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A NON-LINEAR SCHRÖDINGER TYPE FORMULATION OF FLRW SCALAR FIELD COSMOLOGY

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Abstract: We show that the Friedmann-Lemaître-Robertson-Walker equations with scalar field and perfect fluid matter source are equivalent to a suitable non-linear Schrödinger type equation. This provides for an alternate method of obtaining exact solutions of the Einstein field equations for a homogeneous, isotropic universe.

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1. Introduction

Recently, there have been interesting reformulations of Einstein field equations for scalar field cosmologies (both for isotropic and anisotropic models) in terms of generalized types of Ermakov-Milne-Pinney (EMP) equations; see [1], [3], [4], [8], for example. Such equations occur in a variety of physical contexts and in particular have served to provide a link between gravitational and non-gravitational systems, see [5]. We present in this paper an alternate (non-EMP) formulation of homogeneous, isotropic scalar field cosmology. Namely, we provide a formulation in terms of a non-linear Schrödinger type equation. Some applications to exact field solutions are presented, including some string-inspired cosmological solutions.

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One can set up a direct correspondence between EMP solutions and Schrödinger solutions which suggests, for example, that possible connections between this work and that in [5] can be pursued. The Schrödinger equation here also seems to be of some independent interest.

2. Einstein Equations

The Einstein equations for a Friedmann-Lemaître-Robertson-Walker (FLRW) homogeneous, isotropic universe with scalar field ϕ, potential V, and perfect fluid matter source assume the familiar form (for a vanishing cosmological constant)

$$H^2 + \frac{k}{a^2} \overset{(i)}{=} \frac{8\pi G}{3}\rho_T\,, \qquad \frac{\ddot{a}}{a} \overset{(ii)}{=} \frac{-4\pi G}{3}\left(\rho_T + 3p_T\right), \tag{2.1}$$

where the total energy density ρ_T and pressure p_T with matter contributions ρ_m, p_m are given by

$$\rho_T = \rho_\phi + \rho_m\,, \qquad p_T = p_\phi + p_m\,, \tag{2.2}$$

for

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V\circ\phi\,, \qquad p_\phi = \frac{\dot{\phi}^2}{2} - V\circ\phi\,, \tag{2.3}$$

$$\rho_m = \frac{D}{a^n}\,, \qquad p_m = \frac{(n-3)D}{3a^n}\,, \tag{2.4}$$

with D, n = constants ≥ 0, $n \neq 0$, $H \overset{def.}{=} \frac{\dot{a}}{a}$ = the Hubble parameter for the scale factor $a = a(t)$, and $k = 0$, 1, or -1 = the curvature parameter. G is Newton's constant and units are selected so that the speed of light is unity; cf. [7]. Equations (i), (ii) imply the fluid conservation equation

$$\dot{\rho_T} + 3H(\rho_T + p_T) = 0, \tag{2.5}$$

which by definitions (2.3), (2.4) reduces to the Klein-Gordon equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V'\circ\phi = 0 \tag{2.6}$$

of the scalar field ϕ. Note that for $\gamma_m \overset{def.}{=} \frac{n}{3}$ one has the equation of state $p_m = (\gamma_m - 1)\rho_m$ (by (2.4)). For $\gamma_m = 1$ (i.e. $n = 3$), for example, $p_m = 0$, which is the case of a *dust* universe.

There is also an equation of state $p_\phi = (\gamma_\phi - 1)\rho_\phi$ which follows by setting $\gamma_\phi = 2\dot{\phi}^2 \left[\dot{\phi}^2 + 2(V \circ \phi)\right]^{-1}$. γ_ϕ however, unlike γ_m, is a non-constant function of time t.

In this paper we set up a correspondence $(a, \phi, V) \longleftrightarrow u$ between solutions (a, ϕ, V) of the field equations (i), (ii) in (2.1) and solutions u of the following time-independent Schrödinger type equation

$$u''(x) + [E - P(x)]\, u(x) = -\frac{nk}{2} u(x)^{\frac{4-n}{n}} \tag{2.7}$$

with constant energy E and potential $P(x)$. For a flat universe ($k = 0$), or for the special values $n = 2$ or 4 one notes that equation (2.7) is actually a *linear* Schrödinger equation. The correspondence provides an alternate method, for example, of solving the equations in (2.1). In some cases it is convenient to specify the scale factor $a(t)$ a priori and then find the scalar field ϕ and potential V such that (a, ϕ, V) is a solution in (2.1). This can be done, in particular, in our approach as $a(t)$ is sufficient to determine $u(x)$ in (2.7), which by the correspondence determines the desired ϕ and V.

