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THE ACT ON THE DUAL NUMBERS

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Abstract: In this paper, we study that dual number set is a Lie group under the addition of the dual numbers. Furthermore we define act on the dual numbers and we get $\widehat{A}$ which is obtained from the motion of screw by using tangent of the curve.

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1. Introduction

In modern mathematics dual numbers turn up if one poses the problem to find all algebras of rank 2 over a (commutative) field with a characteristic $p \neq 2$. It appears that there are three solutions. One of these is the algebra consisting of the dual elements $a + \varepsilon a^*$ over F and $\varepsilon^2 = 0$. If F is taken to be the field of real numbers, these elements are called dual numbers. Long before abstract algebra was included in the curricula of all mathematical departments, dual numbers had been introduced by William Kingdon Clifford (1845-1879) as o tool for his geometrical investigations. After him, E. Study used dual numbers and dual vectors in research on line geometry and kinematics [8], he devoted special attention to the representation of spears by dual unit vectors. In recent times several authors have been using dual quantities in their investigations

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concerning analysis and synthesis of spatial mechanisms, see [4].

In the Euclidean space E^n of n-dimensions, W. Clifford and James J. McMahon have given a treatment of a rigid body's motion generated by the most general one parameter affine transformation, see [3]. Another treatment was given by H.R. Müller for the same kind of motion, see [7].

1.1. Dual Numbers

A dual numbers can be defined as an ordered pair combining a real part, a and a dual part a^*

$$\widehat{a} = a + \varepsilon a^* ,$$

where ε is the dual with multiplication rule $\varepsilon^2 = 0$. The algebra of dual numbers results from this definition. Two dual numbers are equal if and only if their real and dual parts are equal, respectively. Addition of two dual numbers requires separate addition of their real and dual parts:

$$(a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*).$$

Multiplication of two dual numbers result in,

$$(a + \varepsilon a^*) \,.\, (b + \varepsilon b^*) = ab + \varepsilon(a^* b + ab^*) \,.$$

Hacısalihoğlu (see [5]) presented the definition of division of dual number as follows:

$$\frac{\widehat{a}}{\widehat{b}} = \frac{a}{b} + \varepsilon(\frac{a^*}{b} - \frac{ab^*}{b^2}), \quad b \neq 0.$$

1.2. Dual Vectors and Matrices

An ordered triple of dual numbers $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ is called a dual vector, we write $\widehat{X} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$. The numbers $\widehat{x}_1, \widehat{x}_2, \widehat{x}_3$ are called the coordinates of $\widehat{X}$.

Let $\widehat{X} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ and $\widehat{Y} = (\widehat{y}_1, \widehat{y}_2, \widehat{y}_3)$ be two dual vectors. $\widehat{X} = \widehat{Y}$ if and only if $\widehat{x}_i = \widehat{y}_i$ $(i = 1, 2, 3)$.

Let $\widehat{\lambda}$ be a dual scalar. Multiplication by a dual scalar of dual vector $\widehat{X}$ is defined by

$$\widehat{\lambda}\widehat{X} = \left(\widehat{\lambda}\widehat{x}_1, \widehat{\lambda}\widehat{x}_2, \widehat{\lambda}\widehat{x}_3\right).$$

Inner product and cross-product of two dual vectors are defined as follows:

$$\left\langle \widehat{X}, \widehat{Y} \right\rangle = \widehat{x}_1\widehat{y}_1 + \widehat{x}_2\widehat{y}_2 + \widehat{x}_3\widehat{y}_3 \,,$$

$$\widehat{X} \times \widehat{Y} = (\widehat{x}_2\widehat{y}_3 - \widehat{x}_3\widehat{y}_2, \widehat{x}_3\widehat{y}_1 + \widehat{x}_1\widehat{y}_3, \widehat{x}_1\widehat{y}_2 + \widehat{x}_2\widehat{y}_1) ,$$

respectively (see [8]).

If $\widehat{X} \neq 0$ the norm $\left\|\widehat{X}\right\|$ of $\widehat{X}$ is defined by $\left\langle \widehat{X}, \widehat{X} \right\rangle^{\frac{1}{2}}$.

