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PRIMITIVE LIFTING IN FREE NILPOTENT LIE ALGEBRAS

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Abstract: Let F_n be the free nilpotent Lie algebra of rank n and $L_{n,k}$ be the free n generator nilpotent Lie algebra of class k. We show that, for $1 \leq m \leq n$, every IA-system of m elements of $L_{n,k}$ can be lifted to a primitive system of m elements of $L_{n,k}$. In particular we establish primitive lifting in $L_{n,k}$ of a single element of $L_{n,k}$ modulo $\gamma_{c+1}(L_{n,k})$. We also present an automorphism of $F_n/\gamma_{c+1}(F_n)$ which cannot be lifted to an automorphisms of $F_n/\gamma_{c+1}(F_n)'$.

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1. Introduction

Let F_n be a free Lie algebra with free generating set $X = \{x_1, ..., x_n\}$ over the field K of characteristic 0. Let $L = L_{n,k} = \langle x_1, ..., x_n \rangle = F_n/\gamma_k(F_n)$ be the free nilpotent Lie algebras of rank $n \geq 2$, where $\gamma_k(F_n)$ is the k-th term of the lower central series of F_n. Denote by F_n' the commutator subalgebra $[F_n, F_n]$ of F_n. Let $W = \{w_1, ..., w_m\}$, $m \leq n$ be a set in F_n. The set W is said to be primitive system if it can be included in some free generating set of F_n. Primitivity of a given system W can be algorithmically decided. Such an algorithm for free groups is due to Whitehead [3]. For free Lie algebras there is also a criteria which characterizes primitivity related to Jacobian matrices [6]. The similar criteria was shown for free color Lie superalgebras by A.A. Mikhalev and A.A. Zolotky [4]. The primitivity criteria for free nilpotent and free polynilpotent Lie

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algebras was investigated by V. Shpilrain and Umirbaev. An automorphism of $L_{n,k}$ is called tame if it can be induced by an automorphism of F_n; one can also say that a generating system of $L_{n,k}$ can be lifted to a generating system of F_n. The question of lifting automorphism (generating systems) is naturally related to the problem of finding appropriate necessary and/or sufficient condition(s) for an endomorphism of the free Lie algebra F_n to be an automorphism. In Section 3 we investigate the sets which can be lifted to a primitive systems of $L_{n,k}$. In Section 4, first we give an automorphism of a free Lie algebra $F_n/\gamma_{c+1}(F_n)$ which cannot be lifted an automorphism of a free Lie algebra $F_n/\gamma_{c+1}(F_n)^{'}$. Then we show that also there are automorphisms of the Lie algebra $F_n/\gamma_2(F_n')$ of rank 3, which cannot be lifted to an automorphism of $F_n/\gamma_2(F_n'')$ of rank 3.

2. Preliminaries

Let $U(F_n)$ denote the universal enveloping algebra for F_n. Following [2] we denote by $\frac{\partial g}{\partial x_i}$ the right Fox derivative of g for $g \in U(F_n)$. Let $\varepsilon : U(F_n) \rightarrow K$ be the augmentation map, Δ be the kernel of this augmentation map Δ is $U(F_n)$-module freely generated by the set X and every element $u \in \Delta$ is uniquely expressible in the form $u = \sum \frac{\partial u}{\partial x_i} x_i$.

Let $W = \{w_1, ..., w_m\}$, $m \leq n$. Jacobian matrix of the set W is defined as $J\ (W) = (\frac{\partial w_i}{\partial x_i})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ over $U(F_n)$.

Theorem 2.1. (see [6]) *Let R be an ideal of F_n, and $Y = \{y_1, ..., y_n\}$ be a set of elements of F_n. Then the Lie algebra $F_n/R^{'}$ is generated by images of $Y = \{y_1, ..., y_n\}$ if and only if the Jacobian matrix $J(Y)$ has an inverse over $U(F_n)$.*

