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LINEAR PRESERVER OF MATRIX MAJORIZATION

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Abstract: An $n \times m$ matrix A is said to be *matrix majorized from the left* by an $n \times m$ matrix B, and write $A \prec_\ell B$, if there exists an $n \times n$ row stochastic matrix R such that $A = RB$. Let M_{nm} denote the linear space of all real $n \times m$ matrices. An operator $T : M_{nm} \longrightarrow M_{nm}$ is said to be a preserver of $\prec_\ell$ if $TX \prec_\ell TY$ whenever $X \prec_\ell Y$ and $X, Y \in M_{nm}$. It is shown that a linear operator $T : M_{nm} \longrightarrow M_{nm}$ preserves $\prec_\ell$ if and only if there exist an $n \times n$ permutation matrix $P \neq I$, an $m \times m$ real matrix L, and real numbers a and b with $ab \leq 0$, such that $TX = (aI + bP)XL$ for all $X \in M_{nm}$ and, if $n \neq 2$, $ab = 0$. Moreover, if T satisfies the extra condition $X \prec_{ell} Y$ whenever $TX \prec_\ell TY$, then $ab = 0$ for all n and $aI + bP$ and L are invertible.

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1. Introduction

Throughout the paper M_{nm} denotes the collection of all $n \times m$ real matrices. In case $n = m$, we write M_n for M_{nn}. A matrix $R \in M_n$ is called a *row stochastic matrix* if its entries are nonnegative and the sum of the entries on each row is $+1$. The collection of all $n \times n$ row stochastic matrices is denoted by $\mathcal{RS}(n)$.

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Also, $\mathcal{P}(n)$, $\mathbb{R}^n$ and $\mathbb{R}_m$ denote the collections of all $n \times n$ permutation matrices, $n \times 1$ column vectors, and $1 \times m$ row vectors, respectively. For $X \in M_{nm}$, the notation $R(X) \subset \mathbb{R}_m$ will denote the collection of all rows of X; also, $X^t \in M_{mn}$ denotes the transpose of $X \in M_{nm}$.

Let $A, B \in M_{nm}$. Then A is said to be *matrix majorized from the left* by B, and write $A \prec_\ell B$, if $A = RB$ for some $R \in \mathcal{RS}(n)$. The left matrix majorization has been already considered in the reference [6] as *weak matrix majorization*. Let $T : M_{nm} \longrightarrow M_{nm}$ be a mapping. We say T is a *preserver* of $\prec_\ell$ if $TX \prec_\ell TY$ whenever $X \prec_\ell Y$; it is called a strong preserver of $\prec_\ell$ if it further satisfies $X \prec_\ell$ whenever $TX \prec_\ell TY$.

For more information on this type of matrix majorization as well as other types of matrix majorization, we refer the reader to [2] and [6]. In particular, the authors proved in [3] that if $T : M_n \longrightarrow M_n$ strongly preserves $\prec_\ell$, then there exist a permutation matrix $P \in \mathcal{P}(n)$ and an invertible matrix $L \in M_n$ such that $T(X) = PXL$ for all $X \in M_n$. Linear operators that preserve other types of majorization are characterized in [1, 3, 4].

In the present paper, we will show that, if $T : M_{nm} \longrightarrow M_{nm}$ preserves $\prec_\ell$, then $T(X) = (aI + bP)XL$ for all $X \in M_{nm}$, where $L \in M_m$, I is the $n \times n$ identity matrix, $P \neq I$ is an $n \times n$ permutation matrix, a and b are real numbers such that $ab \leq 0$, and, if $n \neq 2$, $ab = 0$. We conclude this section by fixing the notations $\min X$ and $\max X$ for the minimum and the maximum of the entries of any column vector X, respectively.

In the present paper, we are solely dealing with the matrix majorization $\prec_\ell$ and, hence, we may enjoy the convenience of replacing $\prec_\ell$ with the simpler notation $\prec$ throughout the remainder of the paper.

2. Case $m = 1$

In this section we study linear operators $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which preserve $\prec$ (note that $\mathbb{R}^n = M_{n1}$). As an immediate consequence of the definition we can state the following lemmas.

