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SOME SUBCLASSES OF BOUNDE FUNCTIONS OF THE CLASS OF $\alpha-$CONVEX FUNCTIONS

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Abstract: In this paper we determine some classes of bounded functions, subsets of the class of $\alpha-$convex functions, using the properties of Hardy spaces.

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1. Introduction

Let $\mathcal{H}(U)$ denote the class of holomorphic functions in the unit disk U of the complex plane $\mathbb{C}$ and

$$\mathcal{A} = \left\{ f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + ... \right\}$$

Let S denote the subset of $\mathcal{A}$ consisting of univalent functions, S^* and K denote the subsets of S consisting of starlike and convex functions, respectively.

In 1969 P.T. Mocanu, combined the conditions of starlikeness and convexity, introduced the concept of $\alpha-$convexity, see [5]. Let α be a real number and $f \in \mathcal{A}$ is analytic in U, then f is said to be $\alpha-$convex functions if and only if

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)}\right] \geq 0, \ \ z \in U\,.$$

Let M_α denote the class of $\alpha-$convex functions. It is obvious that $M_0 = S^*$

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and $M_1 = K$.

Furthermore it can be shown that if $\alpha, \beta \in \mathbb{R}$ so that $0 \leq \frac{\beta}{\alpha} < 1$ then $M_\alpha \subset M_\beta$. In particular, $M_\alpha \subset S^*$, $\alpha \in \mathbb{R}$.

For $0 \leq \beta < 1$, we shall denote by $K(\beta)$ the class of convex functions satisfying the inequality

$$Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \beta, \ \ z \in U,$$

in other words, $K(\beta)$ is the class of convex functions of order β. Then, for $0 \leq \alpha \leq \beta < 1$ we have the inclusions

$$K(\beta) \subseteq K(\alpha) \subseteq K(0) = K.$$

In this paper we determine some subclasses of $\alpha-$convex functions which are bounded functions in the unit disk U, using the properties of Hardy spaces.

2. Preliminaries

For f analytic in U and $z = re^{i\theta}$, we denote

$$M_p(r, f) = \begin{cases} \left(\dfrac{1}{2^\pi}\displaystyle\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}}, & 0 < p < \infty\,, \\ \sup\limits_{0\leq\theta\leq 2\pi} \left|f\left(re^{i\theta}\right)\right|, & \text{for } p = \infty\,. \end{cases}$$

A function f analytic in U is said to be of Hardy space H^p, $0 < p \leq \infty$, if $M_p(r, f)$ remains bounded as $r \to 1^-$. H^∞ is the space of bounded analytic functions in the unit disk.

We shall need the following lemmas to prove our results.

Lemma 2.1. (see [5]) *The functions f is in M_α, $\alpha > 0$, if and only if there exists a starlike functions F such that*

$$f(z) = \left[\frac{1}{\alpha}\int_0^z \left[F(t)\right]^{\frac{1}{\alpha}} t^{-1} dt\right]^\alpha. \tag{2.1}$$

Lemma 2.2. (see [1]) *If $f \in H^p$, and $F(z) = \int_0^z f(t)dt$, then:*
(i) $F \in H^{\frac{p}{1-p}}$, for $0 < p < 1$;
(ii) $F \in H^\infty$, for $p \geq 1$.

Lemma 2.3. (see [3]) *If $f \in M_\alpha$ then:*
(i) $f \in H^\infty$, for $|\alpha| > 2$;
(ii) if $f \neq k_\tau^\alpha$, where

$$k_\tau^\alpha(z) = \begin{cases} \left(\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}\left(1-e^{i\tau}t\right)^{-\frac{2}{\alpha}}dt\right)^\alpha, \alpha > 0\,, \\ \frac{z}{\left(1+e^{i\tau}z\right)^2}, \qquad -2 \leq \alpha \leq 0\,, \end{cases} \tag{2.2}$$

then $f \in H^p$, where

$$p = \begin{cases} \infty, \text{ if } |\alpha| = 2\,, \\ \frac{1}{2-\alpha} + \varepsilon, \text{ if } 0 \leq \alpha < 2\,, \\ \frac{1}{2} + \varepsilon, \text{ if } -2 < \alpha < 0\,, \end{cases}$$

and $\varepsilon = \varepsilon(f) > 0$;
(iii) $k_\tau^\alpha \in H^p$ for all $p < p(\alpha)$, where

$$p(\alpha) = \begin{cases} \infty, \text{ for } \alpha = 2\,, \\ \frac{1}{2-\alpha}, \text{ for } 0 \leq \alpha < 2\,, \\ \frac{1}{2}, \text{ for } -2 \leq \alpha < 0\,. \end{cases}$$

