

**A MULTISYMPLECTIC GEOMETRY AND
A MULTISYMPLECTIC SCHEME FOR
MAXWELL'S EQUATIONS**

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Abstract: In this paper the self-adjointness of Maxwell's equations with variable coefficients ε and μ is discussed. For these equations three different Lagrangian forms are presented. Using Legendre's transformation, a multisymplectic Bridges' form is obtained. Based on the multisymplectic structure, the multisymplectic conservation law of the system is derived. A nine-point Preissman multisymplectic scheme, preserving the multisymplectic conservation law, is deduced for Maxwell's equations in an inhomogeneous, isotropic and lossless medium. A numerical example illustrates the results.

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1. Introduction

Transient electromagnetic field problems are interesting and important in many applications of modern technology. The Lagrangian density for electromagnetism on a fixed space in time, X , and the existence of the Euler-Lagrange equation were studied (Gotay, Isenberg and Marsden [4]). In this paper, we will treat this problem as an inverse variational problem. As a complete depiction of the behavior of the electromagnetic field, Maxwell's equations have a very interesting structure. The Lagrangian density and the multisymplectic form of Bridges (Bridges and Derks [2]) are derived using different techniques.

The multisymplectic method is used to describe the dynamics, given by an Euler-Lagrange system, in the terms of the *finite*-dimensional space of fields at a given event in the time-space. Some Euler-Lagrange systems can be formalized as multisymplectic Hamiltonian forms (Marsden, Patrick and Shkoller [5], Bridges and Reich [3], Bridges and Derks [2]), which manifest some characteristics of the original systems. So we discretize multisymplectic Hamiltonian forms directly, in order to simulate the original systems.

Determining whether a given nonlinear partial differential equation (PDE) is an Euler-Lagrange equation of some variational problem, is a fundamental task in finite-element methods and in the nonlinear wave theory. In this paper the straightforward result of Atherton and Homsy (see [1]) is used to discuss the self-adjointness of Maxwell's equations in different representations. The same equations can be given in different forms and show different potentialities. Consequently, the equations can be expressed as different variational problems. So, we choose an appropriate form of Maxwell's equations, to derive its Lagrangian form and then to rewrite it in a multisymplectic Hamiltonian form (see [5], [3], [2]) by using Legendre's transformation.

This paper is organized in four parts. In Section 2, the necessary and sufficient conditions of the self-adjointness for a first-order form of Maxwell's equations in an inhomogeneous, isotropic and lossless medium is derived and two Lagrangians are given. In Section 3, the equations are presented in a second-order Lagrangian form by introducing two vector potential functions. A complex multisymplectic Hamiltonian form and a multisymplectic conservation law are given as well as an ordinary conservation law. In Section 4, as an

application, a multisymplectic scheme is constructed and a numerical simulation is shown.

2. The Lagrangian Formalism for Maxwell's Equations

First some basic concepts in variational problems are introduced. Let $X \subset \mathbb{R}^m$ be an open subset with smooth boundary ∂X , and let M a function space. We consider a differential equation with a solution vector of dependent variables $u_i = u_i(x_1, x_2, \dots, x_m) \in M$, $i = 1, 2, \dots, n$, and a vector $N(u)$ of differential operators $N^i(u)$, $i = 1, 2, \dots, n$, such that

$$N(u) = 0. \quad (1)$$

The existence of a variational principle for the differential equation (1) is equivalent to determining, whether or not an operator is a potential operator. According to a theorem of Vainberg (Vainberg [6]), where N should be a potential operator, it is necessary and sufficient that the Frechét derivative of the operator N is symmetric.

Let N be an operator, defined in an appropriate function space M , then N is a potential operator if $N'_u = N'^*_u$, where

$$N'_u \phi = \lim_{h \rightarrow 0} \frac{N(u + h\phi) - N(u)}{h}.$$

and N'^*_u is the adjoint operator of N'_u . The adjoint operator is always denoted by star. For a potential operator the potential is given by

$$F = \int_X u \int_0^1 N(\lambda u) d\lambda dv. \quad (2)$$

We assume that the Frechét derivative $N'_u \phi$ of N exists with the arbitrary function ϕ belonging to M .

Definition 1. The Frechét derivative generates a matrix N'_u with elements $N'(i, j)$, which are the derivatives of the operators N^i with respect to u_j . Then

$$N'_u = (N'(i, j))_{i,j=1}^n$$

is called the *differential operator matrix* of the vectorial operator N .

