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NEIGHBORHOOD CONDITIONS AND HAMILTONIAN PATHS IN GRAPHS

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Abstract: Let NC=min{ | N(u) $\cup$ N(v) |:u,v $\in$ V(G),uv $\notin$ E(G)}. In 1989 Faudree et al obtained that if G is a 3-connected graph of order n with NC $\geq$ (2n+1)/3, then G is Hamilton-connected graph. In this paper we prove the further result that if G is a 3-connected graph of order n with NC $\geq$ 2n/3, then G is Hamilton-connected graph.

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1. Introduction

In this paper, we consider only finite undirected graph G without loops or multiple edges. The vertex-set, edge-set, minimum degree and independence number of graph G are denoted by V(G), E(G), δ (G) and α (G), respectively. If H and C are subsets of V(G) or subgraphs of G, we denote by N_C (H) the set of vertices in C which are adjacent to some vertex of H. In particular, when C=G and H={u}, we set N_G ({u})=N(u) and d(u)= | N(u) |. We denote the distance between two vertices u and v in G by d(u,v). A path of order m denoted by P_m:x_1 x_2 $\ldots x_m$. Set N_{Pm} (u)={x_{i+1} $\in$ V(P_m): x_i $\in$ N_{Pm} (u)}, N^-_{Pm} (u)={x_{i-1} $\in$ V(P_m): x_i $\in$ N_{Pm} (u)}. $N^{\pm}_{Pm}$ (u)=N^+_{Pm} (u) $\cup$ N^-_{Pm} (u), where subscripts are taken modulo m.

In 1987 Faudree et al showed neighborhood union condition NC=min{ | N(u) $\cup$ N(v) |: u,v $\in$ V(G), uv $\notin$ E(G)}.

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In 1989 Faudree et al [2] obtained the Hamilton-connected graphs with neighborhood unions.

Theorem 1. (see Faudree et al [2]) *If G is a 3-connected graph of order n with NC $\geq$ (2n+1)/3, then G is Hamilton-connected graph.*

In [5] we considered Hamilton-connected graphs with neighborhood unions NC $\geq$ n- δ. In [4] Wei studied Hamilton-connected graphs with conditions $\{$d(u)+d(v)+d(w)- $|$ N(u) $\cap$ N(v) $\cap$ N(w) $|$: uv,vw,wu $\notin$ E(G)$\}$ $\geq$ n+1.

Now we prove the following results.

Theorem 2. *If G is a 3-connected graph of order n with NC $\geq$ 2n/3, then G is Hamilton-connected graph.*

2. The Proofs of Theorems

Proof of Theorem 2. Assume, to the contrary, that G is not Hamilton-connected graph. Then there exist two vertices x,y in G such that not Hamilton (x,y)-path of end-vertices in x and y. We denote a longest (x,y)-path by P_m $=x_1 x_2 \ldots x_m$ (where $x=x_1$, $y=x_m$).

Then we claim:

(1) If there exists some vertex u of G-P_m is adjacent to x_1 and x_m, then d(u) $\leq$ (n+1)/3.

(2) If there exists some vertex u of G-P_m is not adjacent to x_1 and x_m, then d(u) $\leq$ (n-2)/3.