Lemma 2.4. (see [2]) *If $f \in K(\beta)$ is not of the form*

$$f(z) = a + b(1 - ze^{i\tau})^{2\beta-1}, \quad \beta \neq \frac{1}{2}\,,$$
$$f(z) = a + b\log(1 - ze^{i\tau}), \quad \beta = \frac{1}{2}\,,$$

for some complex a, b and real τ, then:
(i) if $0 \leq \beta < \frac{1}{2}$, then there exists $\varepsilon = \varepsilon(f) > 0$ such that $f \in H^{\frac{1}{1-2\beta}+\varepsilon}$;
(ii) if $\beta \geq \frac{1}{2}$, then $f \in H^\infty$.

3. Main Results

Theorem 3.1. *If $f \in M_\alpha$ then:*
(i) if $|\alpha| > 2$, then f is bounded;
(ii) If $|\alpha| = 2$ and $f \neq k_\tau^\alpha$, where k_τ^α is defined by (2.2) then f is bounded.

Proof. (i) From Lemma 2.3 we have that if $f \in M_\alpha,\ |\alpha| > 2$ then $f \in H^\infty$. Hence we obtain that f is bounded

(ii) If $f \in M_\alpha$ with $|\alpha| > 2$ and $f \neq k_\tau^\alpha$ then $f \in H^\infty$. Namely f is bounded. □

In other words, M_α is a class of bounded functions for all α, $|\alpha| > 2$, and for $|\alpha| = 2$ it is also a class of bounded functions, without the Koebe functions.

For $\alpha > 2$ this result was obtained by S.S. Miller [4] using distortion properties.

In the following results we suppose that if $f_1, f_2 \in M_\alpha$ then exists $f_1 \circ f_2$.

Theorem 3.2. *If $1 \leq \alpha_i < 2,\ f_i \in M_{a_i},\ f_i \neq k_\tau^{\alpha_i},\ i \in \{1,2\}$ ($k_\tau^{\alpha_i}$ given by (2.2)), then the composite function $f_1 \circ f_2$, if it exists, is α_1-convex and bounded.*

Proof. Let $f_i \in M_{a_i},\ 1 \leq \alpha_i < 2,\ i \in \{1,2\}$, then from Lemma 2.1, there exists $F_i \in S^*$ such that:

$$f_i(z) = I_i\,(z) = I_i(F_i)(z) = \left(\frac{1}{\alpha_i}\int_0^z [F_i(t)]^{\frac{1}{\alpha_i}} t^{-1} dt\right)^{\alpha_i}.$$

Hence we have:

$$(f_1 \circ f_2)\,(z) = (I_1 \circ I_2)\,(z) = \left(\frac{1}{\alpha_1}\int_0^z [I_2(t)]^{\frac{1}{\alpha_1}} t^{-1} dt\right)^{\alpha_1}.$$

It is well-known that all α-convex functions are univalent and starlike. Hence we have that $I_2\,(z)$ is starlike. By Lemma 2.1 we obtain that

$$\left(\frac{1}{\alpha_1}\int_0^z [I_2(t)]^{\frac{1}{\alpha_1}} t^{-1} dt\right)^{\alpha_1} \quad \text{is } \alpha_1\text{-convex.}$$

That is $f_1 \circ f_2$ is α_1-convex.

Now, we will prove that $f_1 \circ f_2$ is bounded.

