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EXISTENCE AND DECAY OF SOLUTIONS OF A DAMPED KIRCHHOFF EQUATION

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Abstract: This paper is concerned with the study of local and global solutions of the mixed problem for the damped Kirchhoff equation

$$u''(t) + M\big(|A^{1/2}\, u(t)|^2\big)Au(t) + \delta u'(t) = 0, \quad t > 0\,,$$

where A is an unbounded self-adjoint operator, $A \geq 0$, of a real separable Hilbert space with norm $|u|$, M a real function with $M(\lambda) \geq m_0 > 0$ and δ a positive real number. The exponential decay of solutions is obtained when $A \geq \beta I$, β positive real number.

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1. Introduction

Let Ω be an open bounded set of $\mathbb{R}^n$ with boundary Γ and let $M(\lambda)$ be a function with the property $M(\lambda) \geq m_0$, m_0 positive constant.

Consider the following quasilinear mixed problem:

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$$\left| \begin{array}{l} u''(x,t) + M\left(\displaystyle\int_\Omega |\nabla u(y,t)|^2\, dy\right)(-\Delta u(x,t)) = 0, \quad x \in \Omega,\ t > 0\,, \\ u(x,t) = 0, \quad x \in \Gamma,\ \ t \geq 0\,, \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad x \in \Omega\,, \end{array} \right. \tag{P$_1$}$$

where $u' = \frac{\partial u}{\partial t}\cdot$ The equation in (P_1) when $n = 1$, Ω an open interval $(0, L)$ and $M(\lambda) = m_0 + m_1\lambda$, was introduced by Kirchhoff [11] in the study of the small transversal vibrations of an elastic stretched string. See also Lions [14].

When u^0 and u^1 are smooth functions Greenberg and Hu [9], Bernstein [3], Pohozaev [28], Lions [15], Nishihara [26] and Arosio and Spagnolo [2], have obtained global solutions of (P_1). But when $u^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u^1 \in H_0^1(\Omega)$ and $M(\lambda)$ is an arbitrary bounded below smooth function, only local solutions of (P_1) have been obtained. Among these results, we can mention the works of Ebihara et al [8], Medeiros and the second author [19], Arosio and Garavaldi [1], Rivera [30] and Yamada [31]. In the particular case $M(\lambda) = 1\big/(a + b\lambda)^2$ (a, b positive real numbers) Pohozaev [29] obtained a global solution of (P_1). The existence of global solutions of (P_1) when u^0, u^1 lie in the preceding Sobolev spaces and $M(\lambda)$ is arbitrary, is an open problem. To overcome this difficulty it is introduced damping terms in the equation of (P_1). We note that an extensive list of references on the problem (P_1) can be seen in Medeiros et al [21].

We formulate the damping problem in an abstract framework. Let H be a real infinite-dimensional separable Hilbert space and let A be an unbounded self-adjoint operator of H with $A \geq \beta I$, β positive real number. The norm of H is denoted by $|u|$ and is considered δ a positive real number. An abstract formulation of the problem (P_1) with damping is the following:

$$\left| \begin{array}{l} u''(t) + M\big(|A^{1/2}\, u(t)|^2\big) Au(t) + \delta\, A^\alpha\, u'(t) = 0, \quad t > 0\,, \\ u(0) = u^0, \quad u'(0) = u^1\,, \end{array} \right. \tag{P$_2$}$$

where α is a non-negative real number. The physical deduction of the equation of (P_2) with $\alpha = 0$ can be seen in Narashima [25].

Consider A^{-1} compact. Then the existence of global solutions of (P_2) were studied, among others, by Medeiros and the second author [19] for the case $0 < \alpha \leq 1$; by Nishihara [26], Hosoya and Yamada [10], Koumou-Patcheu [13] and Medeiros et al [22], for the case $\alpha = 0$ and u^0, u^1 belong to a ball whose radius depends on δ.

When A^{-1} is not necessarily compact, the existence of global solutions of (P_2) was analyzed by Matos and Pereira [18] for the case $\alpha = 1$ and by Clark [4] for the case $0 < \alpha \leq 1$. Local solutions of (P_2) ($\delta = 0$) have been obtained by Dickey [7], Crippa [5] and by Matos [17] when $A \geq 0$. In the last paper the author applies the Dixmier-Von Neumann's Diagonalization Theorem (cf. Dixmier [6] and Lions and Magenes [16]) in order to set the problem (P_2) in the context of hilbertian integral and in this formulation to solve it. After that he returns to the original problem.

In the present paper we study the existence of local and global solutions of the problem (P_2) with $A \geq 0$ and $(A+I)^{-1}$ not necessarily compact. The exponential decay of solutions is obtained when $A \geq \beta I$, β positive real number. In our study of existence of solutions we use directly the spectral properties of A instead of writing the problem in the setting of the hilbertian integral as in [17]. The study is completed using the Galerkin procedure and the Arzela-Ascoli Theorem to pass to the limit in the nonlinear part. In the decay of solutions, we apply a method introduced by Komornik and Zuazua [12] (cf. also Zuazua [32] and [33]).

2. Notations and Main Results

By H is represented a real infinite-dimensional separable Hilbert space whose scalar product and norm are denoted, respectively, by (u,v) and $|u|$. By A an unbounded self-adjoint operator of H with $A \geq 0$, i.e., $(Au,u) \geq 0$ for all $u \in D(A)$, $D(A)$ domain of A is represented.

Let $(E_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral family of A. Consider a non-negative real number α. We have that the self-adjoint operator A^α has by domain

$$D(A^\alpha) = \left\{ u \in H; \int_0^\infty \lambda^{2\alpha} \, d(E_\lambda u, u) < \infty \right\}$$

and verifies

$$|A^\alpha u|^2 = \int_0^\infty \lambda^{2\alpha} \, d(E_\lambda u, u), \quad \forall \, u \in D(A^\alpha).$$

The space $D(A^\alpha)$ with the scalar product

$$(u,v)_{D(A^\alpha)} = (u,v) + (A^\alpha u, A^\alpha v)$$

is a Hilbert space. Also, for τ positive real number,

$$D\big((A + \tau\, I)^\alpha\big) = \left\{ u \in H; \int_0^\infty (\lambda + \tau)^{2\alpha} \, d(E_\lambda u, u) < \infty \right\}$$

and

$$|(A + \tau\, I)^\alpha \, u|^2 = \int_0^\infty (\lambda + \tau)^{2\alpha} \, d(E_\lambda u, u), \quad \forall \, u \in D\big((A + \tau\, I)^\alpha\big).$$

The space $D\big((A+\tau I)^\alpha\big)$ with the scalar product

$$(u,v)_{D\left((A+\tau I)^\alpha\right)} = \big((A+\tau I)^\alpha u, (A+\tau I)^\alpha v\big)$$

is a Hilbert space

We have, for $\alpha \geq 1/2$, $D(A^\alpha) = D\big((A+\tau I)^\alpha\big)$ and

$$\left(\frac{1}{2}\right)^{2\alpha-1} ||u||^2_{D\left((A+\tau I)^\alpha\right)} \leq |A^\alpha u|^2 + \tau^{2\alpha}|u|^2 \leq 2||u||^2_{D\left((A+\tau I)^\alpha\right)}, \quad \forall\, u \in D(A^\alpha). \tag{2.1}$$

Let $A \geq \beta I$ be with β positive constant, that is, $(Au,u) \geq \beta(u,u)$, for all $u \in D(A)$. If $\alpha \geq \gamma \geq 0$ then $D(A^\alpha) \hookrightarrow D(A^\gamma)$ and

$$|A^\gamma u| \leq \beta^{\gamma-\alpha}\,|A^\alpha u|, \quad \forall\, u \in D(A^\alpha). \tag{2.2}$$

The spectral properties of A can be seen in [23].

We denote by δ a positive real number and by $M(\xi)$ a C^1-function defined on $[0,\infty[$ satisfying

$$M(\xi) \geq m_0 > 0 \qquad (m_0 \text{ constant})\,.$$

With the above considerations we has the following results.