The test functional of the self-adjointness is given as

$$S(\psi, \phi) = \int_X \psi N'_u \phi dv = \int_X \psi_i N'(i, j) \phi_j dv.$$

From Vainberg's Theorem (see [6]) the condition of self-adjointness reads

$$S(\psi, \phi) = S(\phi, \psi) + \int_X \operatorname{div} P dv,$$

if the boundary condition is set to be free, i.e., P vanishes on ∂X . The following definition is useful.

Definition 2. The operator $N'(i, j)$ with free boundary conditions is called *self-adjoint*, if

$$N'(i, j) = N'(j, i)^* \quad \text{for all } i, j. \quad (3)$$

A certain variational problem can lead to more than one Euler-Lagrange equation and to different boundary conditions. Hence, the Lagrangian and the boundary condition are not unique in the inverse problem of finding the corresponding functional for a Euler-Lagrange equation. The requirement for free boundary conditions (see [1]) is too strict for applications. Compare with multi-symplectic methods (see [3], [2]), which need the verification of the self-adjointness of a given system, to find one of its Lagrangians in a first step.

Example. Consider the wave equation

$$u_{tt} - u_{xx} = 0,$$

then the self-adjointness condition reads (see [1])

$$\frac{\partial N(u)}{\partial u_2} \Big|_{\partial X} = 0, \quad \text{where } (t, x) \in X \text{ and } u_2 = u_{xx}, u_{tt},$$

which is not satisfied independently of the boundary condition. But actually it keeps a Lagrangian form. This is the motivation for Definition 2.

2.1. Bundle Space and Covariant Hamiltonian

Let us consider a k -th order functional of maps $u : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}^6$, $u = [H_1, H_2, H_3, E_1, E_2, E_3]$, for Maxwell's equations. Here $\mathbb{R}^{1,3}$ is a $(1+3)$ -dimensional time-space domain, denoted by coordinates x_j , $j = 1, \dots, 4$. \mathbb{R}^6 is the target space, denoted by coordinates u^i , $i = 1, \dots, 6$, additionally, the space $T_u \mathbb{R}^6 \otimes T_x^* \mathbb{R}^{1,3}$ is denoted by \mathfrak{R} . The functional is defined by means of a Lagrangian L : it is a function on the space $\mathbb{R}^{1,3} \times \mathbb{R}^6 \times \mathfrak{R}$, i.e. L is a smooth function defined on the bundle over $\mathbb{R}^{1,3} \times \mathbb{R}^6$ with fiber over (x, u) equal to \mathfrak{R} . We denote this bundle by $T\mathbb{R}^6 \otimes_{\mathbb{R}^{1,3} \times \mathbb{R}^6} T^*\mathbb{R}^{1,3}$. Hence, the functional is defined as

$$\mathcal{L} = \int_{\mathbb{R}^{1,3}} L(x_j, u^i(x), u_\nu^i(x)) \omega,$$

where ω is some volume 4-form on $\mathbb{R}^{1,3}$, $u_\nu^i = \frac{\partial u^i}{\partial x_{j_1} \dots \partial x_{j_\nu}}$, $\nu = 1, 2, \dots, k$. Legendre's transformations is defined as

$$\pi_i^\nu = \frac{\partial L}{\partial u_\nu^i}(x_j, u^i(x), u_\nu^i(x)),$$

which leads to the covariant Hamiltonian function \mathcal{H} on the multisymplectic manifold $T^*\mathbb{R}^6 \otimes_{\mathbb{R}^{1,3} \times \mathbb{R}^6} T\mathbb{R}^{1,3}$, as

$$\mathcal{H}(x_j, u^i(x), \pi_i^\nu) = \pi_i^\nu u_\nu^i - L(x_j, u^i(x), u_\nu^i(x)). \quad (4)$$

2.2. Conditions of Self-Adjointness and Two Lagrangians

Now we consider Maxwell's equations in an inhomogeneous, isotropic, and lossless medium

$$\begin{aligned} \mu \frac{\partial H}{\partial t} + \nabla \times E &= -K, \\ -\nabla \times H + \varepsilon \frac{\partial E}{\partial t} &= -J, \end{aligned} \quad (5)$$

where $E = [E_1, E_2, E_3]^T$ is the electric field strength, and $H = [H_1, H_2, H_3]^T$ is the magnetic field strength. Here ε is the permittivity, μ is the permeability, with both ε and μ scalar functions in space and time. J is the external electric-current density, and K is the external magnetic-current density. System (5) can be arranged in the form

$$\mathcal{G} \cdot \mathcal{Z} = \mathcal{F}, \quad (6)$$

where $\mathcal{Z} = \mathcal{Z}(x, y, z, t)$ is the field vector, consisting of the components of the electric field strength E and the magnetic field strength H as