Since $I_2(F_2) \in M_{\alpha_2}$ then, from Lemma 2.3, we have that $I_2(F_2) \in H^{\frac{1}{2-\alpha_2}}$. Hence $[I_2(F_2)]^{\frac{1}{\alpha_1}} \in H^{\frac{\alpha_1}{2-\alpha_2}}$. It is easy to remark that $f(z)$ and $z^{-1}f(z)$ belong to the same Hardy space. Then

$$[I_2(F_2)(t)]^{\frac{1}{\alpha_1}}\, t^{-1} \in H^{\frac{\alpha_1}{2-\alpha_2}}.$$

Applying Lemma 2.2 we obtain:

$$\int_0^z [I_2(F_2)]^{\frac{1}{\alpha_1}}\,(t)\, t^{-1} dt \in H^p,$$

where $p=\infty$, because $\frac{\alpha_1}{2-\alpha_2}\geq 1$. Hence we have

$$\left(\frac{1}{\alpha_1}\int_0^z [I_2(F_2)]^{\frac{1}{\alpha_1}}(t)t^{-1}dt\right)^{\alpha_1}\in H^\infty .$$

Therefore $I_1(F_1)\circ I_2(F_2)$ is bounded, namely $f_1\circ f_2$ is bounded. □

Remark 3.1. Analogously, we can show that $f_2\circ f_1$ is bounded.

In particular, for all $f\in M_\alpha,\ \alpha\in[1,2),\ f\circ f$ is bounded.

Corollary 3.1. *If an α-convex function, $\alpha\in[1,2)$, can be written as a composite function of two α-convex functions $f_1,f_2\in[1,2)$, then f is bounded.*

We have seen before that for $\alpha=1,\ M_1=K,\ K$ is the class of convex functions. This class has a important subclass of bounded functions.

Theorem 3.3. *If $\frac{1}{2}\leq\beta<1$ and $f\in K(\beta)$ is not of the form*

$$f(z)=a+b(1-ze^{i\tau})^{2\beta-1},\quad \beta\neq\frac{1}{2},$$
$$f(z)=a+b\log(1-ze^{i\tau}),\quad \beta=\frac{1}{2},$$

$a,b\in\mathbb{C},\ \tau\subset\mathbb{R}$, *then f is bounded.*

Proof. If $f\in K(\beta)$ and $\beta\geq\frac{1}{2}$ then from Lemma 2.4 we have that $f\in H^\infty$. In other words, f is bounded. □

Hence we have that $K(\beta)$ is a class of bounded functions for $\frac{1}{2}\leq\beta<1$.

Theorem 3.4. *If $\alpha_i\in(0,1),\ f_i\in M_{\alpha_i},\ f_i\neq k_\tau^{\alpha_i},\ i\in\{1,2,...,n\}$ and if*

$$\alpha_1+\alpha_2+...+\alpha_{n-1}<2 \text{ and } \alpha_1+\alpha_2+...+\alpha_n\geq 2,$$

then

$$f_n\circ f_{n-1}\circ...\circ f_2\circ f_1$$

is bounded.

Proof. Let $f_i\in M_{\alpha_i},\ \alpha_i\in(0,1)$, then exists $F_i\in S^*$ such that:

$$f_i(z)=I_i(F_i)(z)=\left(\frac{1}{\alpha_i}\int_0^z F_i^{\frac{1}{\alpha_i}}(t)\,t^{-1}dt\right)^{\alpha_i},$$

For $f_1\in M_{\alpha_1}$ and $f_2\in M_{\alpha_2}$ we have

$$f_2\circ f_1\in H^{\frac{1}{2-(\alpha_1+\alpha_2)}} .$$

Indeed, $f_1 \in M_{\alpha_1}$ implies $f_1 \in H^{\frac{1}{2-\alpha_1}}$. Namely $I_1(F_1) \in H^{\frac{1}{2-\alpha_1}}$. Hence

$$\left[I_1(F_1)^{\frac{1}{\alpha_2}}\right] \in H^{\frac{\alpha_2}{2-\alpha_1}}$$

and

$$[I_1(F_1(t))]^{\frac{1}{\alpha_2}}\, t^{-1} \in H^{\frac{\alpha_2}{2-\alpha_1}}.$$

From Lemma 2 we have that:

$$\int_0^z [I_1(F_1(t))]^{\frac{1}{\alpha_2}}\ t^{-1}dt \in H^p,$$

when

$$p = \frac{\dfrac{\alpha_2}{2-\alpha_1}}{1-\dfrac{\alpha_2}{2-\alpha_1}} = \frac{\alpha_2}{2-(\alpha_1+\alpha_2)}.$$