Let $W = \{w_1, ..., w_m\} \subset F_n$, $m \leq n$. We say that $W = \{w_1, ..., w_m\}$ is primitive mod $\gamma_c(F_n)$ if for some choice of $v_1, ..., v_m \in \gamma_c(F_n)$, the corresponding set $\{w_1 + v_1, ..., w_m + v_m\}$ is primitive, or equivalently the set $\{w_1 + \gamma_c(F_n), ..., w_m + \gamma_c(F_n)\}$ of cosets can be extended to some free generating set for $F_n/\gamma_c(F_n)$. Let $W = \{w_1, ..., w_m\} \subset F_n$, $m \leq n$. We say that $W = \{w_1, ..., w_m\}$ can be lifted (via $\gamma_c(F_n)$) to a primitive system mod $\gamma_k(F_n)$ for $k > c$ if and only if there exists $v_1, ..., v_m \in \gamma_c(F_n)$ such that the corresponding set $\{w_1 + v_1, ..., w_m + v_m\}$ is primitive mod $\gamma_c(F_n)$. A set of the form $\{x_1 + u_1, ..., x_m + u_m\}$, $u_i \in L_{n,k}^{'}$ will be called an IA-system. We say that an IA-system $\{x_1 + u_1, ..., x_m + u_m\}$, $u_i \in L_{n,k}^{'}$, $m \leq n$ is IA-primitive system if it can be extended to an IA-primitive system of $L_{n,k}$ of the form

$\{x_1 + u_1, ..., x_m + u_m, x_{m+1} + u_{m+1}, ..., x_n + u_n\}$, $u_i \in L'_{n,k}$.

The following lemma is an immediate consequence of the definitions

Lemma 2.2. *Let J be an ideal of $U(F_n)$ and $u \in \Delta$. Then $u \in J\Delta$ if and only if $\frac{\partial u}{\partial x_i} \in J$, $i = 1, .., n$.*

Now we need the following technical lemmas.

Lemma 2.3. *Let R be an ideal of F_n and $u \in \Delta$. Then $u \in \Delta_R \Delta$ if and only if $u \in \gamma_2(R)$.*

Lemma 2.4. *Let R be an ideal of F_n and $u \in \Delta$. Then $u \in \gamma_{c+1}(F_n)$ if and only if $u \in \Delta_R \Delta^c$.*

Corollary 2.5. *Let $u \in \gamma_{c+1}(F_n)$, then $\frac{\partial u}{\partial x_i} \in \Delta_{F_n} \Delta^{c-1}$.*

Proof. If $u \in \gamma_{c+1}(F_n)$ by Lemma 2.4 $u \in \Delta_{F_n} \Delta^c$ and by Lemma 2.2 $\frac{\partial u}{\partial x_i} \in \Delta_{F_n} \Delta^{c-1}$. □

Remark 2.6. If $w \in \gamma_{c+1}(F_n)$, then $w \in \Delta_{F_n} \Delta^c$. $\Delta_{F_n} \Delta^c = \Delta_{F_n} \Delta^{c-1} \Delta$, then $\frac{\partial w}{\partial x_i} \in \Delta_{F_n} \Delta^{c-1}$

3. Primitive Lifting in Free Nilpotent Lie Algebras

Let $W = \{w_1, ..., w_m\}$ be an arbitrary primitive system in $L_{n,k}$. Then it consists of elements of the form $w_i = \sum \alpha_i x_i + u_i$, $\alpha_i \in K$, $u_i \in L'_{n,k}$. It is clear that there exists an automorphism α of $L_{n,k}$ such that $\alpha(W) = \{x_1 + u_1, ..., x_m + u_m\}$, $u_i \in L'_{n,k}$. Thus primitive lifting of systems reduces to primitive lifting of IA-systems.

Theorem 3.1. *If an IA-system $\{x_1 + u_1, ..., x_m + u_m\}, m \leq n, u_i \in L'_{n,k}$ is primitive in $L_{n,k}$ then it is IA-primitive.*

Proof. Let $\{y_1, ..., y_m, z_{m+1}, ..., z_n\}$ be free generating set of $L_{n,k}$, where $y_i = x_i + u_i, i = 1, ..., m, z_j = \sum_{k=1}^{n} \alpha_{jk} x_k + v_j, j = m+1, ..., n, \alpha_{jk} \in K, v_j \in L'_{n,k}$ Let ϕ be an automorphism of $L_{n,k}$ such that

$$\phi : \left\{ z_j \to z_j - \sum \alpha_{jk} y_k, j = m+1, ..., n, y_j \to y_j, n \leq j \leq m \right\}.$$