Lemma 2.1. (see [6]) *For each $X, Y, Z \in M_{nm}$ the following assertions are true.*

(1) $X \prec X$. Also, $X \prec 0$ if and only if $X = 0$.

(2) If $X \prec Y$ and $Y \prec Z$, then $X \prec Z$.

(3) If $X = [X_1, \dots, X_m] \prec Y = [Y_1, \dots, Y_m]$ and if $i_1, i_2, \dots, i_k \in \{1, 2, \dots, m\}$, then $[X_{i_1}, \dots, X_{i_k}] \prec [Y_{i_1}, \dots, Y_{i_k}]$ (repetition of columns is allowed).

(4) If $X \prec Y$ and $B \in M_{mp}$ for some natural number p, then $XB \prec YB$.

(5) If $X \prec Y$ and $P, Q \in \mathcal{P}(n)$, then $PX \prec QY$.

In the following, for $A \in M_{nm}$, $R(A) \subset \mathbb{R}_m$ denotes the set of all distinct rows of A, and $\mathrm{co}\,(R(A))$ denotes its convex hull.

Lemma 2.2. (see [6]) *Let $A \in M_{nm}$. Then*

$$\{X \in M_{nm} : X \prec_\ell A\} = \{X \in M_{nm} : R(X) \subset \mathrm{co}\,(R(A))\}.$$

In particular, if $X, Y \in M_{n1} = \mathbb{R}^n$, then $X \prec Y$ if and only if $\min Y \leq \min X \leq \max X \leq \max Y$.

Theorem 2.3. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator. Then T preserves $\prec_\ell$ if and only if $T(X) = (aI + bP)X$ for all $X \in \mathbb{R}^n$, where $P \neq I$ is an $n \times n$ permutation matrix, and $a, b \in \mathbb{R}$ are such that $ab \leq 0$ and, if $n \neq 2$, $ab = 0$. In case, if $n \neq 2$, then $aI + bP = cQ$ for some $c \in \mathbb{R}$ and, hence, $T(X) = QXK$ for some $K \in M_m$.*

Proof. To prove the sufficiency, let a, b, and P be as in the statement of the theorem. Let $R \in \mathcal{RS}(n)$ and $X \in \mathbb{R}^n$ be arbitrary. If $n \neq 2$, then $aI + bP = cQ$ for some $c \in \mathbb{R}$ and some $Q \in \mathcal{P}(n)$ and, hence, $(aI + bP)RX = cQRX = S(aI + bP)X$, where $S = QRQ^{-1} \in \mathcal{RS}(n)$. If $n = 2$, then, letting

$$R = \begin{bmatrix} r & 1-r \\ s & 1-s \end{bmatrix} \text{ and } S = (a-b)^{-1} \begin{bmatrix} ar + bs - b & a - ar - bs \\ as + br - b & a - as - br \end{bmatrix},$$

it follows that $r, s \in [0, 1]$, $S \in \mathcal{RS}(n)$, and $(aI + bP)RX = S(aI + bP)X$. Thus, in both cases, the linear mapping $X \longmapsto (aI + bP)X$ preserves $\prec$.