Hence

$$\left(\frac{1}{\alpha_2}\int_0^z [I_1(F_1(t))]^{\frac{1}{\alpha_2}}\, t^{-1}dt\right)^{\alpha_2} \in H^p,$$

where

$$p = \frac{1}{2-(\alpha_1+\alpha_2)}\,.$$

Suppose that

$$(f_{n-1}\circ f_{n-2}\circ \ldots \circ f_2 \circ f_1) \in H^p\,,$$

where

$$p = \frac{1}{2-(\alpha_1+\alpha_2+\ldots+\alpha_{n-1})}$$

and denote with

$$G = f_{n-1}\circ f_{n-2}\circ \ldots \circ f_2 \circ f_1.$$

Then

$$(f_n \circ G)(z) = \left(I_n(F_n)\circ G\right)(z) = I_n(G)(z) = \left(\frac{1}{\alpha_n}\int_0^z G^{\frac{1}{\alpha_n}}(t)t^{-1}dt\right)^{\alpha_n}.$$

Since $G \in H^p$, then

$$G^{\frac{1}{\alpha_n}} \in H^{\alpha_n p},\ \alpha_n p = \frac{\alpha_n}{2-(\alpha_1+\alpha_2+\ldots+\alpha_{n-1})}\,.$$

Also $[G(t)]^{\frac{1}{\alpha_n}}\, t^{-1} \in H^{\alpha_n p}$. From Lemma 2.2 we obtain

$$\int_0^z G^{\frac{1}{\alpha_n}}(t)t^{-1}dt \in H^p,$$

where

$$p = \begin{cases} \dfrac{1}{2-(\alpha_1+\alpha_2+...+\alpha_n)}, & \text{if } \alpha_1\alpha_2+...+\alpha_n < 2\,, \\ \infty\,, & \text{if } \alpha_1+\alpha_2+...+\alpha_n \geq 2\,. \end{cases}$$

From hypothesis we have that $\alpha_1+\alpha_2+...+\alpha_n \geq 2$ and we obtain $p=\infty$. Hence

$$\left(\frac{1}{\alpha_n}\int_0^z G^{\frac{1}{\alpha_n}}(t)t^{-1}dt\right)^{\alpha_n} \in H^\infty\,.$$

Therefore $I_n(F_n)\circ G$ is bounded and $f_n\circ f_{n-1}\circ ...\circ f_2\circ f_1$ is bounded. □

For $\alpha_1=\alpha_2=...=\alpha_n=\alpha$ and $f_1=f_2=...=f_n=f$ we obtain the following result.

Corollary 3.2. *If $\alpha\in(0,1)$, $f\in M_\alpha$, $f\neq k_\tau^\alpha$ then there exists $n\in\mathbb{N}^*$ such that $\underbrace{f\circ f\circ\ldots\circ f}_{n}$ is bounded. Indeed, if f is bounded then $n=1$. If $f\in M_\alpha$ and f is unbounded then $f\in H^{\frac{1}{2-\alpha}}$. From Theorem 3.4 we have that*

$$\underbrace{f\circ f\circ\ldots\circ f}_{n} \in H^p,\ \text{where } p = \begin{cases} \dfrac{1}{2-n\alpha}, & \text{if } 2 > n\alpha\,, \\ \infty, & \text{if } 2\leq n\alpha\,. \end{cases}$$

Since $\alpha\in(0,1)$ then there exists $n\in\mathbb{N}$ such that

$$\frac{2}{n}\leq\alpha<\frac{2}{n-1}\,.$$

In this case

$$\underbrace{f\circ f\circ\ldots\circ f}_{n}\in H^\infty,$$

therefore $\underbrace{f\circ f\circ\ldots\circ f}_{n}$ is bounded.

References

[1] L.P. Duren, *Theory of H^p Spaces*, Academic Press, New York and London (1970).

[2] P.J. Eenigenburg, F.R. Keogh, The Hardy class of some univalent functions and their derivatives, *Michigan Math. J.*, **17** (1970), 335-346.

[3] P.J. Eenigenburg, S.S. Miller, The H^p classes for $\alpha-$convex functions, *Proc. Amer. Math. Soc.*, **38**, No. 3 (1973), 558-562.

[4] S.S. Miller, Distorsion properties of alpha-starlike functions, *Proc. Amer. Math. Soc.*, **38**, No. 2 (1973), 311-318.

[5] P.T. Mocanu, Une propriété de convexité généralisée dans la théorie de la représentation conforme, *Mathematica, Cluj.*, **11**, No. 34 (1969), 127-133.