

REDUCED PROJECTIVE CURVES  
WITH NEGATIVE COTANGENT SHEAF

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**Abstract:** Here we study reduced projective curves (mainly nodal ones) whose cotangent sheaf (modulo its torsion) is negative.

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1. Negative Cotangent Sheaf

For any algebraic scheme  $X$  defined over an algebraically closed field  $\mathbb{K}$  (resp. reduced complex analytic space  $X$ ) let  $TX$  denote its tangent sheaf, i.e. the dual of the sheaf  $\Omega_X^1$  of germs of regular (resp. holomorphic) 1-forms on  $X$ . It is well-known that  $\Omega_X^1$  is locally free if and only if  $X$  is smooth. We are interested in the case in which  $TX$  is locally free, but  $X$  is singular. Here, following [1], we study the case  $\dim(X) = 2$ ,  $X$  projective or compact and  $TX$  is rather positive. We consider only the case  $\dim(X) = 1$ , but we show that taking a dual approach (i.e. looking at the negativity of the cotangent sheaf) may give interesting results. Now assume that  $X$  is an integral projective curve. We always assume  $\text{Sing}(X) \neq \emptyset$ . Let  $\pi : Y \rightarrow X$  denote its normalization. If  $A$  is a rank one torsion free sheaf on  $X$ , then  $\pi^*(A)$  has torsion if and only if  $A$  is not locally free, while  $\pi^{[*]} := \pi^*(A)/\text{Tors}(\pi^*(A))$  is locally free. The sheaf  $\Omega_X^1/\text{Tors}(\Omega_X^1)$  is a rank one torsion free sheaf on  $X$  and it would be natural to say that it is negative if  $\deg(\Omega_X^1/\text{Tors}(\Omega_X^1)) < 0$ . We recall that the degree of a rank one torsion-free

sheaf  $F$  on  $X$  may be defined by the Riemann-Roch type formula:  $\deg(F) = \chi(F) - \chi(\mathcal{O}_X)$ . When  $X$  is Gorenstein we have  $\deg(F^*) = -\deg(F)$  (and hence when  $X$  is Gorenstein  $\deg(TX) > 0$  if and only if  $\deg(\Omega_X^1/\text{Tors}(\Omega_X^1)) < 0$ ). Notice that if  $X$  is Gorenstein, then  $\Omega_X^1/\text{Tors}(\Omega_X^1) = TX^*$ . When  $F$  is locally free, then of course  $\deg(F^*) = -\deg(F)$ , even if  $X$  is not Gorenstein. When  $F$  is locally free, then  $\pi^*(F)$  is locally free and  $\deg(\pi^*(F)) = \deg(F)$  (and in particular  $\deg(F) > 0$  if and only if  $F$  is ample), but this is not true if  $F$  is not locally free (see Example 1). In this paper we will heavily use this well-known fact. If  $\text{char}(\mathbb{K}) = 0$ , then  $TX$  is not locally free, because  $X$  is not normal (see [4]). If  $\text{char}(\mathbb{K}) > 0$  we gave examples of plane curves (and hence of Gorenstein curves) with  $TX$  locally free and very ample (see [1], Section 1).