For necessity, we assume without loss of generality $n \neq 2$. Let $i, j = 1, 2, \ldots, n$. Since $e_i \prec e_j \prec e_i$, it follows that $\min T(e_i)$ and $\max T(e_i)$ are constants a and b independent of i. Also, since $0 \prec e_i$, it follows that $a \leq 0 \leq b$. If $a = b = 0$, then the matrix $[T]$ of T with respect to the standard basis is 0. Otherwise, one may replace T by $-T$, if needed, and assume without loss of generality $a < 0 \leq b$. At this point we assume $n \geq 3$. For a proper subset J of $\{1, 2, \ldots, n\}$, $e_1 \prec \sum_{j \in J} e_j \prec e_1$ and, hence, $a = \min T(\sum_{j \in J} e_j) \leq \max T(\sum_{j \in J} e_j) = b$. Since every column of $[T]$ contains at least one entry equal to a, it follows that, there exists exactly one entry equal to a on each row and on each column of $[T]$. Moreover, every row of $[T]$ cannot contain any other negative entry. Thus $T = aQ + A$ for some $Q \in \mathcal{P}(n)$ and some nonnegative matrix A. Define $\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $\tau = -a^{-1}Q^{-1}T$. Then $[\tau] = -I + B$ and T preserves $\prec$ if and only if τ does so. Hence, we can further assume without loss of generality that $a = -1$, $Q = I$, and $[T] = -I + B$, where B is a nonnegative matrix and has a zero diagonal. Since $b > 0$, it follows from an argument similar to the one given for $a < 0$ that there is exactly one entry on each row and each column that is equal to b, and all other entries are nonpositive. Thus $T = -I + bQ'$, where $Q' \in \mathcal{P}(n)$ is such that $Q'e_i \neq e_i$ $(i = 1, 2, \dots, n)$. Hence, $Q'e_1 = e_i$ for some $i \neq 1$. Let $X = e_1 + \frac{1}{2}e_i$. Then $T(X) = (-I + bQ)X = -e_1 + (-\frac{1}{2} + b)e_i + \frac{b}{2}Q'e_i$ and, hence, $\max T(X) \in \{-1, -\frac{1}{2} + b, \frac{b}{2}, -1 + \frac{b}{2}, 0\}$. Since $e_1 \prec X \prec e_1$, $b = \max TX \in \{-1, -\frac{1}{2} + b, \frac{b}{2}, -1 + \frac{b}{2}, 0\}$; a contradiction. Thus $b = 0$ and the proof is complete in case $n \neq 2$.

Finally, assume $n = 2$ and, without loss of generality, $T(e_1) = ae_1 + be_2$, where $a < 0 \leq b$. Then either $T(e_2) = ae_1 + be_2$ or $T(e_2) = be_1 + ae_2$. In the first case $T(e_1 - e_2) = 0$ and, since $e_1 \prec e_1 - e_2$, $T(e_1) = 0$; a contradiction. Thus, $T = aI + bP$, where $P \neq I$ and $P \in \mathcal{P}(2)$. □

3. The General Case

We are now ready to prove the general case of our main result.

Theorem 3.1. *Let $T : M_{nm} \longrightarrow M_{nm}$ be a linear operator. Then T preserves $\prec$ if and only if $T(X) = (aI + bP)XL$ for all $X \in M_{nm}$, where $L \in M_m$, $P \in \mathcal{P}(n)$, $P \neq I$, a and b are real numbers such that $ab \leq 0$, and, if $n \neq 2$, $ab = 0$.*

Proof. Assume without loss of generality that $T \neq 0$ and that $n \geq 2$. The proof of the sufficiency is the same as in the special case of Theorem 2.3. We now prove the necessity of the condition.

For each $i = 1, 2, \dots, m$, define the linear operators $E_i : \mathbb{R}^n \longrightarrow M_{nm}$ by $E_i(X) = Xe_i^t$ for all $X \in \mathbb{R}^n$ and $E^i : M_{nm} \longrightarrow \mathbb{R}^n$ by $E^i(X) = Xe_i$ for all $X \in M_{nm}$, where $\{e_1, e_2, \dots, e_m\}$ denotes the standard basis of $\mathbb{R}^m$. Then $T_{ij} := E^jTE_i$ preserves $\prec$ for all $i, j = 1, 2, \dots, m$. It is now easy to see that, for all $X = [X_1, \dots, X_m] \in M_{nm}$,

$$T(X) = T([X_1, \dots, X_m]) = [\sum_{i=1}^{m} T_{i1}(X_i), \dots, \sum_{i=1}^{m} T_{im}(X_i)].$$

By Theorem 2.3, each T_{ij} is of the form $a_{ij}I + b_{ij}P_{ij}$ for some $n \times n$ permutation matrix $P_{ij} \neq I$, and some real numbers a_{ij} and b_{ij} such that $a_{ij}b_{ij} \leq 0$. Moreover, if $n \neq 2$, then $a_{ij}b_{ij} = 0$. We proceed in the following steps.