3. Description of the Correspondence $(a, \phi, V) \longleftrightarrow u$

With the above notation in place, and where we set $K^2 \overset{def.}{=} 8\pi G$ for convenience, we can state the main theorem.

Theorem 1. *Let $u(x)$ be a solution of equation (2.7), given E, $P(x)$. Then a solution (a, ϕ, V) of the Einstein equations (i), (ii) in (2.1) can be constructed as follows, where D in (2.4) (which specifies ρ_T, p_T) is chosen to be $\frac{-12E}{n^2K^2}$: First choose functions $\sigma(t)$, $\psi(x)$ such that*

$$\dot{\sigma}(t) = u(\sigma(t)), \quad \psi'(x)^2 = \frac{4}{nK^2} P(x). \tag{3.1}$$

Then one can take

$$a(t) = u(\sigma(t))^{-\frac{2}{n}}, \qquad \phi(t) = \psi(\sigma(t)), \tag{3.2}$$

$$V = \left[\frac{12}{K^2n^2}(u')^2 - \frac{2u^2P}{K^2n} + \frac{12u^2E}{K^2n^2} + \frac{3ku^{\frac{4}{n}}}{K^2}\right] \circ \psi^{-1}. \tag{3.3}$$

Here, in fact, (a, ϕ, V) will also satisfy the equations

$$\dot{\phi}(t)^2 = \frac{-2}{K^2}\left[\dot{H}(t) - \frac{k}{a(t)^2}\right] - \frac{nD}{3a(t)^n}, \tag{3.4}$$

$$V(\phi(t)) \quad = \quad \frac{3}{K^2}\left[H(t)^2 + \frac{\dot{H}(t)}{3} + \frac{2k}{3a(t)^2}\right] + \frac{(n-6)D}{6a(t)^n}\,, \tag{3.5}$$

again $D \overset{def.}{=} \frac{-12E}{n^2K^2}$. Conversely, let (a, ϕ, V) be a solution of equations (i), (ii) in (2.1), with some D given in (2.4) that specifies ρ_T, p_T in (2.2). Similar to (3.1), choose some solution $\sigma(t)$ of the equation

$$\dot{\sigma}(t) = a(t)^{-\frac{n}{2}}. \tag{3.6}$$

Then equation (2.7) is satisfied for

$$E \quad \overset{def.}{=} \quad -\frac{K^2n^2}{12}D\,, \tag{3.7}$$

$$P(x) \quad \overset{def.}{=} \quad \frac{nK^2}{4}a(\sigma^{-1}(x))^n\left[\dot{\phi}(\sigma^{-1}(x))\right]^2\,, \tag{3.8}$$

$$u(x) \quad \overset{def.}{=} \quad a(\sigma^{-1}(x))^{-\frac{n}{2}}. \tag{3.9}$$

Theorem 1 therefore provides for a concrete correspondence $(a, \phi, V) \leftrightarrow u$ between solutions (a, ϕ, V) of the gravitational field equations (i), (ii) in (2.1) and solutions u of the non-linear Schrödinger type equation (2.7). The solutions (a, ϕ, V) also correspond to solutions Y of the generalized Ermakov-Milne-Pinney equation

$$Y'' + QY = \frac{\lambda}{Y^3} + \frac{nk}{2Y^{\frac{(n+4)}{4}}}\,, \tag{3.10}$$

as discussed in [8]; also see [1], [4]. In turn, one can set up a correspondence $Y \leftrightarrow u$ between solutions Y of (3.10) and solutions u of (2.7), and thus obtain the correspondence $(a, \phi, V) \leftrightarrow u$ – and the proof of Theorem 1. However, we prefer to give a *direct* proof of Theorem 1 – one that does not rely on the results in [8], nor on the correspondence $Y \leftrightarrow u$ – although the latter route served as the motivation for the formulation of equation (2.7), and consequently of Theorem 1. It is implicitly assumed in definition (3.3) that the inverse function ψ^{-1} of ψ exists, which would not be the case if $P(x) = 0$. Therefore we first assume that $P(x)$ is not the zero function. The case $P(x) = 0$ will be discussed later.