First, we prove the claim (1). Assume, to the contrary, that claim (1) is not hold, namely, d(u) $\geq$ (n+2)/3. Then let H be a component of G-P_m which u $\in$ V(H). Since G is 3-connected, so let x_{i+1}, x_{j+1} $\in$ N^+_{Pm} (H). By P_m is a longest (x_1, x_m) -path, then there do not exist two adjacent vertices in N^+_{Pm} (u). Otherwise, if x_{h+1}, x_{k+1} are two adjacent vertices of N^+_{Pm} (H). Without loss of generality, say hk, let P_H be a path of H which two end-vertices adjacent to x_h, x_k, respectively, then we obtain (x_1, x_m) -path: x_1 x_2 $\ldots x_h$ Px_k x_{k-1} $\ldots x_{h+1}$ x_{k+1} x_{k+2} $\ldots x_m$ is longer path than P_m, a contradiction. We also have that every vertex of N_H (u) $\cup$ $\{$u$\}$ all is not adjacent to x_{i+1} and x_{j+1}. Otherwise, if v in N_H (u) is adjacent x_{i+1} or x_{j+1}, then we get (x_1, x_m) -path: x_1 x_2 $\ldots x_i$ uvx_{i+1} x_{i+2} $\ldots x_m$ or x_1 x_2 $\ldots x_j$ uvx_{j+1} x_{j+2} $\ldots x_m$ is longer than P_m, a contradiction.

By (1) that vertex x_m of P_m is adjacent to u, thus

$$|N^+_{Pm}(u)| = |N_{Pm}(u)| - |\{x_m\}|.$$

Easily we have

$$|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N^+_{Pm}(u)| - |N_H(u) \cup \{u\}|$$
$$\leq |V(G)| - (|N_{Pm}(u)| - |\{x_m\}|) - |N_H(u) \cup \{u\}| \leq n - |N(u)| + |\{x_m\}| - |\{u\}|$$
$$\leq (2n-1)/3,$$

a contradiction.

Second, we shall prove claim (2). Assume, to the contrary, that claim (2) does not hold, namely, $d(u) \geq (n-1)/3$. Since u does not adjacent to x_1 or x_m. Without loss of generality, say u is not adjacent to x_m. Then we let H be a component of $G\text{-}P_m$ which $u \in H$. Since u is not adjacent to x_m, so $|N^+_{Pm}(u)| = |N_{Pm}(u)|$. By a similar discussion of claim (1), we have $|N(x_{i+1}) \cup N(x_{j+1})| \leq n - |N^+_{Pm}(u)| - |N_H(u) \cup \{u\}| \leq |V(G)| - |N_{Pm}(u)| - |N_H(u) \cup \{u\}| \leq n - |N(H)| - |\{u\}| \leq (2n-1)/3$, a contradiction.

Therefore, claim (1) and (2) are true. Then we consider the following cases:

Case 1. There exists some vertex u in $G\text{-}P_m$ that is adjacent to x_1 and x_m.

Subcase 1.1. x_2 is adjacent to x_1 and x_m.

In this case, by $|N(x_2) \cup N(u)| \geq 2n/3$ with $d(u) \leq (n+1)/3$, we get $|N(x_2)| \geq 2n/3 - d(u) + |\{x_1, x_m\}| \geq (n+4)/3$. Let $x_{i+1} \in N^+_{Pm}(H) \setminus \{x_2\}$ satisfying i+1 is as large as possible. Then for any $x_h \in N_{Pm}(x_2) \setminus \{x_1, x_m, x_{m-1}\}$, since P_m: $x_1 x_2 \ldots x_m$ is a longest (x_1, x_m) -path. (1) When $2 \leq h \leq i$, we have x_{h-1} is not adjacent to u and x_{i+1}. Otherwise, if x_{h-1} is adjacent to u, let P^* be a path of H which two end-vertices are adjacent to x_1 and x_{h-1}, respectively. Then a (x_1, x_m) -path: $x_1 P^* x_{h-1} x_{h-2} \ldots x_2 x_h x_{h+1} \ldots x_m$ is longer than P_m, a contradiction. If x_{h-1} is adjacent to x_{i+1}, then let P be a path of H which two end-vertices are adjacent to x_1 and x_i, respectively. Then (x_1, x_m) -path: $x_1 P x_i x_{i-1} \ldots x_h x_2 x_3 \ldots x_{h-1} x_{i+1} x_{i+2} \ldots x_m$ is longer than P_m, a contradiction. (2) When $h \geq i+1$ with $h \neq m, m-1$, we have that x_{h+1} is not adjacent to u and x_{i+1}. By the choose of x_{i+1}, easily we can see that x_{h+1} is not adjacent to u. If x_{h+1} is adjacent to x_{i+1}. Let P be a path of H which two end-vertices are adjacent to x_1 and x_i, respectively. Then (x_1, x_m) -path: $x_1 P x_i x_{i-1} \ldots x_2 x_h x_{h-1} \ldots x_{i+1} x_{h+1} \ldots x_m$ is longer than P_m, a contradiction. (3) By the similar argument as above , we have that every vertex of $N_{G-Pm}(x_2)$ is not adjacent to u and x_{i+1}. Otherwise, if $v \in N_{G-Pm}(x_2)$ is adjacent to u, then (x_1, x_m) -path $x_1 u v x_2 x_3 \ldots x_m$ is longer than P_m, a contradiction. If $v \in N_{G-Pm}(x_2)$ is adjacent to x_{i+1}. Let P be a path of H which two end-vertices are adjacent to x_1 and x_i, respectively. Then (x_1, x_m) -path: $x_1 P x_i x_{i-1} \ldots x_2 v x_{i+1} x_{i+2} \ldots x_m$ is longer than P_m, a contradiction.

