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A GENERALIZED 'USEFUL' INACCURACY OF ORDER α AND TYPE β AND CODING THEOREMS

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Abstract: Useful inaccuracy measures and mean codeword lengths are well known in the literature of information theorey. In this communication, a new generalized 'useful' inaccuracy of order α and type β has been proposed and coding theorem has been established by considering the said measure and a generalized average 'useful' codeword length. Our motivation for studying this is that it generalizes some results which already existing in the literature.

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1. Introduction

Consider the model given below for a finite random experiment scheme having $(x_1, x_2, ..., x_n)$ as a complete system of events, happening with respective probabilities $P = (p_1, p_2, ..., p_n)$ and credited with utilities $U = (u_1, u_2, ..., u_n)$, $u_i > 0$, $i = 1, 2, ..., n$. Denote

$$\chi = \left[\; x_1 x_2 ....x_n \; p_1 p_2 ......p_n \; u_1 u_{2.........} u_n \; \right] , \qquad (1.1)$$

we call (1.1) as utility information scheme.

Let $Q = (q_1, q_2, ..., q_n)$ be the predicted distribution having the utility distribution $(u_1, u_2, ..., u_n)$. Taneja and Tuteja [14] have suggested and characterized the 'useful' inaccuracy measure

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$$I\left(P;Q;U\right)=-\sum_{i=1}^{n}u_ip_i\log q_i\,. \tag{1.2}$$

By considering weighted mean codeword length, see [6]

$$L\left(U\right)=\frac{\sum\limits_{i=1}^{n}u_ip_il_i}{\sum\limits_{i=1}^{n}u_ip_i}\,. \tag{1.3}$$

Taneja and Tuteja [14] derived the lower and upper bounds on $L\left(U\right)$ interms of $I\left(P;Q;U\right)$.

Bhatia [3] defined the 'useful' average code lengths of order α as

$$L_\alpha\left(U\right)=\frac{1}{D^{\frac{\alpha-1}{\alpha}}-1}\left[1-\sum_{i=1}^{n}p_i\left(\frac{u_i}{\sum\limits_{i=1}^{n}u_ip_i}\right)^{\frac{1}{\alpha}}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right], \tag{1.4}$$

where $\alpha>0\ (\neq 1)$ and $\sum\limits_{i=1}^{n}p_i\leq 1, i=1,2,...,n$ and D is the size of the code alphabet. He also derived the bounds for the 'useful' average code length of order α and is given by

$$I_\alpha\left(P;Q;U\right)=\frac{1}{D^{\frac{\alpha-1}{\alpha}}-1}\left[1-\left(\frac{\sum\limits_{i=1}^{n}u_ip_iq_i^{\alpha-1}}{\sum\limits_{i=1}^{n}u_ip_i}\right)^{\frac{1}{\alpha}}\right], \tag{1.5}$$

where $\alpha>0\,(\neq 1)$ and $p_i\geq 0,\ \sum\limits_{i=1}^{n}p_i\leq 1,\ i=1,2,...,n$ and D is the size of the code alphabet.

Under the condition

$$\sum_{i=1}^{n}p_iq_i^{-1}D^{-l_i}\leq 1\,, \tag{1.6}$$

where D is the size of the code alphabet. Inequality (1.6) is a generalized Kraft's inequality [4]. A code satisfying generalized Kraft's inequality would be termed as personal probability code.

Longo [10], Gurdial and Pessoa [5], Autar and Khan [1], Jain and Tuteja [8], Taneja et al [15], Hooda and Bhaker [7], Bhatia [3] and Singh, Kumar and

Tuteja [11] considerd the problem of 'useful' information measure and used it studying the noiseless coding theorems for sources involving utilities.

In the next section, we shall study some coding theorems for a generalized 'useful' inaccuracy of order α and type β for incomplete probability distribution.

2. Coding Theorems

Consider the function

$$I_\alpha^\beta(P;Q;U) = \frac{1}{D^{\frac{\alpha-1}{\alpha}}-1}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{\frac{1}{\alpha}}\right], \tag{2.1}$$

where $\alpha > 0\,(\neq 1)$, $\beta > 0$, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i \leq 1$, $i = 1, 2, ..., n$ and D is the size of the code alphabet.

(a) For $\alpha \to 1, \beta = 1$ and $\sum_{i=1}^{n} p_i = 1$, the measure (2.1) reduces to Taneja and Tuteja [15] measure of 'useful' inaccuracy.

(b) For $\beta = 1, p_i = q_i$, $\forall\ i = 1, 2, ..., n$, the measure (2.1) reduces to the measure given by Autar and Khan [1] 'useful' information measure.

(c) For $\alpha \to 1, \beta = 1$ and $p_i = q_i$, $\forall\ i = 1, 2, ..., n$, the measure (2.1) reduces to Belis and Guiasu [2] measure of 'useful' information for incomplete probability distribution. Further, when utility aspect of the scheme is ignored, the measure reduces to Shannon [12] measure of entropy.

(d) When the probability distribution is complete and the utility aspect of the scheme is ignored as well as $\alpha \to 1, \beta = 1$. The measure (2.1) becomes Kerridge's [9] measure of inaccuracy. We call (2.1) as generalized 'useful' inaccuracy of order α and type β for incomplete probability distribution.