**Example 1.** Assume  $\text{char}(\mathbb{K}) \neq 2$ . Fix an integer  $g > 0$ . Let  $X$  be an integral projective curve such that  $p_a(X) = g$  and  $X$  has exactly  $g$  singular points, say  $P_1, \dots, P_g$ , such that each of them is either an ordinary node or an ordinary cusp. Hence  $X$  is Gorenstein. If  $X$  has at least one ordinary cusp of  $X$ , then assume  $\text{char}(\mathbb{K}) \neq 3$ . Since  $X$  is Gorenstein, the torsion-free sheaf  $\Omega_X^1/\text{Tors}(\Omega_X^1)$  is reflexive. The natural map  $\Omega_X^1 \rightarrow \omega_X$  shows that  $\Omega_X^1/\text{Tors}(\Omega_X^1)$  is the torsion-free sheaf  $\mathcal{I}_{\{P_1, \dots, P_g\}} \otimes \omega_X$ . Thus  $\Omega_X^1/\text{Tors}(\Omega_X^1)$  is not locally free and it has degree  $g - 2$ . This degree is negative if and only if  $g = 1$ . Furthermore, its dual  $TX$  is not locally free, it has degree  $2 - g$  and, if  $g = 1$ , it contains  $\mathcal{O}_X$  (use [2], Lemma 3.1.7 (a), and that  $X$  is Gorenstein). Hence if  $g = 1$ , then  $h^0(X, TX) = 1$ ,  $\deg(TX) = 1$  and  $TX$  is not spanned. Let  $\pi : Y \rightarrow X$  be its normalization. Hence  $Y \cong \mathbf{P}^1$ . The line bundle  $\pi^{[*]}(\Omega_X^1/\text{Tors}(\Omega_X^1))$  has degree  $-2$ , while the line bundle  $\pi^{[*]}(TX)$  has degree  $2 - 2g$  (use [2], part 2 of Proposition 3.2.4). Notice that if  $g = 1$ , then  $\pi^{[*]}(TX)$  is trivial, while the dual of  $\pi^{[*]}(\Omega_X^1/\text{Tors}(\Omega_X^1))$  is very ample. If  $g = 1$  the torsion-free sheaf  $TX$  is an ample  $\mathbb{Q}$ , i.e. it is associated to a Weil divisor  $P_1$  such that  $\mathcal{O}_X(tP_1)$  (i.e. the sheaf  $TX^{\otimes t}/\text{Tors}(TX^{\otimes t})$ ) is an ample line bundle for all even positive integers. Notice that (under our assumption  $g > 0$ ) there is no line bundle on  $X$  with negative degree containing  $\Omega_X^1/\text{Tors}(\Omega_X^1)$ .

**Example 2.** Assume  $\text{char}(\mathbb{K}) \neq 2$ . Fix integers  $g > 0$  and  $q > 0$ . Let  $X$  be an integral projective curve such that  $p_a(X) = g + q$  and  $X$  has exactly  $g$  singular points such that each of them is either an ordinary node or an ordinary cusp. If  $X$  has at least one ordinary cusp of  $X$ , then assume  $\text{char}(\mathbb{K}) \neq 3$ . As in Example 1 we get that  $\Omega_X^1/\text{Tors}(\Omega_X^1)$  is not locally free,  $\deg(\Omega_X^1/\text{Tors}(\Omega_X^1)) = 2q + g - 2$ ,  $TX$  is not locally free,  $\deg(TX) = -2q - g + 2$ , that the line bundle  $\pi^{[*]}(\Omega_X^1/\text{Tors}(\Omega_X^1))$  has degree  $2q - 2$ , while the line bundle  $\pi^{[*]}(TX)$  has degree  $-2q + 2 - 2g$ .

What happens when the projective curve  $X$  is reduced and connected, but not irreducible? The following result is very easy and well-known (see [5] for the two-dimensional case).

**Proposition 1.** *Let  $X$  be a reduced and connected Gorenstein projective curve. Assume  $\omega_X^*$  ample and that  $X$  is not irreducible. Then  $X$  is isomorphic to a reducible plane conic.*

The negativity of the quotient of the cotangent sheaf by its torsion gives more interesting results and examples.

**Proposition 2.** *Let  $X$  be a reduced projective curve such that all its irreducible components are rational and  $\pi : Y \rightarrow X$  its normalization. Then the dual of  $L := \pi^*(\Omega_X^1)/\text{Tors}(\pi^*(\Omega_X^1))$  is an ample line bundle, i.e.  $\deg(L|C) < 0$  for all connected components  $C$  of  $X$ .*

**Theorem 1.** *Let  $X$  be a reduced and connected nodal projective curve.*

- (a) *If there are  $L \in \text{Pic}(X)$  such that  $\deg(L|T) < 0$  for all irreducible components  $T$  of  $X$  and an injective map  $j : \Omega_X^1/\text{Tors}(\Omega_X^1) \rightarrow L$ , then all irreducible components of  $X$  are smooth and rational.*
- (b) *Assume  $p_a(X) = 0$ . Then there are  $L \in \text{Pic}(X)$  such that  $\deg(L|T) < 0$  for all irreducible components  $T$  of  $X$  and an injective map  $j : \Omega_X^1/\text{Tors}(\Omega_X^1) \rightarrow L$ .*