Step I. Assume $n \geq 3$. Then $a_{ij}b_{ij} = 0$ and, hence, $a_{ij}I + b_{ij}P_{ij} = \ell_{ij}Q_{ij}$ for some $Q_{ij} \in \mathcal{P}(n)$ and some $\ell_{ij} \in \mathbb{R}$ $(i, j = 1, 2, \dots, m)$. Fix i, j such that $\ell_{ij} \neq 0$ and, then, assume without loss of generality that $i = j = 1$. Fix $p, q \in \{1, 2, \dots, m\}$. If $\ell_{pq} = 0$, redefining $Q_{pq} = Q_{11}$ will cause no loss of generality. So assume $\ell_{pq} \neq 0$. We claim $Q_{pq} = Q_{11}$. For convenience write $Q = Q_{11}$ and $\ell = \ell_{11}$.

Let $k \in \{1, 2, 3, \ldots, n\}$ and $p, q \in \{1, 2, \ldots, m\}$ with $(p, q) \neq (1, 1)$. Let $\{\varphi_1, \ldots, \varphi_n\}$ be the standard basis of $\mathbb{R}^n$ and define $\varphi = \varphi_1 + \ldots + \varphi_n$. Let $X = [X_1, \ldots, X_m]$ and $Y = [Y_1, \ldots, Y_m]$ be matrices in M_{nm} such that $X_1 = c\varphi$, $Y_1 = c\varphi_k$, $X_p = d\varphi$ and $Y_p = d\varphi_k$ (if $p \neq 1$), and $X_i = Y_i = 0$, otherwise (the real constants c and d depending on the situation will be determined later). Since $X \prec Y$, $TX \prec TY$ and, hence,

$$\begin{aligned}[(TX)_1, (TX)_q] &= [c\ell\varphi + d\ell_{p1}\varphi, c\ell_{1q}\varphi + d\ell_{pq}\varphi] \\ &\prec [(TY)_1, (TY)_q] = [c\ell Q\varphi_k + d\ell_{p1}Q_{p1}\varphi_k, c\ell_{1q}Q_{1q}\varphi_k + d\ell_{pq}Q_{pq}\varphi_k],\end{aligned}$$

where d is taken to be 0 if $p = 1$. Assume $\ell_{1q} \neq 0$. Letting $c = 1$ and $d = 0$ yields $\varphi = RQ\varphi_k = RQ_{1q}\varphi_k$ for some $R \in \mathcal{RS}(n)$. Since Q_{ij} permutes the standard basis for any i and j, and since a row stochastic matrix has at most one column equal to φ, it follows that $Q\varphi_k = Q_{1q}\varphi_k$. Since k was arbitrary, $Q = Q_{1q}$ for all $q = 1, 2, \ldots, m$. More generally, we have shown that, if $\ell_{ir}\ell_{is} \neq 0$, then $Q_{ir} = Q_{is}$ for any $i, r, s = 1, 2, \ldots, m$.

Next, assume $p \neq 1$ and $\ell_{p1} \neq 0$. Choose c, d such that $(c\ell)(d\ell_{p1}) > 0$. Since $c\ell\varphi + d\ell_{p1}\varphi \prec c\ell Q\varphi_k + d\ell_{p1}Q_{p1}\varphi_k$, it follows that, if $Q\varphi_k \neq Q_{p1}\varphi_k$, then min $\{c\ell, d\ell_{p1}\} \leq c\ell + d\ell_{p1} \leq$ max $\{c\ell, d\ell_{p1}\}$; a contradiction. Thus again $Q = Q_{p1}$ for all $p = 1, 2, \ldots, m$. More generally, we have shown that if $\ell_{ri}\ell_{si} \neq 0$, then $Q_{ri} = Q_{si}$ for any $i, r, s = 1, 2, \ldots, m$.

It remains to prove $Q = Q_{pq}$ in case $p \neq 1$, $q \neq 1$, and $\ell_{p1} = \ell_{1q} = 0$. In this case, letting $c = d = 1$ yields $RQ\varphi_k = RQ_{pq}\varphi_k = \varphi$ for some $R \in \mathcal{RS}(n)$ and, hence, $Q\varphi_k = Q_{pq}\varphi_k$. Since k was arbitrary, $Q = Q_{pq}$.