Theorem 2.1. (Local Solution) *Consider* $u^0 \in D(A)$, $u^1 \in D(A^{1/2})$ *and the positive real number*

$$T_0 = (\ell n\, 2) m_0^{3/2}/2RN^2\,,$$

where

$$|(A+I)^{1/2}\,u^1|^2 + k(u^0)|(A+I)u^0|^2 + |u^1|^2 + k(u^0)|(A+I)^{1/2}\,u^0|^2 < N^2/2\,, \tag{2.3}$$

$$k(u^0) = \max_{0\leq\xi\leq|(A+I)^{1/2}u^0|^2} M(\xi), \quad R = \max_{0\leq\xi\leq N^2/m_0} |M'(\xi)|. \tag{2.4}$$

Then there exists a unique function u *in the class*

$$u \in L^\infty(0,T_0;D(A)), \quad u' \in L^\infty(0,T_0;D(A^{1/2}))\,, \tag{2.5}$$

such that u *satisfies*

$$\left|\begin{array}{l} u'' + M\big(|A^{1/2}u|^2\big)Au + \delta u' = 0 \text{ in } L^\infty(0,T_0;H)\,, \\ u(0) = u^0,\ u'(0) = u^1\,. \end{array}\right. \tag{LP}$$

To state the result on global solutions, we introduce some supplementary notations. In fact,

$$\widehat{M}(\xi) = \int_0^\xi M(\xi)\, d\xi, \quad L^2 = |u^1|^2 + \widehat{M}\big(|(A+I)^{1/2}\, u^0|^2\big)\,, \tag{2.6}$$

$$S = \max_{0\le\xi\le L^2/m_0} |M'(\xi)|, \quad a^2 = \frac{1}{m_0}\, |(A+I)^{1/2}\, u^1|^2 + |(A+I)u^0|^2\,. \tag{2.7}$$

Theorem 2.2. (Global Solution) *Let $u^0 \in D(A)$ and $u^1 \in D(A^{1/2})$ be with*

$$\big(a\, LS/m_0\big) < \delta\,. \tag{H1}$$

Then there exists a unique function u in the class

$$u \in L^\infty(0,\infty; D(A)), \quad u' \in L^\infty(0,\infty; D(A^{1/2})\,, \tag{2.8}$$

such that u is the solution of the problem

$$\left|\begin{array}{l} u'' + M\big(|A^{1/2}u|^2\big)Au + \delta u' = 0 \text{ in } L^\infty(0,\infty;H) \\ u(0) = u^0,\ u'(0) = u^1\,. \end{array}\right. \tag{GP}$$

To study the decay of solutions we assume the following hypothesis on A:

$$A \ge \beta\, I \quad (\beta \text{ positive constant}) \tag{H2}$$

i.e., $(Au,u) \ge \beta(u,u)$ for all $u \in D(A)$.

We consider similar notations to (2.6), (2.7), more precisely,

$$\begin{gathered}(L^*)^2 = |u^1|^2 + \widehat{M}\big(|A^{1/2}u^0|^2\big),\ S^* = \max_{0\le\xi\le(L^*)^2/m_0} |M'(\xi)|, \\ (a^*)^2 = \frac{1}{m_0}\, |A^{1/2}u_1|^2 + |Au^0|^2.\end{gathered} \tag{2.9}$$

With these notations, we take $u^0 \in D(A)$ and $u^1 \in D(A^{1/2}$ satisfying

$$\frac{a^* S^* L^*}{m_0} < N^* = \min\left\{\delta, \frac{\beta\, m_0}{2\big[2\beta^{1/2} m_0^{1/2} + \delta\big]}\right\}. \tag{H3}$$

By applying similar arguments to the ones used in the proof of Theorem 2.2, we obtain a solution u of the problem (GP) with initial data u^0 and u^1 satisfying hypothesis (H3).

The energy associated to the problem (GP) is the following:

$$E(t) = |A^{1/2}\, u'(t)|^2 + M\big(|A^{1/2}\, u(t)|^2\big)|Au(t)|^2, \quad t \geq 0.$$

Theorem 2.3. (Decay of Solutions) *Under hypotheses (H2) and (H3), the solution u of the problem (GL) verifies*

$$E(t) \leq 3\, E(0) \exp\left[-\frac{1}{3}\, \sigma\, t\right], \quad \forall\, t \geq 0\,,$$

where

$$\sigma = N^* - \big(a^* S^* L^* / m_0\big).$$

3. Proof of the Results

We introduce the operators $A_\ell = A + \dfrac{1}{\ell}\, I$, ℓ positive integer number.

In order to obtain a solution u of the problem (LP), we need to study a linear problem. For that, we consider a real function μ satisfying:

$$\mu \in W^{1,\infty}_{\text{loc}}(0,\infty),\ \ \mu(t) \geq m_0 > 0,\ \ \forall\, t \geq 0\ \ (m_0 \text{ constant}). \tag{H4}$$

Proposition 3.1. *Assume hypothesis (H4) and consider $u^0_\ell \in D(A^{3/2}_\ell)$, $u^1_\ell \in D(A_\ell)$. Then there exists a function u_ℓ in the class*

$$u_\ell \in L^\infty_{\text{loc}}\big(0,\infty; D(A^{3/2}_\ell)\big),\ \ u'_\ell \in L^\infty_{\text{loc}}\big(0,\infty; D(A_\ell)\big) \tag{3.1}$$

satisfying

$$\left|\begin{array}{l} u''_\ell + \mu\, A_\ell\, u_\ell + \delta u'_\ell = 0 \ \textit{in}\ L^\infty_{\text{loc}}\big(0,\infty; D(A^{1/2}_\ell)\big)\,, \\ u_\ell(0) = u^0_\ell\,,\ \ u'_\ell(0) = u^1_\ell\,. \end{array}\right. \tag{L}$$

The uniqueness of solutions is obtained in the bigger class

$$\begin{array}{l} u_\ell \in L^\infty_{\text{loc}}\big(0,\infty; D(A_\ell)\big), \quad u'_\ell \in L^\infty_{\text{loc}}\big(0,\infty; D(A^{1/2}_\ell)\big)\,, \\ u''_\ell \in L^\infty_{\text{loc}}(0,\infty; H). \end{array} \tag{3.2}$$

Proposition 3.1 follows by applying the Galerkin approximations and the energy method.

We give a sketch of the proof of Theorem 2.1. First, we approximate u^0 and u^1 by smooth functions u^0_ℓ and u^1_ℓ. Then by Proposition 3.1 and the method of successive approximations, we determine the solution u_ℓ of the problem:

$$\left|\begin{array}{l} u''_\ell + M\big(|A^{1/2}_\ell\, u_\ell|^2\big) A_\ell u_\ell + \delta u'_\ell = 0 \text{ in } L^\infty\big(0, T_0; D(A^{1/2}_\ell)\big)\,, \\ u_\ell(0) = u^0_\ell\,,\ \ u'_\ell(0) = u'_\ell\,. \end{array}\right. \tag{P$_\ell$}$$

Estimates obtained for the solutions u_ℓ, allow us to pass to the limit in the equation of(P$_\ell$). The limit of the nonlinear term follows by applying the Arzela-Ascoli Theorem for real functions and by results of spectral theory of self-adjoint operators.

3.1. Proof of Theorem 2.1

We fix a real number $\eta > 0$ such that

$$\begin{aligned}\big[|(A+I)^{1/2}\,u^1|^2+\eta\big]+k_\eta(u^0)\big[|(A+I)u^0|^2+\eta\big]+\big[|u^1|^2+\eta\big]\\ +k_\eta(u^0)\big[|(A+I)^{1/2}\,u^0|^2+\eta\big]<N^2/2\end{aligned} \tag{3.3}$$

where

$$k_\eta(u^0)=\max_{0\le\xi\le|(A+I)^{1/2}u^0|^2+\eta} M(\xi).$$

This is possible by (2.3) and (2.4). Let (u^0_ℓ) and (u^1_ℓ) be sequences of vectors of $D\big((A+I)^{3/2}\big)$ and $D(A+I)$, respectively, verifying

$$u^0_\ell\to u^0 \text{ in } D(A+I) \text{ and } u^1_\ell\to u^1 \text{ in } D\big((A+I)^{1/2}\big). \tag{3.4}$$

We divide the proof of Theorem 2.1 in two parts.

3.1.1. First Part: Existence of Solutions of (P$_\ell$)

In this part we shall show that there exists a unique function u in the class

$$u_\ell\in L^\infty\big(0,T_0;D(A_\ell^{3/2})\big),\quad u'_\ell\in L^\infty\big(0,T_0;D(A_\ell)\big)\,, \tag{3.5}$$

such that u is the solution of the problem (P$_\ell$).