The elements of matrix $\widehat{A}$, which are dual numbers, is called a dual matrix:

$$\widehat{A} = (\widehat{a}_{ij}) = \left(a_{ij} + \varepsilon a^*_{ij}\right) .$$

If $\widehat{A}\widehat{A}^T = \widehat{A}^T\widehat{A} = I$, then $\widehat{A}$ is a dual orthogonal matrix, where I stands for the unit matrix.

Theorem 1.1. *Let D be the set of dual numbers. Then D is a Lie group under the addition operation.*

Proof. Let D be the set of dual numbers, i.e. $D = \{\widehat{a} : \widehat{a} = a + \varepsilon a^*, a, a^* \in \mathbb{R}\}$. Then $(D, +)$ is a group. The map

$$\Psi : (D, +) \to (\mathbb{R}^2, +)$$

is a isomorphism. Thus, D is a differentiable manifold. The map

$$\alpha : D \times D \to D , \qquad (\widehat{a}, \widehat{b}) \to \widehat{a} + \left(-\widehat{b}\right)$$

is differentiable. Then D is a Lie group.

Let D be a vector space over field $\mathbb{R}$. Then it is a Lie algebra and denote is d. Indeed, Lie bracket operator which defined by

$$[,] : D \times D \to D , \qquad (\widehat{a}, \widehat{b}) \to \left[\widehat{a}, \widehat{b}\right] = \widehat{a}\widehat{b} - \widehat{b}\widehat{a}$$

Then D is Lie algebra. In addition it is Abelian because it is $\left[\widehat{a}, \widehat{b}\right] = 0$. □

Theorem 1.2. *Let D be a Lie group. Then it is acts on itself on the left by automorphism*

$$\mu : D \times D \to D , \qquad (\widehat{a}, \widehat{b}) \to \mu(\widehat{a}, \widehat{b}) = \widehat{a} + \widehat{b} + (-\widehat{a}) .$$

Proof. For $0 \in D$ and all $\widehat{a}$, $\widehat{b}$, $\widehat{c} \in D$:

(i) $\mu(0, \widehat{a}) = \widehat{a}$.

(ii) $\mu(\widehat{a} + \widehat{b}, \widehat{c}) = \left(\widehat{a} + \widehat{b}\right) + \widehat{c} + (-\left(\widehat{a} + \widehat{b}\right)) = C$.

and

$$\mu(\widehat{a}, \mu(\widehat{b}, \widehat{c})) = \widehat{a} + \mu\left(\widehat{b}, \widehat{c}\right) + (-\widehat{a}) = \widehat{a} + (\widehat{b} + \widehat{c} + (-\widehat{b})) + (-\widehat{a}) = \widehat{c}.$$

Thus:

$$\mu(\widehat{a}+\widehat{b},\widehat{c}) = \mu(\widehat{a},\mu(\widehat{b},\widehat{c}))\,. \qquad \square$$

Theorem 1.3. *Let A be orthogonal matrix and β be a curve on A. Then the motion of screw using by tangent of the curve is defined as follows*

$$\widehat{A} = \begin{bmatrix} 1 & \varepsilon t_3 & -\varepsilon t_2 \\ \varepsilon\,(t_2\sin\theta - t_3\cos\theta) & \cos\theta - \varepsilon t_1\sin\theta & \sin\theta + \varepsilon t_1\cos\theta \\ \varepsilon\,(t_2\cos\theta + t_3\sin\theta) & -\sin\theta - \varepsilon t_1\cos\theta & \cos\theta - \varepsilon t_1\sin\theta \end{bmatrix}.$$

Proof. Let A be defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

$T\,(t_1,t_2,t_3)$ is tangent of β. For $a_1 = (1,0,0)$, $a_2 = (0,\cos\theta,\sin\theta)$, $a_3 = (0,-\sin\theta,\cos\theta)$, we obtain:

$$a_1^* = T\times a_1 = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ t_1 & t_2 & t_3 \\ 1 & 0 & 0 \end{vmatrix} = \overrightarrow{j}\,t_3 - \overrightarrow{k}\,t_2 = (0,t_3,-t_2)\ ,$$

$$\begin{aligned} a_2^* &= T\times a_2 \\ &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ t_1 & t_2 & t_3 \\ 0 & \cos\theta & \sin\theta \end{vmatrix} = \overrightarrow{i}\,(t_2\sin\theta - t_3\cos\theta) - \overrightarrow{j}\,t_1\sin\theta + \overrightarrow{k}\,t_1\cos\theta\,, \end{aligned}$$

$$\begin{aligned} a_3^* = T\times a_3 &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ t_1 & t_2 & t_3 \\ 0 & -\sin\theta & \cos\theta \end{vmatrix} \\ &= \overrightarrow{i}\,(t_2\cos\theta + t_3\sin\theta) - \overrightarrow{j}\,t_1\cos\theta - \overrightarrow{k}\,t_1\sin\theta\,. \end{aligned}$$