Summing up, we have shown that $Q_{ij} = Q$ for all $i, j = 1, 2, \ldots, m$. Thus, for $n \geq 3$, we observe that if $X = [X_1, \ldots, X_m] \in M_{nm}$, then

$$\begin{aligned}T(X) = T([X_1, \ldots, X_m]) &= [\sum_{i=1}^{m} \ell_{i1}Q_{i1}X_i, \ldots, \sum_{i=1}^{m} \ell_{im}Q_{im}X_i] \\ &= Q[\sum_{i=1}^{m} \ell_{i1}X_i, \ldots, \sum_{i=1}^{m} \ell_{im}X_i] = QXL,\end{aligned}$$

where $L = [\ell_{ij}]$.

Step 2. Assume $n = 2$. In this case $T_{ij} = a_{ij}I + b_{ij}P$ with $a_{ij}b_{ij} \leq 0$ for $i, j = 1, 2, \ldots, m$, where $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Again, here, we assume without loss of generality that $T_{11} \neq 0$ and that $T_{pq} = \ell_{pq}T_{11}$ whenever $T_{pq} = 0$. Now, assume $T_{pq} \neq 0$ for some $(p, q) \neq (1, 1)$. Let $\{\varphi_1, \varphi_2\}$ be the standard basis of $\mathbb{R}^2$ and $\varphi = \varphi_1 + \varphi_2$. Choose $X = [X_1, \ldots, X_m]$ and $Y = [Y_1, \ldots, Y_m]$ in M_{2m} be such that $X_1 = c\varphi$, $Y_1 = c\varphi_1$,$X_p = d\varphi$ and $Y_p = d\varphi_1$ (if $p \neq 1$), and $X_i = Y_i = 0$, otherwise (the real numbers c and d will be determined later). Since $X \prec Y$, $T(X) \prec T(Y)$ and, in particular, $[(T(X))_1, (T(X))_q] \prec [(T(Y))_1, (T(Y))_q]$. Hence

$$[cT_{11}\varphi + dT_{p1}\varphi, cT_{1q}\varphi + dT_{pq}\varphi] = R[cT_{11}\varphi_1 + dT_{p1}\varphi_1, cT_{1q}\varphi_1 + dT_{pq}\varphi_1]$$

for some $R \in \mathcal{RS}(2)$, where $d = 0$ if $p = 1$. If $[\lambda, 1-\lambda]$ is the first row of R, then

$$(ca_{11} + da_{p1}) + (cb_{11} + db_{p1}) = \lambda(ca_{11} + da_{p1}) + (1-\lambda)(cb_{11} + db_{p1}),$$

$$(ca_{1q} + da_{pq}) + (cb_{1q} + db_{pq}) = \lambda(ca_{1q} + da_{pq}) + (1-\lambda)(cb_{1q} + db_{pq})$$

(note that λ depends on c, d). Letting $c = 1$ and $d = 0$ yields $a_{11}b_{1q} = a_{1q}b_{11}$ and, hence, $T_{1q} = \ell_{1q}T_{11}$ for some $\ell_{1q} \in \mathbb{R}$. Similarly, if $c = 0$ and $d = 1$, it follows that, $T_{p1} = fT_{pq}$ for some $f \in \mathbb{R}$. Hence, letting $c = 1$ and $d = t$, it follows that:

$$(a_{11} + tfa_{pq}) + (b_{11} + tfb_{pq}) = \lambda(a_{11} + tfa_{pq}) + (1-\lambda)(b_{11} + tfb_{pq}),$$

$$(\ell_{1q}a_{11} + ta_{pq}) + (\ell_{1q}b_{11} + tb_{pq}) = \lambda(\ell_{1q}a_{11} + ta_{pq}) + (1-\lambda)(\ell_{1q}b_{11} + tb_{pq}).$$