Inequality (3.3), convergence (3.4) and spectral properties of A enables us to determine $\ell_0(\eta)$ such that for $\ell\ge\ell_0(\eta)$, we have

$$|A_\ell^{1/2}\,u^1_\ell|^2+k_\ell(u^0_\ell)|A_\ell u^0_\ell|^2+|u^1_\ell|^2+k_\ell(u^0_\ell)|A_\ell^{1/2}\,u^0_\ell|^2<N^2/2\,, \tag{3.6}$$

where

$$k_\ell(u^0_\ell)=\max_{0\le\xi\le|A_\ell^{1/2}u^0_\ell|^2} M(\xi). \tag{3.7}$$

Fix an integer number $\ell\ge\ell_0(\eta)$. Consider the set $\mathcal{M}$ of the vectorial functions $v\colon\ ]0,T_0[\to H$ satisfying

$$v\in L^\infty(0,T_0;D(A_\ell)),\ v'\in L^\infty(0,T_0;D(A_\ell^{1/2})),\ v''\in L^\infty(0,T_0;H)\,,$$

$$v(0) = u_\ell^0\,,\ v'(0) = u_\ell^1\,,$$

and

$$\sup_{0\le\xi\le T_0}\Big[|A_\ell^{1/2}v'(t)|^2 + m_0|A_\ell v_\ell(t)|^2 + 2\delta\int_0^t |A_\ell^{1/2}v'(s)|^2\,ds \qquad (3.8)$$
$$+|v'(t)|^2 + m_0|A_\ell^{1/2}v(t)|^2| + 2\delta\int_0^t |v'(s)|^2\,ds\Big] \le N^2.$$

We define by induction a sequence $(u_{\ell,\nu})$ of solutions of the problems:

$$\left|\begin{array}{l} u''_{\ell,\nu} + M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}|^2\big)A_\ell\,u_{\ell,\nu} + \delta u'_{\ell,\nu} = 0 \text{ in } L^\infty(0,T_0;H)\,,\\ u_{\ell,\nu}(0) = u_\ell^0\,,\ u'_{\ell,\nu}(0) = u_\ell^1\,. \end{array}\right. \qquad (\mathrm{P}_{\ell,\nu})$$

We begin with $u_{\ell,1}$ solution of the problem

$$\left|\begin{array}{l} u''_{\ell,1} + M\big(|A_\ell^{1/2}\,u_\ell^0|^2\big)A_\ell\,u_{\ell,1} + \delta u'_{\ell,1} = 0 \text{ in } L^\infty(0,T_0;H)\,,\\ u_{\ell,1}(0) = u_\ell^0\,,\ u'_{\ell,1}(0) = u_\ell^1\,. \end{array}\right. \qquad (\mathrm{P}_{\ell,1})$$

Estimates obtained when, first, we take the scalar product of H in both sides of equation $(\mathrm{P}_{\ell,1})_1$ with $2u'_{\ell,1}$ and then with $2A_\ell\,u'_{\ell,1}$ allow us to show that $u_{\ell,1}\in\mathcal{M}$.

We suppose that the solution $u_{\ell,\nu-1}$ of the problem $(\mathrm{P}_{\ell,\nu-1})_1$, $\nu\ge 2$, belongs to $\mathcal{M}$. We shall show that the solution $u_{\ell,\nu}$ of $(\mathrm{P}_{\ell,\nu})$ belongs also to $\mathcal{M}$. In fact, taking the scalar product of H in both sides of equation $(\mathrm{P}_{\ell,\nu})_1$ with $2u'_{\ell,\nu}$ and integrating from 0 to t, we obtain:

$$|u'_{\ell,\nu}(t)|^2 + M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}(t)|^2\big)|A_\ell^{1/2}\,u_{\ell,\nu}(t)|^2 + 2\delta\int_0^t |u'_{\ell,\nu}(s)|^2\,ds$$
$$= \int_0^t 2M'\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}(s)|^2\big)\big(A_\ell^{1/2}\,u_{\ell,\nu-1}(s), A_\ell^{1/2}\,u'_{\ell,\nu-1}(s)\big)|A_\ell^{1/2}\,u_{\ell,\nu}(s)|^2\,ds$$
$$+ |u_\ell^1|^2 + M\big(|A_\ell^{1/2}\,u_\ell^0|^2\big)|A_\ell^{1/2}\,u_\ell^0|^2. \qquad (3.9)$$

Since $u_{\ell,\nu-1}\in\mathcal{M}$, we have:

$$|A_\ell^{1/2}\,u_{\ell,\nu-1}(s)|^2 \le N^2/m_0\,, |M'\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}(s)|^2\big)| \le R\,,$$
$$2|(A_\ell^{1/2}\,u_{\ell,\nu-1}(s), A_\ell^{1/2}\,u'_{\ell,\nu-1}(s)| \le 2N^2/m_0^{1/2}.$$

Taking into account the last inequalities in (3.9), applying the Gronwall inequality and noting that $M(\xi)\ge m_0$, we get:

$$|u'_{\ell,\nu}(t)|^2 + m_0|A_\ell^{1/2}\,u_{\ell,\nu}(t)|^2 + 2\delta\int_0^t |u'_{\ell,\nu}(s)|^2\,ds$$

$$\leq \big(|u_\ell^1|^2 + k_\ell(u_\ell^0)|A_\ell^{1/2}\, u_\ell^0|^2\big) \exp\big(2RN^2T_0/m_0^{3/2}\big), \quad \forall\, t \in [0,T_0]\,, \tag{3.10}$$

where $k_\ell(u_\ell^0)$ were defined in (3.7).

We note that $\mu(t) = M\big(|A_\ell^{1/2}\, u_{\ell,\nu-1}(t)|^2\big)$ belongs to $W^{1,\infty}(0,T_0)$. As $u_\ell^0 \in D(A_\ell^{1/2})$, $u_\ell^1 \in D(A_\ell)$ and $u_{\ell,\nu}$ is the solution of the problem $(\mathrm{P}_{\ell,\nu})$, we obtain by Proposition 3.1 that

$$u_{\ell,\nu} \in L^\infty\big(0,T_0; D(A_\ell^{3/2})\big) \text{ and } u_{\ell,\nu}' \in L^\infty(0,T_0; D(A_\ell)).$$

This enables us to take the scalar product of H in both sides of equation $(\mathrm{P}_{\ell,\nu})_1$ with $2A_\ell\, u_{\ell,\nu}'$. Then integrating from 0 to t, we find:

$$\begin{aligned}&|A_\ell^{1/2}\, u_{\ell,\nu}'(t)|^2 + M\big(|A_\ell^{1/2}\, u_{\ell,\nu-1}(t)|^2\big)|A_\ell\, u_{\ell,\nu}(t)|^2 + 2\delta\int_0^t |A_\ell^{1/2}\, u_{\ell,\nu}'(s)|^2\, ds\\ &= \int_0^t 2M'\big(|A_\ell^{1/2}\, u_{\ell,\nu-1}(s)|^2\big)\big(A_\ell^{1/2}\, u_{\ell,\nu-1}(s), A_\ell^{1/2}\, u_{\ell,\nu-1}'(s)|A_\ell\, u_{\ell,\nu}(s)|^2\, ds\\ &\quad + |A_\ell^{1/2}\, u_\ell^1|^2 + M\big(|A_\ell^{1/2}\, u_\ell^0|^2\big)|A_\ell\, u_\ell^0|^2.\end{aligned}$$

By arguments similar to those used to obtain (3.10), we deduce from this equality that

$$\begin{aligned}&|A_\ell^{1/2}\, u_{\ell,\nu}'(t)|^2 + m_0|A_\ell\, u_{\ell,\nu}(t)|^2 + 2\delta\int_0^t |A_\ell^{1/2}\, u_{\ell,\nu}'(s)|^2\, ds\\ &\leq \big(|A_\ell^{1/2}\, u_\ell^1|^2 + k_\ell(u_\ell^0)|A_\ell\, u_\ell^0|^2\big)\exp\big(2RN^2T_0/m_0^{3/2}\big),\ \forall\, t\in[0,T_0].\end{aligned} \tag{3.11}$$

Adding both sides of inequalities (3.10) and (3.11), taking into account inequality (3.6) and the value of T_0, we obtain:

$$\begin{aligned}&|A_\ell^{1/2}\, u_{\ell,\nu}'(t)|^2 + m_0|A_\ell\, u_{\ell,\nu}(t)|^2 + 2\delta\int_0^t |A_\ell^{1/2}\, u_{\ell,\nu}'(s)|^2\, ds\\ &+|u_{\ell,\nu}'(t)|^2 + m_0|A_\ell^{1/2}\, u_{\ell,\nu}(t)|^2 + 2\delta\int_0^t |u_{\ell,\nu}'(s)|^2\, ds \leq N^2,\\ &\qquad\qquad \forall\, t\in[0,T_0],\ \nu\geq 2,\ \ell\geq \ell_0(\eta).\end{aligned} \tag{3.12}$$

So $u_{\ell,\nu} \in \mathcal{M}$.