Using ϕ the free generating set $\{y_1, ..., y_m, z_{m+1}, ..., z_n\}$ transformed to a free generating set of the form $\{y_1, ..., y_m, z'_{m+1}, ..., z'_n\}$, where z'_j are of the new form given by $z_j^{'} = \sum\limits_{k=m+1}^{n} \alpha_{jk} x_k + v_j^{'}$. Since every $z_j^{'}$ is primitive elements $\alpha_{jj} \neq 0$. Let us define the automorphism θ_k such that

$$\theta_k : \{z_{m+1} \to z_{m+1} - \alpha_{(m+1)k}.\alpha_{(m+2)k}^{-1}.z_{m+2}^{'},\ \alpha_{(m+2)k} \neq 0, z_j \to\ z_j\},$$

where $k = m+2, ..., n$ and $j \geq m+1$. Applying θ_k to the free generating set $\{y_1, ..., y_m, z_{m+1}^{'}, ..., z_n^{'}\}$ successively it can be transformed to a free generating system $\{y_1, ..., y_m, z_{m+1}^{''}, z_{m+2}^{'}, ..., z_n^{'}\}$, where $z_{m+1}^{''} = \beta_{m+1} x_{m+1} + v_{m+1}^{'}$, $\beta_{m+1} \in K, v_{m+1}^{'} \in L_{n,k}$. Similarly the generators $z_{m+1}^{'}, ..., z_n^{'}$ can be transformed to generators of the form $z_j^{''} = \beta_j x_j + v_j^{'}$, $\beta_j \in K$, $v_j^{'} \in L_{n,k}$. So we obtain $\{y_1, ..., y_m, z_{m+1}^{''}, z_{m+2}^{''}, ..., z_n^{''}\}$. Let φ be automorphism such that

$$\varphi : \{z_j^{''} \to \beta^{-1} z_j^{''}, j = m+1, ..., n, y_i \to y_i, i = 1, ..., n\},$$

where $\beta_j^{-1} \in K$. Applying φ to the set $\{y_1, ..., y_m, z_{m+1}^{''}, z_{m+2}^{''}, ..., z_n^{''}\}$ completes the proof. □

Lemma 3.2. *Every automorphisms of $L_{n,k-1}$ can be extended to an automorphism of $L_{n,k}$.*

Proof. Let $\phi \in \operatorname{Aut} L_{n,k-1}$ such that $\phi(x_i) = \sum \alpha_{ij} x_j + u_i$, where $u_i \in L_{n,k-1}^{'}$. Consider the mapping $\psi : L_{n,k} \to L_{n,k}$ which is defined as $\psi(x_i) = \sum \alpha_{ij} x_j + u_i + w_i$, where $w_i \in \gamma_{k-1}(F_n)/\gamma_k(F_n)$. Since the linear part of $\psi(x_i)$ is invertible, then ψ is an automorphism and $\phi(x_i \equiv \psi(x_i)$ modulo $\gamma_{k-1}(L_{n,k})$. Let $u \in L_{n,k}$ and $u = u_1 + ... + u_{k-2} + u_{k-1} + \gamma_k(F_n)$, where u_i is of length i. Since for $i = 1, 2, ...k-2$ $\phi(u_i) = \psi(u_i)$, ψ is an extension of ϕ. □

Lemma 3.3. *Let $1 \leq m \leq n$ be fixed. For each $c \geq 2$ $(2 \leq c < k-1)$ every IA-system of the form $\{x_1 + v_1, ..., x_m + v_m, x_{m+1}, ..., x_t\}$ with $v_i \in \gamma_c(L_{n,k})$ can be lifted to a primitive system of $L_{n,k}$ of the form $\{x_1 + v_1 + w_1, ..., x_m + v_m + w_m, x_{m+1}, ..., x_t\}$ with $w_i \in \gamma_{c+1}(L_{n,k})$.*

Proof. Define an automorphism $\psi \in \operatorname{Aut}(L_{n,k}/\gamma_{c+1}(L_{n,k}))$ such that

$$\psi : \{x_i \to x_i + v_i, i = 1, ..., m, x_j \to x_j, j = m+1, ..., t\}.$$

ψ can be extended an automorphism $\psi^{'} \in \operatorname{Aut}(L_{n,k}/\gamma_{c+2}(L_{n,k}))$ by the Lemma 3.2. If we define $\psi^{'}$ such that

$$\psi^{'} : \{x_i \to \psi(x_i) + w_i, i = 1, ..., m, x_j \to \psi(x_j), j = m+1, ..., t\},$$

then we complete the proof. □

Lemma 3.4. *For $1 \le m \le n$, $c \ge 2$ every IA-system $\{x_1 + u_1, ..., x_m + u_m, x_{m+1}, ..., x_t\}$ with $u_i \in L'_{n,k}$ can be lifted to a primitive system of $L_{n,k}$ of the form $\{x_1 + u_1 + w_1, ..., x_m + u_m + w_m, x_{m+1}, ..., x_t\}$ with $w_i \in \gamma_{c+1}(L_{n,k})$.*