$$\mathcal{Z} = [H_1, H_2, H_3, E_1, E_2, E_3]^T,$$

simplified into $\mathcal{Z} = [H, E]^T$. Similarly $\mathcal{F} = \mathcal{F}(x, y, z, t)$ is the source vector composed of the external electric current source J and the external magnetic current source K , i.e., $\mathcal{F} = [J, -K]^T$. \mathcal{G} is the operator matrix given by

$$\mathcal{G} = \begin{pmatrix} \nabla \times & -\varepsilon \partial_t \\ \mu \partial_t & \nabla \times \end{pmatrix}, \quad (7)$$

where each element of \mathcal{G} is a third-order matrix such as

$$\nabla \times = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \quad \varepsilon(\mu) \partial_t = \varepsilon(\mu) \cdot \begin{pmatrix} \partial_t & 0 & 0 \\ 0 & \partial_t & 0 \\ 0 & 0 & \partial_t \end{pmatrix}.$$

Because \mathcal{G} is a linear operator matrix, i.e., each element of \mathcal{G} is a linear operator, its differential operator matrix denoted as \mathcal{G}' equals \mathcal{G} itself. So the adjoint operator matrix of \mathcal{G}' is

$$\mathcal{G}'^* = \begin{pmatrix} (\nabla \times)^T & \varepsilon \partial_t + \varepsilon_t \\ -\mu \partial_t - \mu_t & (\nabla \times)^T \end{pmatrix}. \quad (8)$$

It is obvious that the self-adjointness condition (3) is that $\mathcal{G}'^T = \mathcal{G}'^*$, so we must have $\varepsilon = \mu$, and the two coefficients are independent on the time variable t . The system in vacuum of course satisfies $\mathcal{G}'^T = \mathcal{G}'^*$, hence we obtain via integral (2) the first Lagrangian form.

Proposition 1. *Systems satisfying the self-adjointness condition (3) have a Lagrangian given by*

$$L = \frac{1}{2} \langle H, \nabla \times H \rangle + \frac{1}{2} \langle E, \nabla \times E \rangle - \mu \langle H, E_t \rangle - \langle H, J \rangle + \langle E, K \rangle, \quad (9)$$

where $\varepsilon = \mu$.

Instead of (7) consider another form of the operator \mathcal{G}

$$\mathcal{G}_1 = \begin{pmatrix} \frac{1}{\varepsilon} \nabla \times & -\partial_t \\ \partial_t & \frac{1}{\mu} \nabla \times \end{pmatrix},$$

where the source function \mathcal{F} is changed into

$$\mathcal{F}_1 = \left[\frac{1}{\varepsilon} J, -\frac{1}{\mu} K \right]^T,$$

and the dependent variable \mathcal{Z} remains unchanged. But the differential operator matrix of \mathcal{G}_1 is not a self-adjoint operator matrix, since

$$\mathcal{G}_1'^* = \begin{pmatrix} A + \frac{1}{\varepsilon} (\nabla \times)^T & \partial_t \\ -\partial_t & B + \frac{1}{\mu} (\nabla \times)^T \end{pmatrix} \neq \mathcal{G}_1'^T, \quad (10)$$

where A and B are two third-order matrices given by

$$A = \begin{pmatrix} 0 & (\frac{1}{\varepsilon})_z & -(\frac{1}{\varepsilon})_y \\ -(\frac{1}{\varepsilon})_z & 0 & (-\frac{1}{\varepsilon})_x \\ (\frac{1}{\varepsilon})_y & (-\frac{1}{\varepsilon})_x & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & (\frac{1}{\mu})_z & -(\frac{1}{\mu})_y \\ -(\frac{1}{\mu})_z & 0 & (-\frac{1}{\mu})_x \\ (\frac{1}{\mu})_y & (-\frac{1}{\mu})_x & 0 \end{pmatrix}.$$

Obviously if $\mathcal{G}_1'^T = \mathcal{G}_1'^*$, ε and μ are free of space variables, additionally such form excludes the condition of $\varepsilon = \mu = 0$. So we get the second Lagrangian of (5) via integral (13) given later.