For the proof of Theorem 1 we first establish the salient formulas (3.4), (3.5), given E, $P(x)$, $u(x)$ in (2.7). By (3.2), $u \circ \sigma \overset{def.}{=} a^{-\frac{n}{2}}$ which differentiated in conjunction with (3.1) gives $(u' \circ \sigma)(u \circ \sigma) = -\frac{n}{2}a^{-\frac{n}{2}-1}\dot{a} \overset{def.}{=} -\frac{n}{2}(u \circ \sigma)H$. That is, $u' \circ \sigma = -\frac{n}{2}H$ which we also differentiate to obtain $(u'' \circ \sigma)(u \circ \sigma) = -\frac{n}{2}\dot{H}$ (again by (3.1)); i.e.,

$$u \circ \sigma = a^{-\frac{n}{2}}, \quad u' \circ \sigma = -\frac{n}{2}H, \quad (u'' \circ \sigma)(u \circ \sigma) = -\frac{n}{2}\dot{H}. \tag{3.11}$$

From equation (2.7), $Pu^2 = u''u + Eu^2 + \frac{nk}{2}u^{\frac{4}{n}}$ so that by (3.11)

$$(Pu^2) \circ \sigma = -\frac{n}{2}\dot{H} + Ea^{-n} + \frac{nk}{2}a^{-2}. \tag{3.12}$$

Using (3.11), (3.12) we now obtain by definitions (3.2), (3.3),

$$\begin{aligned} V \circ \phi = \frac{12}{K^2n^2}\left(-\frac{n}{2}H\right)^2 - \frac{2}{K^2n}\left(-\frac{n}{2}\dot{H} + Ea^{-n} + \frac{nk}{2}a^{-2}\right) \\ + \frac{12E}{K^2n^2}a^{-n} + \frac{3k}{K^2}a^{-2} \\ = \frac{3H^2}{K^2} + \frac{\dot{H}}{K^2} + \frac{D}{a^n}\left(\frac{n}{6} - 1\right) + \frac{2k}{K^2}a^{-2} \quad (\text{since} D \overset{def.}{=} -\frac{12E}{n^2K^2}), \end{aligned}$$

which is equation (3.5) as desired. Again by (3.1), (3.2), $\dot{\phi} = (\psi' \circ \sigma)(u \circ \sigma) \Rightarrow \dot{\phi}^2 \overset{def.}{=} \frac{4}{nK^2}(P \circ \sigma)(u \circ \sigma)^2 =$ (by (3.12)) $\frac{4}{nK^2}\left(-\frac{n}{2}\dot{H} + Ea^{-n} + \frac{nk}{2}a^{-2}\right) = -\frac{2}{K^2}\dot{H} - \frac{nD}{3}a^{-n} + \frac{2k}{K^2}a^{-2}$, which is equation (3.4).

With equations (3.4), (3.5) now established, we can compute ρ_T, p_T in (2.2), using (2.3):

$$\begin{aligned} \rho_\phi &= -(\dot{H} - ka^{-2})K^{-2} - \frac{nD}{6}a^{-n} + 3(H^2 + \frac{\dot{H}}{3} + \frac{2k}{3}a^{-2})K^{-2} \\ &\quad + \frac{(n-6)}{6}Da^{-n} \ \ (\text{by (2.3), (3.4), (3.5)}) \\ &= (3ka^{-2} + 3H^2)K^{-2} - Da^{-n} \\ &\overset{def.}{=} (3ka^{-2} + 3H^2)K^{-2} - \rho_m \quad (\text{by (2.4)}) \\ \Rightarrow \rho_T &\overset{def.}{=} \rho_\phi + \rho_m \\ &\overset{(i)'}{=} 3(ka^{-2} + H^2)K^{-2}, \end{aligned}$$