Proof. We will prove the lemma by induction on $c \ge 2$. It is clear for $c = 2$. For the inductive step let $\{x_1 + u_1, ..., x_m + u_m, x_{m+1}, ..., x_t\}$ with $u_i \in L'_{n,k}$ be already transformed to $\{x_1 + u_1 + u_{1,c}, ..., x_m + u_m + u_{m,c}, x_{m+1}, ..., x_t\}$ with $u_{i,c} \in \gamma_c(L_{n,k})$. In this case the system can be lifted to $\{x_1 + u_1 + u_{1,c} + u_{1,c+1}, ..., x_m + u_m + u_{m,c} + u_{m,c+1}, x_{m+1}, ..., x_t\}$ with $u_{i,c+1} \in \gamma_{c+1}(L_{n,k})$. Put $g_i = x_i + u_i + u_{i,c} + u_{i,c+1}, i = 1, .., m, g_i = x_i, j = m+1, ..., t$. Thus there exist $\alpha \in \operatorname{Aut} L_{n,k}$ such that

$$\alpha : \{x_i + u_i \to g_i, i = 1, .., m, x_j \to x_j, j = m+1, ..., t\} .$$

Then

$$\alpha^{-1} : \{g_i \to x_i + u_i, i = 1, .., m, x_j \to x_j, j = m+1, ..., t\}$$

and for $i = 1, .., m$

$$\alpha^{-1}(x_i + u_i + u_{i,c}) = \alpha^{-1}(g_i - u_{i,c+1}) = x_i + u_i + \alpha^{-1}(-u_{i,c+1}) = x_i + u_i + v_i$$

for some $v_i \in \gamma_{c+1}(L_{n,k})$, and

$$\alpha^{-1}(x_k) = x_k .$$

Then the system transforms to the primitive system of the form $\{x_1 + u_1 + v_1 + u_{1,c+1}, ..., x_m + u_m + v_m + u_{m,c+1}, x_{m+1}, ..., x_t\}$. Put $w_i = v_i + u_{i,c+1}, i = 1, ..., m$. Thus we obtain the primitive system of the form $\{x_1 + u_1 + w_1, ..., x_m + u_m + w_m, x_{m+1}, ..., x_t\}$. This completes the proof of the lemma. □

Lemma 3.5. *For $1 \le m \le n$, $c \ge 2$ every IA-system $\{x_1 + v_1, ..., x_m + v_m\}$ with $v_i \in \gamma_2(L_{n,k})$ can be lifted (via $\gamma_{c+1}(L_{n,k})$) to a primitive system of $L_{n,k}$.*

Proof. For $m = 1$ there is nothing to prove. By the induction hypothesis $\{x_2 + v_2, ..., x_m + v_m\}$ can be lifted to a primitive system of $L_{n,k}$, so by the above theorem it can be lifted to a primitive IA-system. Thus there is an IA-automorphism $\alpha \in \operatorname{Aut}(L_{n,k})$ such that

$$\alpha : \{x_i + v_i + w_i \to x_i, x_1 + v_1 \to x_1 + u_1\} ,$$

where $i = 2, .., m$, $w_i \in \gamma_{c+1}(L_{n,k})$ and $u_1 \in \gamma_2(L_{n,k})$. Thus α transforms the system $\{x_2 + v_2, ..., x_m + v_m\}$ to $\{x_1 + u_1, x_2 + w'_2, ..., x_m + w'_m\}, w'_i \in \gamma_{c+1}(L_{n,k})$. Thus the problem reduces to lifting (via $\gamma_{c+1}(L_{n,k})$) of a system of the form $\{x_1 + v'_1, ..., x_m\}, v'_1 \in \gamma_2(L_{n,k})$, to a primitive system of $L_{n,k}$ by the above lemma, is the case. □

As a corollary to Lemma 3.4 and Lemma 3.5 we obtain the following important lemma.