Proposition 2. *Systems satisfying the self-adjointness condition (3) have a Lagrangian given by*

$$L = \frac{1}{2\varepsilon} \langle H, \nabla \times H \rangle + \frac{1}{2\mu} \langle E, \nabla \times E \rangle - \langle H, E_t \rangle - \frac{1}{\varepsilon} \langle H, J \rangle + \frac{1}{\mu} \langle E, K \rangle . \quad (11)$$

Remarks. This Lagrangian seems equivalent to the Lagrangian (9), but in fact the above cases actually deal with two different systems because of different restrictions for the coefficients ε and μ in their self-adjointness conditions. Each Lagrangian is a first order system. Legendre’s transformations are not necessary in constructing the multisymplectic Hamiltonian form. In fact, for linear first-order systems, which are self-adjoint, the multisymplectic Hamiltonian form can be obtained directly.

3. Multisymplectic Hamiltonian Forms for Maxwell’s Equations

Introducing two vector functions U and V satisfying $U_t = E$ and $V_t = H$, the system (5) can be rewritten in form (6) with different operator matrix \mathcal{G} , different dependent variable \mathcal{Z} and source function \mathcal{F} as

$$\mathcal{G} = \begin{pmatrix} \mu \partial_t^2 & \nabla \times \partial_t \\ -\nabla \times \partial_t & \varepsilon \partial_t^2 \end{pmatrix}, \quad \mathcal{Z} = [V_1, V_2, V_3, U_1, U_2, U_3]^T, \quad (12)$$

$$\mathcal{F} = -[K, J]^T.$$

3.1. The Bridges’ Form

The matrix \mathcal{G}' of the differential operator equals \mathcal{G} itself. If ε and μ are not dependent on the time variable, then it is easy to demonstrate that \mathcal{G}' is self-adjoint, so the self-adjointness of the form (6) with the representation of (12) is verified. The potential functional (2) of (1) is given by

$$F(V, U) = \int_X \int_0^1 [V, U] \mathcal{G}(\lambda \mathcal{Z}) d\lambda dv$$

$$= \int_X \left(\frac{1}{2} \mu \langle V, V_{tt} \rangle + \frac{1}{2} \langle V, \nabla \times U_t \rangle + \frac{1}{2} \varepsilon \langle U, U_{tt} \rangle - \frac{1}{2} \langle U, \nabla \times V_t \rangle + UJ + VK \right) dv$$

$$= \int_X \left(-\frac{1}{2}\mu \langle V_t, V_t \rangle - \frac{1}{2} \langle V_t, \nabla \times U \rangle - \frac{1}{2}\varepsilon \langle U_t, U_t \rangle + \frac{1}{2} \langle U_t, \nabla \times V \rangle + UJ + VK + \operatorname{div}P \right) dv, \quad (13)$$

where $\langle \cdot, \cdot \rangle$ represents the inner production of vectors. We know the Lagrangian is unique for a given Euler-Lagrange equation except for the difference of a term, $\operatorname{div}P$. From (13) we get the *third Lagrangian* as

$$L = \frac{1}{2}\mu \langle V_t, V_t \rangle + \frac{1}{2} \langle V_t, \nabla \times U \rangle + \frac{1}{2}\varepsilon \langle U_t, U_t \rangle - \frac{1}{2} \langle U_t, \nabla \times V \rangle - UJ - VK, \quad (14)$$

correspondingly the generalized conjugate momentums are

$$P = \frac{\partial L}{\partial V_t} = \mu V_t + \frac{1}{2} \nabla \times U, \quad \frac{\partial L}{\partial \nabla \times V} = -\frac{1}{2} U_t, \\ Q = \frac{\partial L}{\partial U_t} = \varepsilon U_t - \frac{1}{2} \nabla \times V, \quad \frac{\partial L}{\partial \nabla \times U} = \frac{1}{2} V_t.$$

Proposition 3. *The Maxwell's equations can be transformed into the following form*

$$\begin{aligned} \frac{1}{2} \nabla \times U &= P - \mu H, & -\frac{1}{2} \nabla \times V &= Q - \varepsilon E, \\ -P_t - \frac{1}{2} \nabla \times E &= K, & -Q_t + \frac{1}{2} \nabla \times H &= J, \\ V_t &= H, & U_t &= E. \end{aligned} \quad (15)$$

Using formula (4) the covariant Hamiltonian reads

$$\begin{aligned} S &= \langle P, V_t \rangle + \langle Q, U_t \rangle + \langle \frac{\partial L}{\partial \nabla \times V}, \nabla \times V \rangle \\ &+ \langle \frac{\partial L}{\partial \nabla \times U}, \nabla \times U \rangle - L = \langle P, H \rangle + \langle Q, E \rangle \\ &- \frac{1}{2}\mu \langle H, H \rangle - \frac{1}{2}\varepsilon \langle E, E \rangle + UJ + VK. \end{aligned} \quad (16)$$

Denote $Z = [H, E, V, U, P, Q]^T$, then we get the following result.