Estimate (3.12) and equations $(\mathrm{P}_{\ell,\nu})_1$ imply that there exists a subsequence of $(u_{\ell,\nu})$, still denoted by $(u_{\ell,\nu})$, such that

$$u_{\ell,\nu} \longrightarrow u_\ell \quad \text{weak star in} \quad L^\infty(0,T_0; D(A_\ell))$$

$$u'_{\ell,\nu} \longrightarrow u'_\ell \quad \text{weak star in} \quad L^\infty(0,T_0;D(A_\ell^{1/2}))$$
$$u'_{\ell,\nu} \longrightarrow u''_\ell \quad \text{weak star in} \quad L^\infty(0,T_0;H)\,. \tag{3.13}$$

In the sequel we shall prove that

$$M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}|^2\big) \to M\big(|A_\ell^{1/2}\,u_\ell|^2\big) \text{ in } C^0([0,T_0]). \tag{3.14}$$

In fact, we consider the functions $\varphi_{\ell,\nu}(t) = |A_\ell^{1/2}\,u_{\ell,\nu-1}(t)|^2$. As $u_{\ell,\nu-1} \in \mathcal{M}$, we have $\varphi_{\ell,\nu} \in C^0([0,T_0])$ and $(\varphi_{\ell,\nu})$ is uniformly bounded in $C^0([0,T_0])$. Take $t_1, t_2 \in [0,T_0]$, $t_1 < t_2$. Then

$$|\varphi_{\ell,\nu}(t_2) - \varphi_{\ell,\nu}(t_1)|$$
$$\le \big[|A_\ell^{1/2}\,u_{\ell,\nu-1}(t_2)| + |A_\ell^{1/2}\,u_{\ell,\nu-1}(t_1)|\big]\,|A_\ell^{1/2}\,u_{\ell,\nu-1}(t_2) - A_\ell^{1/2}\,u_{\ell,\nu-1}(t_1)|.$$

We have

$$A_\ell^{1/2}\,u_{\ell,\nu-1}(t_2) - A_\ell^{1/2}\,u_{\ell,\nu-1}(t_1) = \int_{t_1}^{t_2} A_\ell^{1/2}\,u'_{\ell,\nu-1}(s)\,ds.$$

From the last two expressions and noting that $u_{\ell,\nu-1} \in \mathcal{M}$, we find:

$$|\varphi_{\ell,\nu}(t_2) - \varphi_{\ell,\nu}(t_1)| \le \frac{2N^2}{m_0^{1/2}}\,|t_2 - t_1|, \quad \forall\,\nu \ge 2.$$

So, the sequence $(\varphi_{\ell,\nu})$ satisfies the hypotheses of Arzela-Ascoli Theorem. Then there exists a subsequence of $(\varphi_{\ell,\nu})$, which is also denoted by $(\varphi_{\ell,\nu})$, and a function $\varphi_\ell \in C^0([0,T_0])$ such that

$$\varphi_{\ell,\nu} \to \varphi_\ell \quad \text{in} \quad C^0([0,T_0]).$$

This implies

$$M(\varphi_{\ell,\nu}) \to M(\varphi_\ell) \quad \text{in} \quad C^0([0,T_0]). \tag{3.15}$$

The next step is to show that $(\varphi_{\ell,\nu})$ is a Cauchy sequence in $C^0\big([0,T_0];\,D(A_\ell^{1/2})\big)$. This and the last convergence will imply convergence (3.14). Let $u_{\ell,\nu}$ and $u_{\ell,\sigma}$ be, respectively, the solutions of the problem $(\mathrm{P}_{\ell,\nu})$ and $(\mathrm{P}_{\ell,\sigma})$. Consider $w_{\nu\sigma} = u_{\ell,\nu} - u_{\ell,\sigma}$. Then $w_{\nu\sigma}$ is solution of the problem

$$\left|\begin{array}{l} w''_{\nu\sigma} + M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}|^2\big)A_\ell\,w_{\nu\sigma} + \delta w'_{\nu\sigma} \\ \quad = \big[M\big(|A_\ell^{1/2}\,u_{\ell,\sigma-1}|^2\big) - M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}|^2\big)\big]A_\ell\,u_{\ell,\sigma}\,, \\ w_{\nu\sigma}(0) = 0,\ w'_{\nu\sigma}(0) = 0\,. \end{array}\right. \tag{$\mathrm{P}_{\ell,\nu,\sigma}$}$$

Taking the scalar product of H in both sides of equation $(\mathrm{P}_{\ell,\nu\sigma})_1$ with $2w'_{\nu\sigma}$, we get:

$$\begin{aligned}
&\frac{d}{dt}\big[|w'_{\nu\sigma}(t)|^2 + M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}(t)|^2\big)|A_\ell^{1/2}\,w_{\nu\sigma}(t)|]^2 + 2\delta|w'_{\nu\sigma}(t)|^2 \\
&\quad = 2M'\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}(t)|^2\big)\big(A_\ell^{1/2}\,u_{\ell,\nu-1}(t), A_\ell^{1/2}\,u'_{\ell,\nu-1}(t)\big)|A_\ell^{1/2}\,w_{\nu\sigma}(t)|^2 \\
&\quad + \big[M\big(|A_\ell^{1/2}\,u_{\ell,\sigma-1}(t)|^2\big) - M\big(|A_\ell^{1/2}\,u_{\ell,\nu-1}(t)|^2\big)\big](A_\ell\,u_{\ell,\sigma}, 2w'_{\nu\sigma}(t)). \quad (3.16)
\end{aligned}$$

Integrating from 0 to t and applying arguments similar to those used to obtain (3.10), we deduce:

$$\begin{aligned}
|w'_{\nu\sigma}(t)|^2 + m_0|A_\ell^{1/2}\,w_{\nu\sigma}(t)|^2 + 2\delta\int_0^t |w'_{\nu\sigma}(s)|^2\,ds \\
\le \frac{2RN^2}{m_0^{3/2}}\int_0^t m_0|A_\ell^{1/2}\,m_{\nu\sigma}(s)|^2\,ds \\
+ \frac{4N^2}{m_0^{1/2}}\int_0^{T_0} |M(\varphi_{\ell,\sigma}(t) - M(\varphi_{\ell,\nu}(t))|dt. \quad (3.17)
\end{aligned}$$

This inequality implies, by the Gronwall inequality and noting that $\big(M(\varphi_{\ell,\nu})\big)$ is a Cauchy sequence in $C^0([0,T_0])$ (cf. (3.15)), that $(u_{\ell,\nu})$ is a Cauchy sequence in $C^0([0,T_0])$. Thus convergence (3.14) is showed.

Convergences (3.13) and (3.14) allow us to pass to the limit in $(\mathrm{P}_{\ell,\nu})$ and to show that u_ℓ is a solution of the problem (P_ℓ).

By noting that $\mu(t) = M\big(|A_\ell^{1/2}\,u_\ell(t)|^2\big)$ belongs to $W^{1,\infty}(0,T_0)$, we deduce by the uniqueness of solutions of Proposition 3.1 that u_ℓ belongs to class (3.5).

