$$

and

$$N(z, Bv, kt) \leq N(z, Sv, t) \Diamond 0 \Diamond N(Sv, Bv, t) \Diamond N(Sv, z, t) \Diamond N(z, Bv, t).$$

Then we have

$$M(z, Bv, kt) \geq M(z, Bv, t) \text{ and } N(z, Bv, kt) \leq N(z, Bv, t).$$

Therefore by Lemma 2, we have $Bv = z$. Thus $Bv = z = Sv$.

Since the pair $\{S,\ A\}$ is weakly compatible therefore S and A commute at their coincidence point i.e., $SAu = ASu$ i.e., $Sz = Az$. Now, we prove that $Sz = z$. By (3.3) with $\alpha = 1$, we have

$$\begin{aligned}M(Az, Bx_{2n+1}, kt) \geq\ & M(Sz, Sx_{2n+1}, t) * M(Az, Sz, t)\\ & * M(Sx_{2n+1}, Bx_{2n+1}, t) * M(Sx_{2n+1}, Az, t) * M(Sz, Bx_{2n+1}, t)\\ \geq\ & M(Az, Sx_{2n+1}, t) * M(Az, Az, t) * M(Sx_{2n+1}, Bx_{2n+1}, t)\\ & * M(Sx_{2n+1}, Az, t) * M(Az, Bx_{2n+1}, t)\end{aligned}$$

and

$$\begin{aligned}N(Az, Bx_{2n+1}, kt) \leq\ & N(Sz, Sx_{2n+1}, t)\Diamond N(Az, Sz, t)\\ & \Diamond N(Sx_{2n+1}, Bx_{2n+1}, t)\Diamond N(Sx_{2n+1}, Az, t)\Diamond N(Sz, Bx_{2n+1}, t)\\ \leq\ & N(Az, Sx_{2n+1}, t)\Diamond N(Az, Az, t)\Diamond N(Sx_{2n+1}, Bx_{2n+1}, t)\\ & \Diamond N(Sx_{2n+1}, Az, t)\Diamond N(Az, Bx_{2n+1}, t).\end{aligned}$$

Taking the limit $n \to \infty$, we have

$$M(Az, z, kt) \geq M(Az, z, t) \text{ and } N(Az, z, kt) \leq N(Az, z, t).$$

Therefore by Lemma 2, we have $Az = z$. Therefore $Az = z = Sz$.

Similarly by definition of weak compatibility it follows that $SBv = BSv$ and so $Sz = SBv = BSv = Bz$. We have already proved that $Sz = z$. Thus $Sz = Bz = z$. Hence $Sz = Az = Bz = z$. Thus z is a common fixed point of S, A and B.

For uniqueness of fixed point let w ($w \neq z$) be another common fixed point of A, B and S. Then by (3.3) with $\alpha = 1$, we write

$$\begin{aligned}M(Aw, Bz, kt) \geq\ & M(Sw, Sz, t) * M(Aw, Sw, t) * M(Sz, Bz, t)\\ & * M(Sz, Aw, t) * M(Sw, Bz, t)\end{aligned}$$

and

$$\begin{aligned}N(Aw, Bz, kt) \leq\ & N(Sw, Sz, t)\Diamond N(Aw, Sw, t)\Diamond N(Sz, Bz, t)\\ & \Diamond N(Sz, Aw, t)\Diamond N(Sw, Bz, t)\end{aligned}$$

which implies that

$$M(w, z, kt) \geq M(w, z, t) \text{ and } N(w, z, kt) \leq N(w, z, t).$$

Therefore by Lemma 2, we have $z = w$. This completes the proof. □

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