We define a bijection f on N(x_{j+1}) as follows: Let v $\in$ N(x_{j+1}),

$$f(u) = \begin{cases} v^+ = x_{k+1}\,, & \text{when } v = x_k \in \{x_1, x_2, \dots x_i\}\backslash\{x_1\}\,, \\ v^- = x_{k-1}\,, & \text{when } v = x_k \in \{x_{i+1}, x_{i+2}, \dots x_m\}\backslash\{x_{m-1}, x_m\}\,, \\ v\,, & \text{when } v \notin V(P_m). \end{cases}$$

From the previous arguments, for any y $\in$ f (N(x_{j+1})), we have yx_{i+1}, yu $\notin$ E(G). Hence we obtain

$$|N(x_{i+1}) \cup N(u)| \le n \\ - (|N_{Pm}(x_2)| - |\{x_1, x_{m-1}, x_m\}|) - |N_{G-Pm}(x_2)| - |\{u, x_{i+1}\}|\,. \quad (*)$$

Then we consider the following:

Since NC $\geq$ 2n/3, so | N(x_2)$\cup$ N(u) | $\geq$ 2n/3, together with claim (1) that d(u) $\leq$ (n+1)/3, we have | N(x_2)| $\geq$ 2n/3-d(u)+ | {x_1, x_m } | $\geq$ (n+5)/3, together with (*). Hence

$$|N(x_{i+1}) \cup N(u)| \le n \\ - (|N_{Pm}(x_2)| - |\{x_1, x_{m-1}, x_m\}|) - |N_{G-Pm}(x_2)| - |\{u, x_{i+1}\}| \le (2n-1)/3,$$

a contradiction.

Subcase 1.2. When x_2 is not adjacent to x_m.

In this case, by | N(x_2)$\cup$ N(u) | $\geq$ 2n/3, clearly we have

$$|N(x_2)| \ge 2n/3 - d(u) + |\{x_1\}| \ge (n+2)/3\,,$$

then by the similar discussion as above we have

$$|N(x_{i+1}) \cup N(u)| \le n - (|N_{Pm}(x_2)| - |\{x_1, x_{m-1}\}|) - |N_{G-Pm}(x_2) \cup \{u, x_{i+1}\}| \\ \le n - |N(x_2)| \le (2n-1)/3,$$

a contradiction.

Case 2. There exists vertex u of G-P_m that is not adjacent to x_1 or x_m.