To finish the first part, we shall prove the uniqueness of solutions of the problem (P_ℓ). In fact, consider $w = u_\ell - z_\ell$. Then by following the lines of the proof of (3.16) and (3.17), we find

$$\begin{aligned}
|w'(t)|^2 + m_0|A_\ell^{1/2}\,w(t)|^2 + 2\delta\int_0^t |w'(s)|^2\,ds \le \frac{2RN^2}{m_0^{3/2}}\int_0^t m_0|A_\ell^{1/2}\,w(s)|^2\,ds \\
+ \int_0^t |M\big(|A_\ell^{1/2}\,z_\ell(s)|^2\big) - M\big(|A_\ell^{1/2}\,u_\ell(s)|^2\big)|\,|(A_\ell z_\ell(s), w'(s)|\,ds.
\end{aligned}$$

Observe that

$$|M\big(|A_\ell^{1/2}\,z_\ell(s)|^2\big) - M\big(|A_\ell^{1/2}\,u_\ell(s)|^2\big)| \le \frac{2RN}{m_0^{1/2}}\,|A_\ell^{1/2}\,w(s)|,$$

$$|(A_\ell z_\ell(s), w'(s))| \leq \frac{N}{m_0^{1/2}}\, |w'(s)|.$$

Combining the last three inequalities and applying the Gronwall inequality, we obtain $w \equiv 0$. So the uniqueness of solutions of (P_ℓ) is proved.

3.1.2. Second Part: Existence of Solutions of (LP)

Inequality (3.12) and convergence (3.13) imply

$$\begin{aligned} &|A_\ell^{1/2}\, u_\ell'(t)|^2 + m_0|A_\ell u_\ell(t)|^2 + 2\delta \int_0^t |A_\ell^{1/2}\, u_\ell'(s)|^2\, ds \\ &+|u_\ell'(t)|^2 + m_0|A_\ell^{1/2}\, u_\ell(t)|^2 + 2\delta \int_0^t |u_\ell'(s)|^2\, ds \leq N^2, \\ &\qquad \forall\, t \in [0, T_0] \text{ and } \ell \geq \ell_0(\eta). \end{aligned} \tag{3.18}$$

Noting that

$$u_\ell(t) = \int_0^t u_\ell'(s)\, ds + u_\ell^0\,, \quad |Au_\ell| \leq |A_\ell u_\ell|, \quad |A^{1/2}\, u_\ell| \leq |A_\ell^{1/2}\, u_\ell|\,,$$

we obtain by (3.18), convergence (3.4) and equation $\mathrm{P}(_\ell)_1$,that

$$(u_\ell), (Au_\ell), (A^{1/2}\, u_\ell), (u_\ell') \text{ and } (u_\ell'') \text{ are bounded in } L^\infty(0, T_0; H). \tag{3.19}$$

This implies that there exists a subsequence of (u_ℓ), still denoted by (u_ℓ), such that

$$u_\ell \to u \text{ and } Au_\ell \to \chi_1 \text{ weak star in } L^\infty(0, T_0; H). \tag{3.20}$$

As A is a closed operator, these convergences imply that $u \in D(A)$ and $Au = \chi_1$, i.e.,

$$Au_\ell \to Au \text{ weak star in } L^\infty(0, T_0; H). \tag{3.21}$$

Analogously

$$A^{1/2}\, u_\ell \to A^{1/2}\, u \text{ weak star in } L^\infty(0, T_0; H). \tag{3.22}$$

Also, by (3.19),

$$u_\ell' \to u' \text{ and } u_\ell'' \to u'' \text{ weak star in } L^\infty(0, T_0; H). \tag{3.23}$$

We obtain, for $v \in H$,

$$|A_\ell^{1/2}\,v - A^{1/2}\,v|^2 = \int_0^\infty \big[(\lambda + \frac{1}{\ell})^{1/2} - \lambda^{1/2}\big]^2\, d(E_\lambda v, v) \\ \leq \frac{1}{\ell}\int_0^\infty d(E_\lambda v, v) = \frac{1}{\ell}\,|v|^2.$$

This equality and boundedness of (u_ℓ) in $C^0([0,T_0;H)$ (cf. (3.19)), give

$$A_\ell^{1/2}\,u_\ell - A^{1/2}\,u_\ell \to 0 \text{ in } C^0([0,T_0];H). \tag{3.24}$$

Analogously,

$$A_\ell u_\ell - A u_\ell \to 0 \text{ in } L^\infty(0,T_0;H). \tag{3.25}$$

Noting that $A_\ell^{1/2}\,u_\ell = A_\ell^{1/2}\,u_\ell - A^{1/2}\,u_\ell + A^{1/2}\,u_\ell$, we obtain by convergence (3.24) and (3.22),

$$A_\ell^{1/2}\,u_\ell \to A^{1/2}\,u \text{ weak star in } L^\infty(0,T_0;H). \tag{3.26}$$

Analogously, by (3.25) and (3.21),

$$A_\ell u_\ell \to Au \quad \text{weak star in} \quad L^\infty(0,T_0;H). \tag{3.27}$$

Next we prove the convergence

$$|A_\ell^{1/2}\,u_\ell|^2 \to |A^{1/2}\,u|^2 \quad \text{in} \quad C^0([0,T_0]). \tag{3.28}$$

For that, we apply the same ideas used to obtain convergence (3.14).

In fact, we consider $\varphi_\ell(t) = |A_\ell^{1/2}\,u_\ell(t)|^2$. Then as in (3.15) and by similar arguments, we deduce by (3.18) that there exists a subsequence of (φ_ℓ), still denoted by (φ_ℓ), and a function φ of $C^0([0,T_0])$ such that

$$\varphi_\ell \to \varphi \quad \text{in} \quad C^0([0,T_0]).$$

Therefore,

$$M(\varphi_\ell) \to M(\varphi) \quad \text{in} \quad C^0([0,T_0]). \tag{3.29}$$

Let u_ℓ and u_k be, respectively, solutions of the problems

$$\left|\begin{array}{l} u_\ell'' + M\big(|A_\ell^{1/2}\,u_\ell|^2\big)A_\ell u_\ell + \delta u_\ell' = 0\,, \\ u_\ell(0) = u_\ell^0\,,\ u_\ell'(0) = u_\ell^1\,, \end{array}\right.$$

$$\left|\begin{array}{l} u_k'' + M\big(|A_k^{1/2}\,u_k|^2\big)A_k u_k + \delta u_k' = 0\,, \\ u_k(0) = u_k^0\,,\ u_k'(0) = u_k^1\,. \end{array}\right.$$

Consider $w_{\ell k} = u_\ell - u_k$. Then from these problems it follows that

$$w''_{\ell k} + M\big(|A_\ell^{1/2}\, u_\ell|^2\big) A_\ell w_{\ell k} + \delta w'_{\ell k} = M\big(|A_k^{1/2}\, u_k|^2\big) A_k u_k - M\big(|A_\ell^{1/2}\, u_\ell|^2\big) A_\ell u_k\,.$$

Taking the scalar product of H in both sides of this equality with $2w'_{\ell k}$, we obtain:

$$\begin{aligned}
&\tfrac{d}{dt}\Big[|w'_{\ell k}|^2 + M\big(|A_\ell^{1/2}\, u_\ell|^2\big)|A_\ell^{1/2}\, w_{\ell k}|^2\Big] + 2\delta |w'_{\ell k}|^2 \\
&= 2M'\big(A_\ell^{1/2}\, u_\ell|^2\big((A_\ell u_\ell, u'_\ell)|A_\ell^{1/2}\, w_{\ell k}|^2 \\
&+M\big(|A_k^{1/2}\, u_k|^2\big((A_k u_k, 2w'_{\ell k}) - M\big(|A_\ell^{1/2}\, u_\ell|^2\big)(A_\ell u_k, 2w'_{\ell k}) \\
&= X_1 + X_2 - X_3\,.
\end{aligned} \tag{3.30}$$