Further, consider a generalized 'useful' mean length credited with utilities and probabilites as

$$L_\alpha^\beta(U) = \frac{1}{D^{\frac{\alpha-1}{\alpha}}-1}\left[1-\sum_{i=1}^{n} p_i^\beta \left(\frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right], \tag{2.2}$$

where $\alpha > 0\,(\neq 1)$, $\beta > 0$, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i \leq 1, i = 1, 2, ..., n$ and D is the size of the code alphabet.

(a) For $\alpha \to 1, \beta = 1$, the measure (2.2) reduces to 'useful' mean length $L(U)$ of the code, given by Guiasu and Picard [6].

(b) when the utility aspect of the scheme is ignored by taking $u_i = 1$, $\forall i = 1, 2, .., n$, $\sum_{i=1}^{n} p_i = 1$ and $\alpha \to 1, \beta = 1$, the mean length (2.2) becomes optimal code length identical to Shannon, see [12].

Now we find the bounds for $L_\alpha^\beta(U)$ in terms of $I_\alpha^\beta(P;Q;U)$ under the condition

$$\sum_{i=1}^{n} p_i^\beta q_i^{-1} D^{-l_i} \le 1\,, \tag{2.3}$$

where D is the size of the code alphabet.

Theorem 2.1. *For all integers D $(D > 1)$, let l_i satisfy (2.3), then the generalized average 'useful' codeword length satisfies*

$$L_\alpha^\beta(U) \ge I_\alpha^\beta(P;Q;U)\ , \tag{2.4}$$

and the equality holds iff

$$l_i = -\log\left(\frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\alpha-1}}\right)\ . \tag{2.5}$$

Proof. By Hölders' inequality, see [13]

$$\sum_{i=1}^{n} x_i y_i \ge \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}\ , \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1, p < 1\,(\neq 0)$ and $x_i, y_i > 0, i = 1, 2, .., n$, we see the equality holds if and only if there exists a positive constant c such that

$$x_i^p = c y_i^q\,. \tag{2.7}$$

Making the substitution

$$p = \frac{\alpha - 1}{\alpha}, \quad q = 1 - \alpha, \quad x_i = p_i^{\frac{\alpha\beta}{\alpha-1}} \left(\frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{\frac{1}{\alpha-1}} D^{-l_i}\,,$$

$$y_i = p_i^{\frac{\beta}{1-\alpha}} \left(\frac{u_i}{\sum\limits_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{1-\alpha}} q_i^{-1}$$

in (2.6), using (2.3) and after making suitable operations we get (2.4) for $\left(D^{\frac{\alpha-1}{\alpha}} - 1\right) \neq 0$ according as $\alpha \neq 1$. □

Theorem 2.2. *For every code with lengths* $\{l_i\}$, $i = 1, 2, ..., n$ *of Theorem 2.1,* $L_\alpha^\beta(U)$ *can satisfy the inequality*

$$L_\alpha^\beta(U) < I_\alpha^\beta(P;Q;U)\, D^{\frac{1-\alpha}{\alpha}} + \frac{1 - D^{\frac{1-\alpha}{\alpha}}}{D^{\frac{\alpha-1}{\alpha}} - 1}. \tag{2.8}$$

Proof. Let l_i be the positive integer satisfying the inequality

$$-\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) \leq l_i < -\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) + 1. \tag{2.9}$$

Consider the intervals

$$\delta_i = \left[-\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right), -\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) + 1 \right] \tag{2.10}$$

of length 1. In every δ_i, there lies exactly one positive integer l_i such that

$$0 < -\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) \leq l_i < -\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) + 1. \tag{2.11}$$

We will first show that the sequences $l_1, l_2, ..., l_n$, thus defined satisfy (2.3). From (2.11), we have

$$-\log\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) \leq l_i,$$

or

$$\left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) \geq D^{-l_i}\,.$$

Multiply both sides by $p_i^{\beta} q_i^{-1}$ and summing over $i = 1, 2, .., n$, we get (2.3). The last inequality of (2.11) gives

$$l_i < -\log \left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right) + 1\,,$$

or

$$D^{-l_i} < \left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right)^{-1} D\,,$$

$$D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} < \left(\frac{u_i q_i^{\alpha}}{\sum\limits_{i=1}^{n} u_i p_i^{\beta} q_i^{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} D^{\frac{1-\alpha}{\alpha}}\,.$$

Multipying both sides by $p_i^{\beta} \left(\frac{u_i}{\sum\limits_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{\alpha}}$ and summing over $i, i = 1, 2, .., n$ and after suitable operations, we get

$$L_{\alpha}^{\beta}(U) < I_{\alpha}^{\beta}(P;Q;U)\, D^{\frac{1-\alpha}{\alpha}} + \frac{1 - D^{\frac{1-\alpha}{\alpha}}}{D^{\frac{\alpha-1}{\alpha}} - 1}\,. \qquad \square \tag{2.12}$$

Remark. For $0 < \alpha < 1$ and since $D \geq 2$, from (2.12), we have $\frac{1-D^{\frac{1-\alpha}{\alpha}}}{D^{\frac{\alpha-1}{\alpha}}-1} > 1$ from which it follows that the upper bound of $L_{\alpha}^{\beta}(U)$ in (2.8) is greater than unity.

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