International Journal of Pure and Applied Mathematics
Volume 32 No. 4 2006, 53-57

THE MEASURE FOR A CLASS OF AFFINE HYPERSURFACES

Caristi Giuseppe[1 §], Giovanni Molica Bisci[2]

[1]Department of Economic and Business Branches of Knowledge
Faculty of Economics
University of Messina
98122 - (Me), ITALY
e-mail: g.caristi@unime.it

[2]Faculty of Engineering, DIMET
University of Reggio Calabria
Via Graziella (Feo di Vito), Reggio Calabria, I-89100, ITALY
e-mail: giovanni.molica@ing.unirc.it

Abstract: In the real affine 9-dimensional space $\mathbf{A}^9$, we show the measurability of the family of reducible hypersurfaces of type $\mathbf{S} = \mathbf{p}_1 \cdot ... \cdot \mathbf{p}_9$, where the components $\mathbf{p}_i$ are hyperplanes passing trough a fixed point.

AMS Subject Classification: 60D05, 52A22
Key Words: integral geometry, Lie groups, invariant varieties

1. Introduction

Problems of measurability in projective spaces or in affine spaces have been studied by many authors, for various families of varieties and different kinds of geometric configurations. The goal of this paper is to show the measurability of the family $\mathcal{F}$ of the reducible hypersurfaces in the real affine space $\mathbf{A}^9$ of the type $\mathbf{S} = \mathbf{p}_1 \cdot ... \cdot \mathbf{p}_9$, where the components $\mathbf{p}_i$ are hyperplanes passing trough a fixed point. Without loss of generality, we can suppose that the fixed point is the origin of coordinates. Hence,

Received: October 21, 2006
© 2006, Academic Publications Ltd.

§Correspondence author

$$\mathbf{S}: \quad \prod_{j=1}^{9}\left(x_1 + A_j^{(1)}x_2 + \dots + A_j^{(8)}x_9\right) = 0 \quad (A_j^{(i)} \in \mathbf{R}),$$

where $\det(A_j^{(\ell)}) \neq 0$ ($\ell = 0, \dots, 8$; $j = 1, \dots, 9$, $A_j^{(0)} := 1$, for $1 \leq j \leq 9$).

We show that the unique invariant integral function of the family $\mathcal{F}$ is given by

$$\phi = \frac{1}{\left(\det A_i^{(\ell)}\right)^9}.$$

For other results on the same subject see also the papers [1], [2] and [4].

2. Preliminaries

Let $\mathbf{A}_n$ be the affine space over the real field and

$$\mathbf{G}_r: \quad x_i' = f_i(x_1, \dots, x_n; a_1, \dots, a_r) \quad (i = 1, \dots, n), \tag{1}$$

a Lie group of transformations on $\mathbf{A}_n$ with $a_1, \dots, a_r$ independent set of essential parameters for $\mathbf{G}_r$.

We assume that the identity of the group $\mathbf{G}_r$ is determined by $a_1 = \dots = a_r = 0$. A function ϕ, solution of the system of partial differential equations

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\xi_j^i(x_1, \dots, x_n)\phi(x_1, \dots, x_n)) = 0, \tag{2}$$

where

$$\xi_j^i(x_1, \dots, x_n) = \left.\frac{\partial x_i'}{\partial a_j}\right|_{\mathbf{a}=\mathbf{0}} \quad (j = 1, \dots, r),$$

is called invariant integral function for the group $\mathbf{G}_r$. M. Stoka, [5] (see also the book [3]) defines measurable a group $\mathbf{G}_r$ with a unique invariant integral function ϕ, up to a constant factor. Let $\mathcal{V}_q$ be a family of p-dimensional varieties defined by $\mathcal{V}_q : F_1(x_1, \dots, x_n; \alpha_1, \dots, \alpha_q) = 0, \dots, F_{n-q}(x_1, \dots, x_n; \alpha_1, \dots, \alpha_q) = 0$ where $\alpha_s \in \mathbf{R}$, $s = 1, \dots, q$ are essential independent parameters and F_k, $k = 1, \dots, (n-p)$ are analytic functions on its domains. Following the definitions and the proof in [5], if $\mathbf{G}_r$ is the maximal invariant group of $\mathcal{V}_q$ one has an associated group of transformations

$$\mathbf{H}_r: \alpha_h' = \varphi_h(\alpha_1, \dots, \alpha_q; a_1, \dots, a_r) \quad (h = 1, \dots, q),$$

and $\mathbf{G}_r$ and $\mathbf{H}_r$ are isomorphic.

If we suppose that $\mathbf{H}_r$ is measurable of invariant integral function $\phi(\alpha_1, ..., \alpha_q)$, the expressions

$$\mu_{\mathbf{G}_r}(\mathcal{V}_q) = \int_{\mathcal{F}(\alpha)} \phi(\alpha_1, ..., \alpha_q) d\alpha_1 \wedge ... \wedge d\alpha_q,$$

where $\mathcal{F}(\alpha) = \{(\alpha_1, ..., \alpha_q) \in \mathbf{R}^q | \ F_k(\mathbf{x}; \alpha_1, ..., \alpha_q) = 0, \ k = 1, ..., n-p\}$ and

$$|\phi(\alpha_1, ..., \alpha_q)| d\alpha_1 \wedge ... \wedge d\alpha_q,$$

are called respectively the measure of the family $\mathcal{V}_q$ and the invariant density respect to the group $\mathbf{G}_r$ of the family $\mathcal{V}_q$. Hence, by definition, $\mathcal{V}_q$ is measurable if there exists a unique non-zero function ϕ.