We have:

$$\begin{aligned}
&M\big(|A_k^{1/2}\, u_k|^2\big)(A_k u_k, 2w'_{\ell k}) - M\big(|A_\ell^{1/2}\, u_\ell|^2\big)(A_\ell u_k, 2w'_{\ell k}) \\
&= M\big(|A_k^{1/2}\, u_k|^2\big)(A_k u_k, 2w'_{\ell k}) - M\big(|A_\ell^{1/2}\, u_\ell|^2\big)(A_k u_k, 2w'_{\ell k}) \\
&+M\big(|A_\ell^{1/2}\, u_\ell|^2\big)(A_k u_k, 2w'_{\ell k}) - M\big(|A_\ell^{1/2}\, u_\ell|^2\big)(A_\ell u_k, 2w'_{\ell k})
\end{aligned}$$

that is

$$X_2 - X_3 = (X_2 - Z) + (Z - X_3).$$

From the boundedness (3.19), it follows that

$$\begin{aligned}
&|X_1| \le C_1\, m_0 |A_\ell^{1/2}\, w_{\ell k}|^2\,, \\
&|X_2 - Z| \le C_2 |M\big(|A_k^{1/2}\, u_k|^2\big) - M\big(|A_\ell^{1/2}\, u_\ell|^2\big)|\,, \\
&|Z - X_3| \le C_3 |(A_k - A_\ell) u_k|\,,
\end{aligned}$$

where C_1, C_2 and C_3 are positive constants independents of ℓ and $t \in [0, T_0]$. Also, by boundedness of (u_k) in $C^0([0,T_0];H)$ (cf. (3.19)),

$$|(A_k - A_\ell)u_k|^2 = \left(\frac{1}{k} - \frac{1}{\ell}\right)^2 |u_k|^2 \le C_4 \left(\frac{1}{k} - \frac{1}{\ell}\right)^2.$$

So

$$|Z - X_3| \le C_4 \left(\frac{1}{k} - \frac{1}{\ell}\right)^2.$$

Integrating (3.30) from 0 to t and taking into account the last inequalities, we obtain:

$$m_0 |A_\ell^{1/2}\, w_{\ell k}|^2 \le C_1 \int_0^t m_0 |A_\ell^{1/2}\, w_{\ell k}|^2\, ds + C_2 \int_0^{T_0} |M(\varphi_k) - M(\varphi_\ell)|\, dt$$

$$+ C_4 T_0 \left(\frac{1}{k} - \frac{1}{\ell}\right)^2 .$$

By Gronwall inequality and noting that $(M(\varphi_\ell))$ is a Cauchy sequence in $C^0([0,T_0])$ (cf. (3.29)), we find from this inequality that

$$A_\ell^{1/2} u_\ell - A_\ell^{1/2} u_k \to 0 \text{ in } C^0([0,T_0];H), \ \ \ell, k \to \infty.$$

We have:

$$A_\ell^{1/2} u_\ell - A_k^{1/2} u_k = \big(A_\ell^{1/2} u_\ell - A_\ell^{1/2} u_k\big) + \big(A_\ell^{1/2} u_k - A^{1/2} u_k\big) + \big(A^{1/2} u_k - A_k^{1/2} u_k\big).$$

As in (3.24), we obtain:

$$\big(A_\ell^{1/2} u_k - A^{1/2} u_k\big) \to 0 \text{ and } \big(A^{1/2} u_k - A_k^{1/2} u_k\big) \to 0 \text{ in } C^0([0,T_0;H) \quad \ell, k \to \infty.$$

It follows from the last three convergence that

$$\big(A_\ell^{1/2} u_\ell\big) \text{ is a Cauchy sequence in } C^0([0,T_0];H).$$

By convergence (3.26), we get then

$$A_\ell^{1/2} u_\ell \to A^{1/2} u \text{ in } C^0([0,T_0];H)\,,$$

that implies

$$M\big(|A_\ell^{1/2} u_\ell|^2\big) \to M\big(|A^{1/2} u|^2\big) \text{ in } C^0([0,T_0];H). \tag{3.31}$$

Convergences (3.23), (3.27), (3.31) and (3.4) allow us to pass to the limit in (P_ℓ) and to show that u is a solution of the problem (LP).

We prove the uniqueness of solutions as in the first part. Thus we finish the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2

We fix real number $\eta > 0$ such that

$$\big(a_\eta\, L_\eta\, S_\eta / m_0\big) < \delta\,, \tag{3.32}$$

where

$$\left|\begin{array}{l} L_\eta^2 = \big[|u^1|^2+\eta\big] + \widehat{M}\big(|(A+I)^{1/2}\,u^0|^2+\eta\big)\,, \\ S_\eta = \max\limits_{0\le\xi\le L_\eta^2/m_0} |M'(\xi)|\,, \\ a_\eta^2 = \dfrac{1}{m_0}\big[|(A+I)^{1/2}u^1|^2+\eta\big] + \big[|(A+I)u^0|^2+\eta\big]\,. \end{array}\right. \tag{3.33}$$

Let (u_ℓ^0) and (u_ℓ^1) be sequences of $D\big((A+I)^{3/2}\big)$ and $D(A+I)$, respectively, such that

$$u_\ell^0 \to u^0 \text{ in } D(A+I) \text{ and } u_\ell^1 \to u^1 \text{ in } D\big((A+I)^{1/2}\big). \tag{3.34}$$

Then there exists $\ell_0(\eta)$ such that for $\ell \ge \ell_0(\eta)$, we have:

$$\left|\begin{array}{l} L_\ell^2 = |u_\ell^1|^2 + \widehat{M}\big(|(A_\ell^{1/2}\,u_\ell^0|^2\big) \le L_\eta^2\,, \\ S_\ell = \max\limits_{0\le\xi\le L_\ell^2/m_0} |M'(\xi)| \le S_\eta\,, \\ a_\ell^2 = \dfrac{1}{m_0}\big[|A_\ell^{1/2}\,u_\ell^1|^2 + |A_\ell\,u_\ell^0|^2 \le a_\eta^2\,. \end{array}\right. \tag{3.35}$$

We consider the problem (P_ℓ) with initial data u_ℓ^0 and u_ℓ^1 satisfying conditions (3.35). The problem (P_ℓ) was given in Subsection 3.1. As the problem (P_ℓ) has uniqueness of solutions, we have that there exists a unique $T_{\max,\ell}$, $T_{\max,\ell}$ a real number or $T_{\max,\ell}$ infinite, where $[0,T_{\max,\ell}[$ is the maximal interval of existence of the solution u_ℓ of the problem (P_ℓ). So

$$\begin{array}{l} u_\ell \in L^\infty_{\mathrm{loc}}\big(0,T_{\max,\ell};D(A_\ell^{3/2})\big), u_\ell' \in L^\infty_{\mathrm{loc}}\big(0,T_{\max,\ell};D(A_\ell)\big)\,, \\ u_\ell'' \in L^\infty_{\mathrm{loc}}\big(0,T_{\max,\ell};D(A_\ell^{1/2})\big)\,, \end{array}$$

and

$$\left|\begin{array}{l} u_\ell'' + M\big(|A_\ell^{1/2}|^2\big)A_\ell u_\ell + \delta u_\ell' = 0 \text{ in } L^\infty_{\mathrm{loc}}\big(0,T_{\max,\ell};D(A_\ell^{1/2})\big)\,, \\ u_\ell(0) = u_\ell^0\,,\;\; u_\ell'(0) = u_\ell^1\,. \end{array}\right. \tag{3.36}$$

We divide the proof of Theorem 2.2 in two parts.

3.2.1. First Part: $T_{\max,\ell}$ is Infinite

To prove that $T_{\max,\ell}$ is infinite it is sufficient to show that the function

$$\psi_\ell(t) = \frac{1}{M(|A_\ell^{1/2}u_\ell(t)|^2)}\,|A_\ell^{1/2}\,u_\ell'(t)|^2 + |A_\ell u_\ell(t)|^2, \quad t\in[0,T_{\max,\ell}[\,, \tag{3.37}$$

is not increasing. For that we need two estimates.