Proposition 4. *The equations (15) can be organized in the multisymplectic form*

$$\mathcal{M}Z_t + \mathcal{K}\nabla \times Z = \nabla_Z S(Z), \quad (17)$$

where the rotation action $\nabla \times Z$ denotes

$$[\nabla \times H, \nabla \times E, \nabla \times V, \nabla \times U, \nabla \times P, \nabla \times Q]^T,$$

and $\mathcal{K}\nabla \times Z$ consists of three components such that

$$\mathcal{K}\nabla \times Z = \mathcal{K}_1 Z_x + \mathcal{K}_2 Z_y + \mathcal{K}_3 Z_z, \quad (18)$$

and

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{pmatrix}, \quad \mathcal{K}_i = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\mathcal{R}_i & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\mathcal{R}_i & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\mathcal{R}_i & 0 & 0 & 0 & 0 \\ \frac{1}{2}\mathcal{R}_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where I is the identity element belonging to $R^{3 \times 3}$.

Proof. This proposition is obvious, because \mathcal{M} is anti-symmetric and with the matrices

$$\mathcal{R}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{R}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

\mathcal{K}_i belongs to $R^{18 \times 18}$ and is anti-symmetric. Additionally, the rotation operator is given by

$$\nabla \times = \mathcal{R}_1 \frac{\partial}{\partial x} + \mathcal{R}_2 \frac{\partial}{\partial y} + \mathcal{R}_3 \frac{\partial}{\partial z},$$

which completes the proof. \square

3.2. Multisymplectic Conservation Laws

The equation (17) is a simplified expression in vector form, and its complete extension is the multisymplectic Hamiltonian system (see [2]). It satisfies the multisymplectic conservation law

$$\frac{\partial}{\partial t} \omega + \nabla \times \kappa = 0, \quad (19)$$

where ω and κ are the presymplectic forms

$$\omega = \frac{1}{2} dZ \wedge \mathcal{M} dZ, \quad \kappa = \frac{1}{2} dZ \wedge \mathcal{K} dZ$$

with $dZ = (dZ_1, dZ_2, \dots)^T$, and for two n -order vector functions $p = (p_1, \dots, p_n)^T$ and $q = (q_1, \dots, q_n)^T$, notation $\dot{\wedge}$ is defined as

$$dp \dot{\wedge} dq = \sum_{i=1}^n dp_i \wedge dq_i.$$

Then from (2) we immediately have the following result.

Proposition 5. *The system (17) satisfies the local multisymplectic conservation law*

$$\frac{1}{2} \nabla \times [dU \dot{\wedge} dH - dV \dot{\wedge} dE] + \frac{\partial}{\partial t} (dV \dot{\wedge} dP + dU \dot{\wedge} dQ) = 0. \quad (20)$$

Now we return to the second case introduced in Section 2. System (20) can be written in the multisymplectic Hamiltonian form

$$\begin{aligned} & \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} H \\ E \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} \mathcal{R}_1 & 0 \\ 0 & \frac{1}{\mu} \mathcal{R}_1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} H \\ E \end{pmatrix} \\ & + \begin{pmatrix} \frac{1}{\varepsilon} \mathcal{R}_2 & 0 \\ 0 & \frac{1}{\mu} \mathcal{R}_2 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} H \\ E \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} \mathcal{R}_3 & 0 \\ 0 & \frac{1}{\mu} \mathcal{R}_3 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} H \\ E \end{pmatrix} = \begin{pmatrix} J/\varepsilon \\ -K/\mu \end{pmatrix}, \end{aligned} \quad (21)$$

where the covariant Hamiltonian is $S = \frac{1}{\varepsilon} \langle J, H \rangle - \frac{1}{\mu} \langle K, E \rangle$. It has the multisymplectic conservation law

$$\nabla \times \left[\frac{1}{\varepsilon} dH \dot{\wedge} dH + \frac{1}{\mu} dE \dot{\wedge} dE \right] + \frac{\partial}{\partial t} (dE \dot{\wedge} dH) = 0. \quad (22)$$