Lemma 3.6. *For $1 \leq m \leq n$, $c \geq 2$ every IA-system $\{x_1+v_1, ..., x_m+v_m\}$ with $v_i \in \gamma_c(L_{n,k})$ can be lifted (via $\gamma_{c+1}(L_{n,k})$) to a primitive system of $L_{n,k}$.*

Now we consider primitive lifting of a single element of $L_{n,k}$.

Theorem 3.7. *Let $g \in L_{n,k}$ and let g be primitive mod $\gamma_{c+1}(L_{n,k})$, $n \geq 3$, $c \geq 2$. Then g can be lifted (via $\gamma_{c+1}(L_{n,k})$) to a primitive element of $L_{n,k}$.*

Proof. We may assume that $g = x_1+u$, $u \in \gamma_{c+1}(L_{n,k})$. Since g is primitive mod $\gamma_{c+1}(L_{n,k})$, g can be completed to the free generating set of $L_{n,k}$ of the form $\{x_1+u, y_2, ..., y_n\}$. Let α define such that

$$\alpha : \{x_1 + u \rightarrow x_1 + u,\ y_i \rightarrow x_i\}\,.$$

By Lemma 3.5 the set of $\{x_1+u, x_2, ..., x_n\}$ can be lifted (via $\gamma_{c+1}(L_{n,k})$) to the primitive system of $L_{n,k}$ of the form $\{x_1+u+w, x_2, ..., x_n\}$ with $w \in \gamma_{c+1}(L_{n,k})$. Then x_1+u+w is a primitive element of $L_{n,k}$. This completes the proof of the theorem. □

4. Non-Tame Automorphisms

In this section, we are going to give some non-tame automorphisms of $L_{n,k}$ for the special cases of n and k. Let R be an ideal of F_n. By Δ_R we denote the ideal of $U(F_n)$ generated by R. It is the kernel of the natural homomorphism $U(F_n) \rightarrow U(F_n/R)$.

Denote $R = \gamma_{c+1}(F_n)$. Note that invertible elements of $U(F_n)/\Delta_R$ are of the form $\alpha + \Delta_R$, where $\alpha \in K$.

Lemma 4.1. (see Bryant and Drensky [8]) *Let $s \geq 2$and let ψ be an endomorphism of $U(F_n)$ such that*

$$\psi(x_1) \equiv x_1 + f(\text{modulo}\Delta^{s+1})$$

$$\psi(x_i) \equiv x_i(\text{modulo}\Delta^{s+1})\,,$$

where f is a homogenous polynomial of total degree s. If ψ is an automorphism of $U(F_n)$ then $\frac{\partial f}{\partial x_1}$ is balanced.

Now we are going to give examples of non-tame automorphisms.

Theorem 4.2. *For any $c \geq 3$ and $n \geq 2$, the following automorphism ϕ of the free Lie algebra $F_n/\gamma_{c+1}(F_n)$ defined as*

$$\phi : x_1 \to x_1 + [[x_1 x_2] x_2] \ ,$$

$$x_i \to x_i, \quad 2 \leq i \leq n \, ,$$

cannot be lifted to an automorphism of $F_n/\gamma_{c+1}(F_n)'$.

Proof. Assume that ϕ can be lifted to an automorphism $\overline{\phi} : F_n/\gamma_{c+1}(F_n)' \to F_n/\gamma_{c+1}(F_n)'$ such that

$$\overline{\phi} : x_1 \to x_1 + [x_1 x_2 x_2] + u_1 \, ,$$

$$x_i \to x_i + u_i, \quad 2 \leq i \leq n \, ,$$

with $u_i \in \gamma_{c+1}(F_n)$. $\overline{\phi}$ can be extend to an automorphism ψ of $U(F_n)$ such that $\psi(x_i) = \overline{\phi}(x_i)$. We know that $\gamma_{c+1}(F_n) \subset \Delta^{c+1}$. Then

$$\psi : x_1 \to x_1 + [[x_1 x_2] \, x_2] \, (\text{modulo} \Delta^{c+1}) \, ,$$

$$x_i \to x_i (\text{modulo} \Delta^{c+1}), \quad c \geq 3 \, .$$

By Lemma 4.1, $[[x_1 x_2] \, x_2]$ will be balanced. But $\frac{\partial [[x_1 x_2] x_2]}{\partial x_1} = x_2 x_2$ is not balanced, which is a contradiction. □

Theorem 4.3. *The automorphism φ of the Lie algebra $L_{n,2} = F_n/\gamma_2(F_n')$ given by*

$$\varphi : x_1 \to x_1 + [[[x_3, x_2] \, , x_1] \, , x_1] \ ,$$

$$x_i \to x_i, \quad i \neq 1 \, ,$$

cannot be lifted to an automorphism of the Lie algebra $F_n/\gamma_2(F_n'')$.