*Proof of Proposition 1.* We outline two different proofs. Let  $T$  be any of the irreducible component of  $X$ . Let  $Z$  be the closure of  $X \setminus T$  in  $X$ . Since  $X$  is reducible and connected,  $T \cap Z \neq \emptyset$ . Since  $\omega_X$  is locally free and ample.  $\omega_X|T$  is a line bundle and  $\deg(\omega_X|T) < 0$ . Since  $\omega_X$  is locally free, it is easy to check that  $\omega_T$  is a subsheaf of  $\omega_X|T$  (this is a particular case of the subadjunction formula given in [3], Ch. III, Ex. 7.2, or [5], Proposition 2.3 and Proposition 2.11). Hence the torsion-free sheaf  $\omega_T$  has negative degree. Thus  $T \cong \mathbf{P}^1$ . Since  $T$  is smooth, it is easy to check that the support of  $\omega_X|T/\omega_T$  is equal to the set  $T \cap Z$ . Since  $\deg(\omega_T) = -2$  and  $\deg(\omega_X|T) < 0$ , we get  $\sharp(T \cap Z) = 1$ . Since this must be true for all irreducible components of  $X$ , we get that  $X$  has a unique singular point,  $P$ , and  $m \geq 2$  irreducible components  $X_1, \dots, X_m$ , all of them smooth, rational and containing  $P$ . We also see that  $X_i$  is quasi-transversal to  $X_j$  for all  $i \neq j$ , i.e.  $X_i \cup X_j$  is a plane conic for all  $i \neq j$ . Let  $F$  be a torsion-free sheaf on  $X$  such that its restriction to every irreducible component of  $X$ . We may define the degree  $\deg(F)$  by the formula  $\deg(F) = \chi(F) - \chi(\mathcal{O}_X)$ . With this definition we may apply duality to get  $\deg(\omega_X) = 2p_a(X) - 2$  as in the case of irreducible curves done in [2]. Since  $\omega_X^*$  is ample, we have  $\deg(\omega_X) \leq -m$ .

Hence we get  $p_a(X) = 0$  (and in particular every irreducible component of  $X$  is smooth and rational) and  $m = 2$ .  $\square$

**Remark 1.** Let  $X$  be a reduced connected curve,  $T$  an irreducible component of  $X$ ,  $\pi : Y \rightarrow X$  its normalization and  $C$  the connected component of  $Y$  such that  $\pi(C) = T$ . The functoriality properties of the sheaf of Kähler differentials gives maps  $j_T : \Omega_X^1|_T \rightarrow \Omega_T^1$  and  $j_C : f^*(\Omega_X^1)|_C \rightarrow \Omega_C^1$  which are an isomorphism over each point of  $X_{reg} \cap T$ . Thus  $j_T$  and  $j_C$  are injective mod torsion. Thus  $\deg((\Omega_X^1|_T)/\text{Tors}(\Omega_X^1|_T)) \leq \deg(\Omega_T^1/\text{Tors}(\Omega_T^1))$ . Furthermore,  $\deg((f^*(\Omega_X^1)|_C)/\text{Tors}(f^*(\Omega_X^1)|_C)) < 0$  if  $C \cong \mathbf{P}^1$ .

*Proof of Proposition 2.* Apply Remark 1.  $\square$

*Proof of Theorem 1.* Part (a) follows from Example 1 and the last sentence of Example 2. Now we will check part (b). Since  $X$  is nodal and connected and  $p_a(X) = 0$ , every irreducible component of  $X$  is smooth and rational and a line bundle on  $X$  is uniquely determined by the degrees of its restriction to the irreducible components of  $X$ . Fix an irreducible component  $T$  of  $X$ . Set  $S := \{P \in \text{Sing}(X) : P \in T_{reg}\}$ . Thus  $S = T \cap E$ , where  $E$  is the closure of  $X \setminus T$  in  $X$ . Notice that  $S$  is a Cartier divisor of  $T$ . We have  $\omega_X|_T = \omega_T(S)$ . Since  $T$  is smooth at each point of  $S$ , a local calculation as in Example 1 shows that  $(\Omega_X^1|_T)/\text{Tors}(\Omega_X^1|_T) = \Omega_T^1/\text{Tors}(\Omega_T^1)$ . Let  $L$  be any line bundle on  $X$  such that  $\deg(L|_T) = -1$  for all irreducible components of  $X$ . Applying several times a Mayer-Vietoris exact sequence we get  $h^0(X, (\Omega_X^1/\text{Tors}(\Omega_X^1)) \otimes L^*) \neq 0$ . Thus there is an injective map  $\Omega_X^1/\text{Tors}(\Omega_X^1) \rightarrow L$ .  $\square$

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