4. A Multisymplectic Scheme for Maxwell's Equations

Maxwell's equations are simulated by many methods. Its first numerical simulation is given by Yee [7] in 1966, which has been improved by more efficient algorithms. In this section, we adopt a multisymplectic scheme to approximate the system. For simplicity, we consider a 1 + 1-dimensional multisymplectic form of (15), and suppose that the external magnetic-current vanished, that is, $K = 0$ and the rotation operator becomes

$$\nabla \times Z = \mathcal{R}_1 Z_x. \quad (23)$$

4.1. A Multisymplectic Scheme

We discretize the equations (15) by using the midpoint scheme in both t and x directions,

$$\begin{aligned}
\mathcal{R}_1 \frac{U_{i+1j+\frac{1}{2}} - U_{ij+\frac{1}{2}}}{2\Delta x} &= P_{i+\frac{1}{2}j+\frac{1}{2}} - \mu_{i+\frac{1}{2}} H_{i+\frac{1}{2}j+\frac{1}{2}}, \\
-\mathcal{R}_1 \frac{V_{i+1j+\frac{1}{2}} - V_{ij+\frac{1}{2}}}{2\Delta x} &= Q_{i+\frac{1}{2}j+\frac{1}{2}} - \varepsilon_{i+\frac{1}{2}} E_{i+\frac{1}{2}j+\frac{1}{2}}, \\
-\frac{P_{i+\frac{1}{2}j+1} - P_{i+\frac{1}{2}j}}{\Delta t} - \mathcal{R}_1 \frac{E_{i+1j+\frac{1}{2}} - E_{ij+\frac{1}{2}}}{2\Delta x} &= 0, \\
-\frac{Q_{i+\frac{1}{2}j+1} - Q_{i+\frac{1}{2}j}}{\Delta t} + \mathcal{R}_1 \frac{H_{i+1j+\frac{1}{2}} - H_{ij+\frac{1}{2}}}{2\Delta x} &= J_{i+\frac{1}{2}j+\frac{1}{2}}, \\
\frac{V_{i+\frac{1}{2}j+1} - V_{i+\frac{1}{2}j}}{\Delta t} &= H_{i+\frac{1}{2}j+\frac{1}{2}}, \\
\frac{U_{i+\frac{1}{2}j+1} - U_{i+\frac{1}{2}j}}{\Delta t} &= E_{i+\frac{1}{2}j+\frac{1}{2}}.
\end{aligned} \tag{24}$$

Proposition 6. *The scheme (24) is a multisymplectic scheme.*

Proof. We can easily show that the obtained scheme preserves the multisymplectic conservation law (19) in discrete space, that is,

$$\frac{\kappa_{i+1j+\frac{1}{2}} - \kappa_{ij+\frac{1}{2}}}{2\Delta x} + \frac{\omega_{i+\frac{1}{2}j+1} - \omega_{i+\frac{1}{2}j}}{\Delta t} = 0, \tag{25}$$

where

$$\begin{aligned}
\kappa_{ij+\frac{1}{2}} &= \mathcal{R}_1 dU_{ij+\frac{1}{2}} \wedge dH_{ij+\frac{1}{2}} - \mathcal{R}_1 dV_{ij+\frac{1}{2}} \wedge dE_{ij+\frac{1}{2}}, \\
\omega_{i+\frac{1}{2}j} &= dV_{i+\frac{1}{2}j+1} \wedge dP_{i+\frac{1}{2}j+1} + dU_{i+\frac{1}{2}j+1} \wedge dQ_{i+\frac{1}{2}j+1}.
\end{aligned}$$

Eliminating the auxiliary variables V, U, P and Q from the Preissman scheme we get a nine-point multisymplectic integrator

$$\begin{aligned}
\mathcal{R}_1 \frac{E_{i+2j+2} + 2E_{i+2j+1} + E_{i+2j} - E_{ij+2} - 2E_{ij+1} - E_{ij}}{4\Delta x} \\
= -\mu_{i+3/2} \frac{H_{i+2j+2} + H_{i+1j+2} - H_{i+2j} - H_{i+1j}}{4\Delta t} \\
- \mu_{i+1/2} \frac{H_{i+1j+2} + H_{ij+2} - H_{i+1j} - H_{ij}}{4\Delta t}, \tag{26}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_1 & \frac{H_{i+2j+2} + 2H_{i+2j+1} + H_{i+2j} - H_{ij+2} - 2H_{ij+1} - H_{ij}}{4\Delta x} \\
& = \varepsilon_{i+3/2} \frac{E_{i+2j+2} + E_{i+1j+2} - E_{i+2j} - E_{i+1j}}{4\Delta t} \\
& \quad + \varepsilon_{i+1/2} \frac{E_{i+1j+2} + E_{ij+2} - E_{i+1j} - E_{ij}}{4\Delta t} \\
& \quad + (J_{i+3/2j+3/2} + J_{i+3/2j+1/2} + J_{i+1/2j+3/2} + J_{i+1/2j+1/2})/2. \quad \square
\end{aligned}$$