Without loss of generality, say u is not adjacent to x_m. By the claim (2) we have d(u) $\leq$ (n-2)/3, and by | N(x_2)$\cup$ N(u) | $\geq$ 2n/3, thus | N(x_2)| $\geq$ 2n/3-d(u)+ | {x_1 } | $\geq$ (n+4)/3. Then by a similar argument as in Case (1):

(I) When u is adjacent to x_1, for any x_h $\in$ N_{Pm} (x_2)$\backslash$ {x_1, x_m }, we have | N(x_{i+1})$\cup$ N(u) | $\leq$ n-(| N_{Pm} (x_2)| $-$ | {x_1, x_m } |) $-$ | N_{G-Pm} (x_2)$\cup$ {u,x_{i+1}}| $\leq$ n- | N(x_2)| $\leq$ (2n-1)/3, a contradiction.

(II) When u is not adjacent to x_1. Without loss of generality, say $x_k \in N_{Pm}$ (u) satisfying k is as small as possible. Similarly, we can have

$$|N(x_{i+1}) \cup N(u)| \leq n - (|N_{Pm}(x_{k+1})| - |\{x_1, x_m\}|) - |N_{G-Pm}(x_{k+1}) \cup \{u, x_{i+1}\}| \leq n - |N(x_{k+1})| \leq (2n-1)/3,$$

a contradiction. □

Theorem 3. *If G is a 2-connected graph of order n with NC ≥ 2n/3, then G is Hamilton-connected graph or $G \in \mathfrak{X}$ (where $\mathfrak{X}$ is the graphs showed in the end of the proof of Theorem 3).*

Proof. Assume, to the contrary, that G is neither Hamilton-connected graph nor $G \notin \mathfrak{X}$. Then there exist two vertices x, y in G without Hamilton (x,y)-path which end-vertices in x and y. We denote a longest (x,y)-path of end-vertices in x and y of G by $P_m = x_1 x_2 \ldots x_m$ (which $x = x_1$, $y = x_m$).

Case 1. There exists a component H of G-P_m satisfying $|N_{Pm}(H)| \geq 2$ with vertex $x_i \in N_{Pm}(H)$ satisfying $x_i \notin \{x_1, x_m\}$.

In this case we can apply the similar discussion as in Theorem 2, and obtain a contradiction.

Case 2. For any component H of G-P_m, $\{x_1, x_m\} = N_{Pm}(H)$.

In this case we claim that G-P_m is connected.

Otherwise, if there exist two components G_1, G_2 in G-P_m. Let $G_3 = \{x_2, x_3, \ldots x_{m-1}\}$,

1) When $|V(G_i)| \geq (n-4)/3$ (where i=1 or 2 or 3).

Without loss of generality, say $|V(G_3)| \geq (n-4)/3$, then let u, v be two vertices in G_1, G_2, respectively. Clearly, we have

$$|N(u) \cup N(v)| \leq n - |V(G_3)| - |\{u, v\}| \leq (2n-1)/3,$$

a contradiction.

2) When $|V(G_i)| \leq (n-5)/3$ (where i=1, 2, 3).

In this case, let u, v be two vertices in G_1, G_2, respectively. Clearly we have

$$|N(u) \cup N(v)| \leq |V(G_1)| + |V(G_2)| - |\{u, v\}| + 2|\{x_1, x_m\}| \leq (2n-1)/3,$$

a contradiction.

Therefore, the claim is true.

Then let G_1 =G-P_m. We claim that the subgraph of vertices number minimum of graphs G[$\{x_2, x_3, \ldots x_{m-1}\}$] and G_1 are complete subgraph. Otherwise, if G_1 is the graph of vertices number minimum and it is not complete subgraph, then let u,v be two nonadjacent vertices of G_1. Hence we have $|N(u) \cup N(v)| \leq$

n- | $\{x_2, x_3, \ldots x_{m-1}\}$ | $-$ | {u,v} | $\leq$ n-(n-2)/2-2 $\leq$ (2n-1)/3, a contradiction. Then let $\mathfrak{X}$ is the graph which every two nonadjacent vertices satisfying NC $\geq$ 2n/3. □

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