Since the sum of two real numbers of the same sign cannot be their convex combinations, it follows that

$$(a_{11} + tfa_{pq})(b_{11} + tfb_{pq}) \leq 0,$$

$$(\ell_{1q}a_{11} + ta_{pq})(\ell_{1q}b_{11} + tb_{pq}) \leq 0,$$

for all $t \in \mathbb{R}$. So, we must have

$$f^2(a_{pq}b_{11} - b_{pq}a_{11})^2 \leq 0,$$

$$\ell_{1q}^2(a_{pq}b_{11} - b_{pq}a_{11})^2 \leq 0.$$

If $f \neq 0$ or $\ell_{1q} \neq 0$, then $a_{pq}b_{11} = b_{pq}a_{11}$ and hence $T_{pq} = \ell_{pq}T_{11}$ for some $\ell_{pq} \in \mathbb{R}$. If $f = \ell_{1q} = 0$, we can let $t = 1$ to obtain $(1-\lambda)a_{11} = -\lambda b_{11}$ and $(1-\lambda)a_{pq} = -\lambda b_{pq}$. Thus $a_{11}b_{pq} = a_{pq}b_{11}$ and, again, $T_{pq} = \ell_{pq}T_{11}$.

Therefore, for $X = [X_1, \ldots, X_m] \in M_{2m}$,

$$T([X_1, \ldots, X_m]) = [\sum_{i=1}^{m} T_{i1}(X_i), \ldots, \sum_{i=1}^{m} T_{im}(X_i)]$$

$$\begin{aligned} &= T_{11}[\sum_{i=1}^{m} \ell_{i1} X_i, \ldots, \sum_{i=1}^{m} \ell_{im} X_i] \\ &= (a_{11} I + b_{11} P) X L, \end{aligned}$$

where $L = [\ell_{ij}]$. Let $a = a_{11}$ and $b = b_{11}$, and observe that $ab \leq 0$. □

We conclude the paper by a generalization of Theorem 5.2 of [3].

Corollary 3.2. *A linear operator $T : M_{nm} \to M_{nm}$ strongly preserves the matrix majorization $\prec$ if and only if there exist a permutation matrix $Q \in \mathcal{P}(n)$ and an invertible matrix L in M_m such that $T(X) = QXL$ for all $X \in M_{nm}$.*

Proof. By Theorem 3.1, there exist $P \in \mathcal{P}(n)$, $L \in M_m$, and $a, b \in \mathbb{R}$ such that $T(X) = (aI + bP)XL$ for all $X \in M_{nm}$, $ab \leq 0$, and $P \neq I$. Moreover, if $n \neq 2$, $ab = 0$. Choose $X \in M_{nm}$ such that $XL = 0$. Then $(aI + bP)XL = 0 \prec 0$ and hence $X \prec 0$. Thus $X = 0$ which implies that L is invertible. Similarly, $(aI + bP)$ is invertible. If $n \neq 2$, then $(aI + bP) = cQ$ for some $Q \in \mathcal{P}(2)$ and some nonzero real c. Replacing L by $c^{-1}L$ yields $T(X) = QXL$ for all $X \in X_{nm}$ and the necessity of the condition is proved in case $n \neq 2$.

Let $n = 2$. Since T is invertible with $T^{-1}(X) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} XL^{-1}$ and, since T^{-1} is a linear preserver of $\prec$, there exist $c, d \in \mathbb{R}$ and $K \in M_m$ such that $cd \leq 0$ and, for all $X \in M_{2m}$,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} XL^{-1} = \begin{bmatrix} c & d \\ d & c \end{bmatrix} XK.$$

Thus

$$X = \begin{bmatrix} ac + bd & ad + bc \\ ad + bc & ac + bd \end{bmatrix} XKL.$$

Now, trying $X = [\varphi_1, 0, \ldots, 0] \in M_{2m}$ yields $ac + bd \neq 0$ and $ad + bc = 0$. These, together with the inequalities $ab \leq 0$ and $cd \leq 0$, imply that either $a = 0$ or $b = 0$. In either case, it follows that $aI + bQ = fP$ for some nonzero $f \in \mathbb{R}$ and some $P \in \mathcal{P}(2)$. Replacing L by $f^{-1}L$, it follows that $T(X) = PXL$ for all $X \in M_{2m}$. □

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