Proposition 7. *The numerical procedures can be described by the following expression*

$$AZ^{j+2} = BZ^{j+1} + CZ^j + \hat{J},$$

where

$$A = \begin{pmatrix} D & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ B_0 & D & A_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_1 & D & A_2 & \cdots & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & B_{s-1} & D & A_s \\ 0 & 0 & 0 & 0 & \cdots & 0 & B_s & D \\ & & \ddots & & & & & \end{pmatrix},$$

$$C = \begin{pmatrix} -D & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ B_0 & -D & A_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_1 & -D & A_2 & \cdots & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & B_{s-1} & -D & A_s \\ 0 & 0 & 0 & 0 & \cdots & 0 & B_s & -D \\ & & \ddots & & & & & \end{pmatrix},$$

$$B = \text{diag}\{-2D, \dots, -2D\} \quad \hat{J} = \begin{pmatrix} a(0) + b(0) \\ a(1) + b(1) \\ \vdots \end{pmatrix},$$

with $a(i) = J_{i+\frac{1}{2}j+\frac{1}{2}} + J_{i+\frac{3}{2}j+\frac{1}{2}}$, $b(i) = J_{i+\frac{1}{2}j+\frac{3}{2}} + J_{i+\frac{3}{2}j+\frac{3}{2}}$, and

$$D = \begin{pmatrix} 0 & \Delta t \mathcal{R}_1 & & & \\ -\Delta t \mathcal{R}_1 & 0 & \Delta t \mathcal{R}_1 & & \\ & -\Delta t \mathcal{R}_1 & 0 & \Delta t \mathcal{R}_1 & \\ & & \ddots & \ddots & \ddots \\ & & & -\Delta t \mathcal{R}_1 & 0 \end{pmatrix}.$$

Set $r = 1, 2, \dots$, then for every $i = 1, 2, \dots, s, \dots$, there are

$$A_i = \begin{pmatrix} e_0 + e_1 & e_1 & 0 & 0 \cdots & 0 & 0 & 0 \\ e_1 & e_1 + e_2 & e_2 & 0 \cdots & 0 & 0 & 0 \\ 0 & e_2 & e_2 + e_3 & e_3 \cdots & 0 & 0 & 0 \\ & & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 \cdots & e_{r-2} & e_{r-2} + e_{r-1} & e_{r-1} \\ 0 & 0 & 0 & 0 \cdots & 0 & e_{r-1} & e_{r-1} + e_r \end{pmatrix},$$

here $e_j = \Delta x \varepsilon_{j+ir+\frac{1}{2}} I \in R^{3 \times 3}$, $j = 1, 2, \dots, r$. B_i is defined similarly, change e_j to $u_j = \Delta x \mu_{j+ir+\frac{1}{2}} I \in R^{3 \times 3}$.

4.2. Numerical Test

To check the integrator (26), we consider a simple example. We choose $\varepsilon = \mu = 1$ and $J = 0$. Furthermore, we take initial conditions as

$$\begin{aligned} H_1(x, 0) = H_3(x, 0) = 0, & \quad H_2(x, 0) = -\sqrt{\frac{\varepsilon}{\mu}} \sin(x), \\ E_1(x, 0) = E_2(x, 0) = 0, & \quad E_3(x, 0) = \sin(x). \end{aligned}$$

With these initial conditions we can get an exact solution

$$\begin{aligned} H_1 = H_3 = 0, & \quad H_2(x, t) = -\sqrt{\frac{\varepsilon}{\mu}} \sin\left(x - \sqrt{\frac{1}{\varepsilon\mu}} t\right), \\ E_1 = E_2 = 0, & \quad E_3(x, t) = \sin\left(x - \sqrt{\frac{1}{\varepsilon\mu}} t\right). \end{aligned}$$

The function H_2 is computed in three rectangles $[0, 2\pi + 3] \times [0, 0.1]$, $[0, 2\pi + 3] \times [0, 1]$, and $[0, 2\pi + 3] \times [0, 10]$ using the grid sizes $\Delta x = (2\pi + 3)/61$ and $\Delta t = 0.01$. Figure 1, Figure 2 and Figure 3 show the numerical result in the first, second and third rectangle, respectively. We see that the solution is positively moving to the x -direction. In Figure 3, the trend of the solution is more manifest.

In Figure 4 the absolute error after 1000 time steps is given by the “error-line”, and the “exact solution-line” is the graph of $\Delta t H_2$, which has a very regular route like the solution graph H_2 . Their vertexes are almost at the zero points of each other. The regular similarity gives ideas on how one could improve the scheme, which will be investigated in future projects.