Proof. Assume that φ can be lifted to an automorphism ψ of $F_n/\gamma_2(F_n'')$ such that

$$\psi : x_1 \to x_1 + [[[x_3, x_2], x_1], x_1] + w_1 \, ,$$

$$x_i \to x_i + w_i, \quad i \neq 1 \, ,$$

where $w_i \in \gamma_2(F_n')$. Then the jacobian matrix $J(\psi)$ of ψ is of the form

$$\begin{bmatrix} 1+[[x_3,x_2],x_1]-x_1[x_3,x_2]+\frac{\partial w_1}{\partial x_1} & x_1 x_1 x_3 + \frac{\partial w_1}{\partial x_2} & -x_1 x_1 x_2 + \frac{\partial w_1}{\partial x_3} & \cdot & \frac{\partial w_1}{\partial x_n} \\ \frac{\partial w_2}{\partial x_1} & 1 + \frac{\partial w_2}{\partial x_2} & \frac{\partial w_2}{\partial x_3} & \cdot & \frac{\partial w_2}{\partial xn} \\ \frac{\partial w_3}{\partial x_1} & \frac{\partial w_3}{\partial x_2} & 1 + \frac{\partial w_3}{\partial x_3} & \cdot & \frac{\partial w_3}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial w_n}{\partial x_1} & \frac{\partial w_n}{\partial x_2} & \frac{\partial w_n}{\partial x_3} & \cdot & 1 + \frac{\partial w_n}{\partial xn} \end{bmatrix} .$$

$U(F_n)/\Delta_{F_n''}$ is an ore domain. So we can reduce $J(\psi)$ to a diagonal form by applying elementary transformations to its rows. Let us say $\overline{J(\psi)}$ to this new matrix. In the end we get an element of the following form in the upper left corner:

$$a_{11} = a(1 + [[x_3, x_2], x_1] - x_1.[x_3, x_2] + \frac{\partial w_1}{\partial x_1}).$$

The Jacobian matrix $J(\psi)$ is invertible. So $\overline{J(\psi)}$ is also invertible over $U(F_n)/\Delta_{F_n''}$. Then the elements on the diagonal of $\overline{J(\psi)}$ are invertible. Therefore there exists an element α of the field K such that

$$a([[x_3, x_2], x_1] - x_1.[x_3, x_2] + \frac{\partial w_1}{\partial x_1}) = \alpha - 1.$$

Hence if $\alpha \neq 1$ then $\alpha - 1$ must be contained in $\Delta_{F_n''}$, which is impossible. If $\alpha = 1$ then

$$a([[x_3, x_2]\,, x_1] - x_1.\,[x_3, x_2] + \frac{\partial w_1}{\partial x_1}) = 0.$$

Since $U(F_n)/\Delta_{F_n''}$ is an integral domain $a \in \Delta_{F_n''}$ or $[[x_3, x_2]\,, x_1] - x_1.\,[x_3, x_2] + \frac{\partial w_1}{\partial x_1} \in \Delta_{F_n''}$. We know that $a \in U(F_n)/\Delta_{F_n''}$. Then $[[x_3, x_2]\,, x_1] - x_1.\,[x_3, x_2] + \frac{\partial w_1}{\partial x_1} \in \Delta_{F_n''}$. We are only interested in monomials of weight 4 from F_n''. We have to consider the following possibilities:

$$w_1 = [[x_i x_1][x_j x_1]], \quad i, j \neq 1\,,$$

and

$$w_1 = [[x_i x_j][x_k x_1]], \quad i, j, k \neq 1\,.$$

Compute $\frac{\partial w_1}{\partial x_1}$ for each case:

$$\frac{\partial w_1}{\partial x_1} = [x_i x_1]x_j - [x_j x_1]x_i$$

and

$$\frac{\partial w_1}{\partial x_1} = [x_i x_j]x_k \neq [[x_3, x_2], x_1] - x_1.[x_3, x_2].$$

It is clear that in both cases $\frac{\partial w_1}{\partial x_1}$ cannot be equal to $[[x_3, x_2], x_1] - x_1.[x_3, x_2]$. So this contradiction completes the proof. □

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