First Estimate

We take the scalar product of H in both sides of $(3.36)_1$ with $2u'_\ell$ and integrate from 0 to t. We have:

$$\begin{gathered}|u'_\ell(t)|^2 + m_0|A_\ell^{1/2}u_\ell(t)|A^2 + 2\delta\int_0^t |u'_\ell(s)|^3 ds \leq |u^1_\ell|^2 \\ +\widehat{M}\big(A_\ell^{1/2}u^0_\ell|^2\big) = L^2_\ell\,,\end{gathered} \tag{3.38}$$

that implies

$$|A_\ell^{1/2}\, u_\ell(t)|^2 \leq \big(L^2_\ell/m_0\big) \leq L^2_\eta/m_0\,, \quad \forall\, t\in [0, T_{\max,\ell}[\,. \tag{3.39}$$

Therefore,

$$|M'\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)| \leq S_\ell \leq S_\eta\,, \quad \forall\, t\in [0, T_{\max,\ell}[\,. \tag{3.40}$$

Second Estimate

We take the scalar product of H in both sides of $(3.36)_1$ with $2A_\ell\, u'_\ell$. We obtain:

$$\begin{aligned}\frac{1}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}\,\frac{d}{dt}\,|A_\ell^{1/2}\, u'_\ell(t)|^2 + \frac{d}{dt}\,|A_\ell\, u_\ell(t)|^2& \\ = -\frac{2\delta}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}\,|A_\ell^{1/2}\, u'_\ell(t)|^2&.\end{aligned}$$

Deriving $\psi_\ell(t)$ and take into account this equality, we find:

$$\psi'_\ell(t) = \left[\frac{\frac{d}{dt}\, M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)} - 2\delta\right]\frac{|A_\ell^{1/2}\, u'_\ell(t)|^2}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}\,.$$

But, by (3.39) and (3.40),

$$\frac{|\frac{d}{dt}\, M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)|}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)} \leq 2S_\eta\,\frac{L_\eta}{m_0^{1/2}}\,\frac{|A_\ell^{1/2}\, u'_\ell(t)|^2}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}\,.$$

Use the notation

$$\gamma_\ell(t) = \frac{2S_\eta\, L_\eta}{m_0^{1/2}}\,\frac{|A_\ell^{1/2}\, u'_\ell(t)|^2}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}\,.$$

Then

$$\psi_\ell'(t) \le [\gamma_\ell(t) - 2\delta] \, \frac{|A_\ell^{1/2}\, u_\ell'(t)|^2}{M\big(|A_\ell^{1/2}\, u_\ell(t)|^2\big)}\,, \quad \forall\, t \in [0, T_{\max,\ell}[\,. \tag{3.41}$$

Our objetive is to show that

$$\gamma_\ell(t) \le 2\delta, \qquad \forall\, t \in [0, T_{\max,\ell}[\,. \tag{3.42}$$

This will imply that $\psi_\ell(t)$ is not increasing.

We have

$$\gamma_\ell(t) \le \frac{2S_\eta\, L_\eta}{m_0}\, \psi_\ell^{1/2}(t), \qquad \forall\, t \in [0, T_{\max,\ell}[\,. \tag{3.43}$$

As $\psi_\ell(0) \le a_\ell^2 \le a_\eta^2$, it follows from (3.32) and this inequality that

$$\gamma_\ell(0) < 2\delta.$$

We prove (3.42) by contradiction. In fact we suppose that there exists $t > 0$ such that $\gamma_\ell(t) > 2\delta$. Define

$$t^* = \inf \big\{ t > 0; \gamma_\ell(t) = 2\delta \big\}.$$

Then

$$\gamma_\ell(t) < 2\delta \text{ in } [0, t^*[\text{ and } \gamma_\ell(t^*) = 2\delta. \tag{3.44}$$

This and (3.41) imply that $\psi_\ell(t)$ is not increasing in $[0, t^*]$. Therefore $\psi_\ell(t^*) \le \psi_\ell(0)$. This inequality and (3.43) give

$$\gamma_\ell(t^*) \le \big(2S_\eta\, L_\eta\, \psi_\ell(0)/m_0\big) < 2\delta.$$

This expression is in contradiction with (3.44). So (3.42) is true.

Function $\psi_\ell(t)$ defined in (3.37) gives the estimates

$$\left|\begin{array}{l} |A_\ell\, u_\ell(t)|^2 \le a_\eta^2\,, \quad \forall\, t \in [0, T_{\max,\ell}[\,, \\ |A_\ell^{1/2}\, u_\ell'(t)|^2 \le a_\eta^2 \Big(\max\limits_{0 \le \xi \le L_\eta^2 / m_0} M(\xi) \Big), \; \forall\, t \in [0, T_{\max,\ell}[\,. \end{array}\right. \tag{3.45}$$

Also, taking the scalar product of H in both sides of $(3.36)_1$ with $2u_\ell$ and integrating from 0 to t, we find

$$\delta |u_\ell(t)|^2 + 2\int_0^t M\big(|A_\ell^{1/2}\, u_\ell(s)|^2\big) |A_\ell^{1/2}\, u_\ell(s)|^2\, ds$$

$$= \delta|u_\ell^0|^2 + 2\int_0^t |u_\ell'(s)|^2\,ds - 2(u_\ell'(t), u_\ell(t)) + 2(u_\ell^1, u_\ell^0)\,,$$

which implies by estimate (3.38) that

$$|u_\ell(t)| \le C, \quad \forall\, t \in [0, T_{\max,\ell}[\text{ and } \ell \ge \ell_0(\eta). \tag{3.46}$$

Equation (3.36) and estimates (3.38) and (3.45) give also

$$|u_\ell''(t)| \le C, \quad \forall\, t \in [0, T_{\max,\ell}[\text{ and } \ell \ge \ell_0(\eta). \tag{3.47}$$

Estimates (3.38), (3.45), (3.46) and Proposition 3.1 allow us to say that $T_{\max,\ell}$ is infinite because if $T_{\max,\ell}$ was finite by Theorem 2.1 we could find a solution of the problem (P_ℓ) with initial conditions $u_\ell(T_{\max,\ell}) \in D(A+I)$ and $u_\ell'(T_{\max,\ell}) \in D(A+I)^{1/2}$ defined in $[0, T_0]$. This gives a solution u_ℓ of (P_ℓ) defined in $[0, T_{\max,\ell} + T_0[$, fact that is in contradiction with the definition of $[0, T_{\max,\ell}[$ (see [24]).

3.2.2. Second Part: Existence of Solution of (GP)

By estimates (3.38), (3.45), (3.46), (3.47) and applying similar arguments to the ones used to obtain convergences (3.26) and (3.27) in the proof of Theorem 2.1, we have that there exists a subsequence of (u_ℓ), which is also denoted by (u_ℓ), and a function u such that

$$\left|\begin{array}{l} u_\ell \to u,\ u_\ell' \to u',\ u_\ell'' \to u_\ell''\,, \\ \qquad \text{weak star in } L^\infty(0,\infty; H)\,, \\ A_\ell^{1/2}\, u_\ell \to A^{1/2}\, u,\ A_\ell^{1/2}\, u_\ell' \to A^{1/2}\, u',\ A_\ell u_\ell \to Au \\ \qquad \text{weak star in } L^\infty(0,\infty; H)\,. \end{array}\right. \tag{3.48}$$

By the last three convergences and by reasoning similar to the one used to obtain (3.28), we prove that there exists a subsequence $(u_{\ell(1)})$ of (u_ℓ) such that

$$M\big(|A_{\ell(1)}^{1/2}\, u_{\ell(1)}|^2\big) A_{\ell(1)}\, u_{\ell(1)} \to M\big(|A^{1/2}\, u|^2\big) Au \text{ weak star in } L^\infty(0,1;H).$$

Analogously, there exists a subsequence $(u_{\ell(2)})$ of $(u_{\ell(1)})$ such that

$$M\big(|A_{\ell(2)}^{1/2}\, u_{\ell(2)}|^2\big) A_{\ell(2)}\, u_{\ell(2)} \to M\big(|A^{1/2}\, u|^2\big) Au \text{ weak star in } L^\infty(0,2;H).$$

We repeat the process for $3, 4, \ldots, n, \ldots$. Take the diagonal subsequence $(u_{\ell(\ell)})$, still denoted by (u_ℓ). Then

$$M\big(|A_\ell^{1/2}\, u_\ell|^2\big) A_\ell\, u_\ell \to M\big(|A^{1/2}\, u|^2\big) Au \text{ weak star in } L^\infty_{\mathrm{loc}}(0,\infty;H). \tag{3.49}$$

Convergences (3.48) and (3.49) allow us to take the limit in (3.36) and to prove that u belongs to class (2.8) and u is a solution of (GP). The uniqueness of solutions follows as in the proof of Theorem 2.1. So we conclude the proof of Theorem 2.2.