For the multisymplectic form (21) there exist many schemes to simulate the Maxwell’s equations and to preserve the multisymplectic conservation law (22),

Fig.1: t-number=20

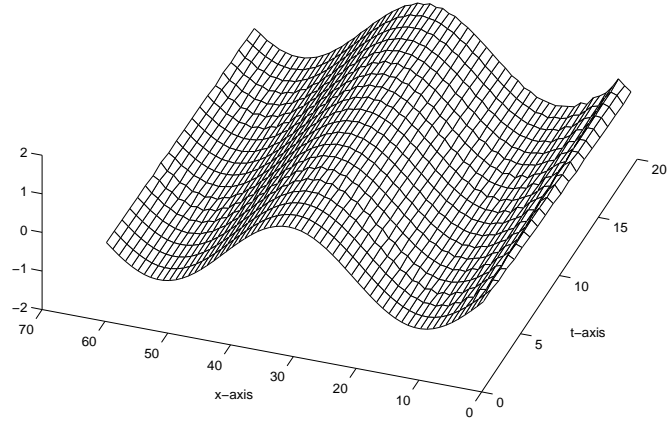


Figure 1: 20 time steps

Fig.2: t-number=100

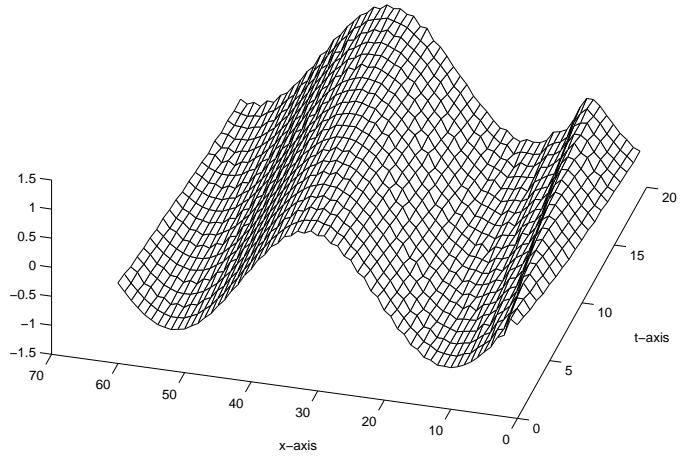


Figure 2: 100 time steps

such as the leap-frog scheme, which is not presented here. For cases of more general ε and μ , the discussion for the multisymplecticity is similar.

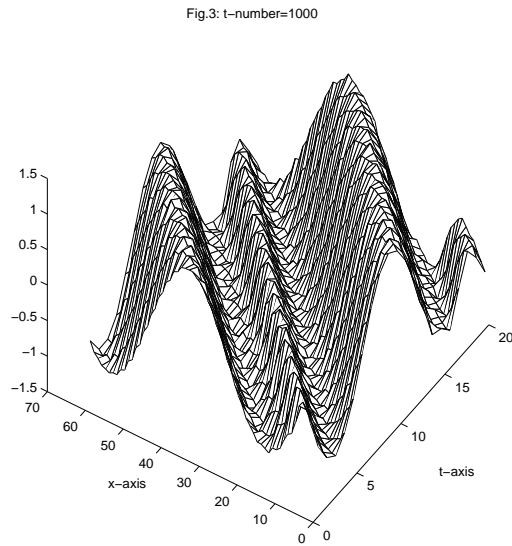


Figure 3: 1000 time steps

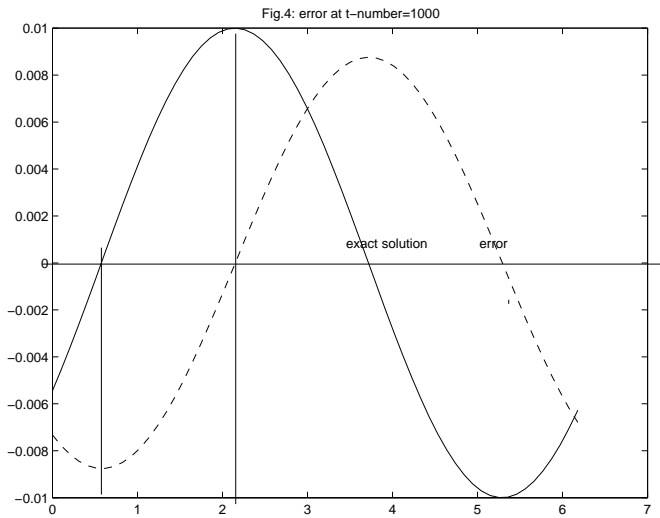


Figure 4: Error after 1000 time steps

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