3. Main Result

The parameters space of the family $\mathcal{F}$ is of dimension 72 and coordinates $A_j^{(i)} \in \mathbf{R}$ with $1 \leq i \leq 8$ and $j = 1, ..., 9$. The maximal group of invariance of the family is

$$\mathbf{G}_{9^2} : \ x_r = \sum_{s=1}^{9} \alpha_r^s x_s^{'}, \quad r = 1, 2, ..., 9\,, \tag{3}$$

with $\alpha_r^s \in \mathbf{R}$ and $\det(\alpha_r^s) \neq 0$.

Acting by $\mathbf{G}_{9^2}$ on $\mathbf{S}$ we obtain

$$\mathbf{S}' : \quad \prod_{j=1}^{9} \left(x_1 + [A_j^{(1)}]' x_2 + ... + [A_j^{(8)}]' x_9 \right) = 0,$$

where $[A_j^{(i)}]' \in \mathbf{R}$, with

$$\mathbf{H}_{9^2} : \qquad [A_j^{(i)}]' = \frac{\alpha_1^j + \sum_{l=1}^{8} A_j^{(l)} \alpha_l^j}{\alpha_1^1 + \sum_{l=1}^{8} A_j^{(l)} \alpha_l^1}, \tag{4}$$

where $1 \leq i \leq 8$; $j = 1, ..., 9$ and $\alpha_1^1 + \sum_{l=1}^{4} A_j^{(l)} \alpha_l^1 \neq 0$, $\forall j = 1, ..., 9$.

Theorem 1. *The family of reducible hypersurfaces in* $\mathbf{A}^9$ *of the type*

$$\mathbf{S} : \quad \prod_{j=1}^{9} \left(x_1 + A_j^{(1)} x_2 + ... + A_j^{(8)} x_9 \right) = 0,$$

where $\det(A_j^{(\ell)}) \neq 0$, $(\ell = 0, ..., 8;\ j = 1, ..., 9,\ A_j^{(0)} := 1,$ *for* $1 \leq j \leq 9)$ *is measurable and its unique invariant integral function is given by*

$$\phi = \frac{1}{\left(\det A_i^{(\ell)}\right)^9}.$$

Proof. By direct calculations we give the following Deltheil's system:

$$\sum_{j=1}^{9}\sum_{i=1}^{8} A_j^{(i)} \frac{\partial \phi}{\partial A_j^{(i)}} = -36\phi, \quad \sum_{j=1}^{9} \frac{\partial \phi}{\partial A_j^{(i)}} = 0, \quad i = 1, ..., 8\,,$$

$$\sum_{j=1}^{9} A_j^{(i)} \frac{\partial \phi}{\partial A_j^{(i)}} = -9\phi, \quad \sum_{j=1}^{9} A_j^{(i)} \frac{\partial \phi}{\partial A_j^{(k)}} = 0, \quad i, k = 1, ..., 8;\ k \neq i.$$

Finally

$$\sum_{j=1}^{9} \left(A_j^{(h)}\right)^2 \frac{\partial \phi}{\partial A_j^{(h)}} + \sum_{j=1}^{9} A_j^{(h)} A_j^{(k)} \frac{\partial \phi}{\partial A_j^{(k)}} = -9 \sum_{j=1}^{9} A_j^{(h)} \phi,$$

$$\sum_{j=1}^{9} \left(A_j^{(k)}\right)^2 \frac{\partial \phi}{\partial A_j^{(k)}} + \sum_{j=1}^{9} A_j^{(h)} A_j^{(k)} \frac{\partial \phi}{\partial A_j^{(h)}} = -9 \sum_{j=1}^{9} A_j^{(k)} \phi,$$

for every pair (h, k) with $h \neq k = 1, ..., 8$.

The unique non-zero solution (up to a constant factor) is

$$\phi = \frac{1}{\left(\det A_i^{(\ell)}\right)^9}. \qquad \square$$

References

[1] G. Caristi, G. Molica Bisci, The measure for a family of hypersurfaces with multiple components, *Int. J. Contemp. Math. Sci.*, **1**, No. 10 (2006), 463-467.

[2] G. Caristi, G. Molica Bisci, On the solution of the Deltheil's system associated to a family of hypersurfaces, *Far East J. Math. Sci.*, To Appear.

[3] A.M. Mathai, *An Introduction to Geometrical Probability: Distributional Aspects with Applications*, Gordon and Breach Science Publischers (1999).

[4] G. Santoro, Sulla misurabilità della famiglia di superfici cubiche spezzate in tre piani passanti per un punto fisso, *Atti Acc. Peloritana dei Pericolanti Classe S.M.F.N.*, **LXVIII** (1990).

[5] M. Stoka, *Géométrie Intégrale*, Mém. Sci. Math., **165**, Gauthier-Villars, Paris (1968).