3.3. Proof of Theorem 2.3

We introduce similar notations to (3.33) and (3.35). Fix a real number $\eta > 0$ such that

$$\big(a_\eta^* \, L_\eta^* \, S_\eta^*/m_0\big) < N^* \,, \tag{3.50}$$

where

$$\begin{aligned}
&(L_\eta^*)^2 = [|u^1|^2 + \eta] + \widehat{M}\big(|A^{1/2}\, u^0|^2 + \eta\big),\\
&S_\eta^* = \max_{0\le\xi\le(L_\eta^*)^2/m_0} |M'(\xi)|, \quad (a_\eta^*)^2 = \frac{1}{m_0}\,\big[|A^{1/2}\, u^1|^2 + \eta\big] + \big[|Au^0|^2 + \eta\big].
\end{aligned}$$

Let (u_ℓ^0) and (u_ℓ^1) be sequences of $D(A^{3/2})$ and $D(A)$, respectively, such that

$$u_\ell^0 \to u^0 \text{ in } D(A) \text{ and } u_\ell^1 \to u^1 \text{ in } D(A^{1/2}).$$

Then there exists $\ell_0(\eta)$ such that for $\ell \ge \ell_0(\eta)$, we get

$$\begin{aligned}
&(L_\ell^*)^2 = |u_\ell^1|^2 + \widehat{M}\big(|A_\ell^{1/2}\, u_\ell^0|^2\big) \le (L_\eta^*)^2\,,\\
&S_\ell^* = \max_{0\le\xi\le(L_\ell^*)^2/m_0} |M'(\xi)| \le S_\eta^*\,,\\
&(a_\ell^*)^2 = \frac{1}{m_0}\,|A_\ell^{1/2}\, u_\ell^1|^2 + |A_\ell\, u_\ell^0|^2 \le (a_\eta^*)^2\,.
\end{aligned}$$

Consider the problem

$$\left|\begin{aligned}
&u_\ell'' + M\big(|A^{1/2}\, u_\ell|^2\big) Au_\ell + \delta u_\ell' = 0 \text{ in } L^\infty(0,\infty; D(A^{1/2}))\,,\\
&u_\ell(0) = u_\ell^0\,,\ u_\ell'(0) = u_\ell^1\,.
\end{aligned}\right. \tag{P_ℓ^*}$$

In similar way to first and second part of the proof of Theorem 2.2, we obtain for all $t \in [0,\infty[$:

$$\left|\begin{aligned}
&|u_\ell'(t)|^2 + m_0|A^{1/2}\, u_\ell(t)|^2 + 2\delta \textstyle\int_0^t |u_\ell'(s)|^2\, ds \le (L_\ell^*)^2 \le (L_\eta^*)^2\,,\\
&|M'\big(|A^{1/2}\, u_\ell(t)|^2\big)| \le S_\ell^* \le S_\eta^*\,,\\
&\frac{1}{M\big(|A^{1/2}\, u_\ell(t)|^2\big)}\,|A^{1/2}\, u_\ell'(t)|^2 + |Au_\ell(t)|^2 \le (a_\ell^*)^2 \le (a_\eta^*)^2\,,
\end{aligned}\right. \tag{3.51}$$

and the convergences,

$$\left|\begin{array}{l} u_\ell \to u,\ A^{1/2}\,u_\ell' \to A^{1/2}\,u',\ Au_\ell \to Au\,, \\ \qquad \text{weak star in } \ L^\infty(0,\infty;H)\,, \\ M\big(|A^{1/2}\,u_\ell|^2\big)Au_\ell \to M\big(|A^{1/2}\,u|^2\big)Au \\ \qquad \text{weak star in } \ L^\infty_{\text{loc}}(0,\infty;H)\,. \end{array}\right. \tag{3.52}$$

Consider the energy $E_\ell(t)$ of (P_ℓ^*):

$$E_\ell(t) = |A^{1/2}\,u_\ell'(t)|^2 + M\big(|A^{1/2}\,u_\ell(t)|^2\big)|Au_\ell(t)|^2, \quad t \geq 0.$$

Take the scalar product of H in both sides of (P_ℓ^*) with $2Au_\ell'$ and consider estimates (3.51). One has

$$\frac{d}{dt}\,E_\ell(t) \leq \big(2S_\eta^*\,L_\eta^*\,a_\eta^*/m_0\big)M\big(|A^{1/2}\,u_\ell(t)|^2\big)|Au_\ell(t)|^2 - 2\delta|A^{1/2}\,u_\ell'(t)|^2\,,$$

that is,

$$\frac{d}{dt}\,E_\ell(t) \leq 2b_\eta\,M\big(|A^{1/2}\,u_\ell(t)|^2\big)|Au_\ell(t)|^2 - 2\delta|A^{1/2}\,u_\ell'(t)|^2,\ t \geq 0 \qquad \big(b_\eta = \big(2S_\eta^*\,L_\eta^*\,a_\eta^*/m_0\big)\big). \tag{3.53}$$

Introduce the functional

$$\rho_\ell(t) = \big(u_\ell'(t), Au_\ell(t)\big) + \frac{\delta}{2}\,|A^{1/2}\,u_\ell(t)|^2,\ \ t \geq 0\,,$$

and for $\varepsilon > 0$, consider the perturbed energy

$$E_\varepsilon(t) = E_\ell(t) + \varepsilon\rho_\ell(t), \quad t \geq 0. \tag{3.54}$$

We have, by noting that $|A^{1/2}\,u_\ell(t)| \leq \beta^{-1/2}\,|Au_\ell(t)|$ (see (2.2)):

$$|\rho_\ell(t)| \leq \frac{1}{\beta^{1/2}m_0^{1/2}}\,M^{1/2}\big(|A^{1/2}\,u_\ell(t)|^2\big)|Au_\ell(t)|\,|A^{1/2}\,u_\ell'(t)| + \frac{\delta}{2\beta m_0}\,M\big(|A^{1/2}\,u_\ell(t)|^2\big)|Au_\ell(t)|^2 \leq \left(\frac{2\beta^{1/2}m_0^{1/2}+\delta}{2\beta m_0}\right)E_\ell(t).$$

So

$$|\rho_\ell(t)| \leq C^*\,E_\ell(t), \quad \forall\,t \geq 0 \quad \big(C^* = \big(2\beta^{1/2}m_0^{1/2}+\delta\big)/2\beta m_0\big)\big).$$

By (3.54) we obtain then

$$(1-\varepsilon\,C^*)E_\ell(t) \leq E_\varepsilon(t) \leq (1+\varepsilon\,C^*)E_\ell(t)\,,$$

that implies, for $0 < \varepsilon \leq 1/2C^*$,

$$\frac{1}{2}\, E_\ell(t) \leq E_\varepsilon(t) \leq \frac{3}{2}\, E_\ell(t), \quad \forall\, t \geq 0. \tag{3.55}$$

We have

$$\frac{d}{dt}\, \rho_\ell(t) = |A^{1/2}\, u_\ell'(t)|^2 - M\big(|A^{1/2}\, u_\ell(t)|^2\big)|Au_\ell(t)|^2.$$

Then, by (3.53),

$$\frac{d}{dt}\, E_\varepsilon(t) \leq (2b_\eta - \varepsilon) M\big(|A_\ell^{1/2}(t)|^2\big)|Au_\ell(t)|^2 - (2\delta - \varepsilon)|A^{1/2}\, u_\ell'(t)|^2. \tag{3.56}$$

Take $\tau_\eta = (N^* - b_\eta)\big/2$ and ε_1, ε_2 in the conditions

$$2b_\eta < \varepsilon_1 < 2b_\eta + \tau_\eta\,, \;\; 2b_\eta + 2\tau_\eta < \varepsilon_2 < 2b_\eta + 3\tau_\eta\,.$$

Note that $0 < \varepsilon_2 < 1/2C^*$ and

$$\varepsilon_2 - 2b_\eta > \varepsilon_2 - \varepsilon_1 > \tau_\eta \text{ and } 2\delta - \varepsilon_2 > \tau_\eta\,.$$

Then take into account these inequalities in (3.56), we obtain:

$$\frac{d}{dt}\, E_{\varepsilon_2}(t) \leq -\tau_\eta\, E_\ell(t).$$

Then inequality (3.55) gives

$$\frac{d}{dt}\, E_{\varepsilon_2}(t) \leq -\frac{2}{3}\, \tau_\eta\, E_{\varepsilon_2}(t)$$

which implies

$$E_{\varepsilon_2}(t) \leq \big[\exp\big(-\frac{2}{3}\, \tau_\eta\, t\big)\big] E_{\varepsilon_2}(0).$$

Combining this expression with (3.55), we find

$$E_\ell(t) \leq 3\left(\exp\left[-\frac{1}{3}\,(N^* - b_\eta)t\right]\right) E_\ell(0).$$

Making $\eta \to 0$ and using convergence (3.52) we obtain the result.

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