Then

$$A^* = \begin{bmatrix} 1 & t_3 & -t_2 \\ t_2\sin\theta - t_3\cos\theta & -t_1\sin\theta & t_1\cos\theta \\ t_2\cos\theta + t_3\sin\theta & -t_1\cos\theta & -t_1\sin\theta \end{bmatrix}.$$

Thus we have

$$\widehat{A} = A + \varepsilon A^*$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & t_3 & -t_2 \\ t_2\sin\theta - t_3\cos\theta & -t_1\sin\theta & t_1\cos\theta \\ t_2\cos\theta + t_3\sin\theta & -t_1\cos\theta & -t_1\sin\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \varepsilon t_3 & -\varepsilon t_2 \\ \varepsilon\left(t_2\sin\theta - t_3\cos\theta\right) & \cos\theta - \varepsilon t_1\sin\theta & \sin\theta + \varepsilon t_1\cos\theta \\ \varepsilon\left(t_2\cos\theta + t_3\sin\theta\right) & -\sin\theta - \varepsilon t_1\cos\theta & \cos\theta - \varepsilon t_1\sin\theta \end{bmatrix}.$$

Definition 1.1. Let A be orthogonal 3×3 matrix and A^* be defined by

$$A^* = \begin{bmatrix} a_1^* \\ a_2^* \\ a_3^* \end{bmatrix}.$$

Then

$$OD(3) = \left\{\widehat{A} : \widehat{A} = A + \varepsilon A^*, A, A^* \in \mathbb{R}_3^3\right\}$$

is called orthogonal dual matrix space.

Theorem 1.4. *Let $OD(3)$ be orthogonal dual matrix space. Then it is a Lie group under the addition operator.*

Proof. Let

$$OD(3) = \left\{\widehat{A} : \widehat{A} = A + \varepsilon A^*, A, A^* \in \mathbb{R}_3^3\right\}$$

be orthogonal dual matrix space and $(OD(3), +)$ is a group under addition operator. Then the map

$$\gamma : (OD(3), +) \to (\mathbb{R}^9, +)$$

is a isomorphism. Thus $OD(3)$ is a differentiable manifold. Then ϕ defined by

$$\phi : OD(3) \times OD(3) \to OD(3)\,, \qquad (\widehat{A}, \widehat{B}) \to \widehat{A} + (-\widehat{B})$$

is differentiable. Then it is a Lie group.

Theorem 1.5. *Let $OD(3)$ be a Lie group. Then it acts on itself on left by automorphism*

$$\theta : OD(3) \times OD(3) \to OD(3)\,, \qquad (\widehat{A}, \widehat{B}) \to \widehat{A} + \widehat{B} + (-\widehat{A})\,.$$

Proof. For $0 \in OD(3)$ and all $\widehat{A}, \widehat{B}, \widehat{C} \in OD(3)$

(i) $\theta(0, \widehat{A}) = \widehat{A}$,

(ii) $\theta(\widehat{A}+\widehat{B},\widehat{C}) = \left(\widehat{A}+\widehat{B}\right) + \widehat{C} + (-\left(\widehat{A}+\widehat{B}\right)) = \widehat{C}$,
and $\theta(\widehat{A},\mu(\widehat{B},\widehat{C})) = \widehat{A} + \theta\left(\widehat{B},\widehat{C}\right) + (-\widehat{A}) = \widehat{A} + (\widehat{B}+\widehat{C}+(-\widehat{B})) + (-\widehat{A}) = \widehat{C}$.
Thus:

$$\theta(\widehat{A}+\widehat{B},\widehat{C}) = \theta(\widehat{A},\mu(\widehat{B},\widehat{C})). \qquad \square$$

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