which is exactly equation (i) in (2.1). Similarly, by (2.3), (3.4), (3.5),

$$p_\phi = -(\dot{H} - ka^{-2})K^{-2} - \frac{n}{6}Da^{-n} - 3(H^2 + \frac{\dot{H}}{3}$$

$$\begin{aligned}
&+\frac{2k}{3}a^{-2})K^{-2}-\frac{(n-6)}{6}Da^{-n}\\
&=-2\dot{H}K^{-2}-ka^{-2}K^{-2}-3H^2K^{-2}+\frac{(3-n)}{3}Da^{-n}\\
&\overset{def.}{=}-(2\dot{H}+ka^{-2}+3H^2)K^{-2}-p_m\ \text{(by (2.4))}\\
&\Rightarrow p_T\overset{def.}{=}p_\phi+p_m=-(2\dot{H}+ka^{-2}+3H^2)K^{-2},
\end{aligned}$$

which with $(i)'$ gives $\rho_T+3p_T=3(ka^{-2}+H^2)K^{-2}-3(2\dot{H}+ka^{-2}+3H^2)K^{-2}=-6(H^2+\dot{H})K^{-2}=-6\frac{\ddot{a}}{a}K^{-2}$, which is exactly equation (ii) in (2.1). The arrow $u\longrightarrow(a,\phi,V)$ has therefore been established in one direction.

For the other direction $(a,\phi,V)\longrightarrow u$, with (a,ϕ,V) a solution of the equations in (2.1), we must show that $u(x)$ defined in (3.9) solves equation (2.7), for the data E, $P(x)$ defined in (3.7), (3.8), with $\sigma(t)$ defined by (3.6). For convenience let $g(x)=\sigma^{-1}(x)$ denote the inverse function of $\sigma(t)$: $\sigma(g(x))=x\Rightarrow\dot{\sigma}(g(x))g'(x)=1$. That is,

$$u(x)g'(x)=1 \tag{3.13}$$

since by (3.6), (3.9) this product is $a(g(x))^{-\frac{n}{2}}g'(x)=\dot{\sigma}(g(x))g'(x)$. Then

$$\begin{aligned}
u'(x)&=-\frac{n}{2}a(g(x))^{-\frac{n}{2}-1}\dot{a}(g(x))g'(x)\\
&\overset{def.}{=}-\frac{n}{2}u(x)H(g(x))g'(x)\\
&=-\frac{n}{2}H(g(x))\\
\Rightarrow u''(x)&=-\frac{n}{2}\dot{H}(g(x))g'(x)\\
&=-\frac{n}{2}\left[\frac{\ddot{a}(g(x))}{a(g(x))}-H(g(x))^2\right]g'(x),
\end{aligned}$$

where by (2.1) the bracket here is

$$\begin{aligned}
&-\frac{K^2}{6}\left[\rho_T(g(x))+3p_T(g(x))\right]+\frac{k}{a(g(x))^2}-\frac{K^2}{3}\rho_T(g(x))\\
&=-\frac{K^2}{2}\left[\rho_T(g(x))+p_T(g(x))\right]+ku(x)^{\frac{4}{n}}\ \ \text{(i.e. } u(x)^{\frac{-2}{n}}\overset{def.}{=}a(g(x)))\\
&=-\frac{K^2}{2}\left[\dot{\phi}(g(x))^2+\frac{nD}{3a(g(x))^n}\right]+ku(x)^{\frac{4}{n}}\ \ \text{(by(2.2), (2.3), (2.4))}\\
&=-\frac{K^2}{2}\left[\frac{4P(x)}{nK^2}u(x)^2+\frac{nD}{3}u(x)^2\right]+ku(x)^{\frac{4}{n}},\ \text{by(3.8), (3.9).}
\end{aligned}$$

That is, we see that

$$u''(x) = -\frac{n}{2}g'(x)\left\{-\frac{K^2}{2}\left[\frac{4P(x)}{nK^2}u(x) + \frac{nD}{3}u(x)\right]u(x) + ku(x)^{\frac{4}{n}-1}u(x)\right\}$$
$$= P(x)u(x) + \frac{n^2K^2}{12}Du(x) - \frac{n}{2}ku(x)^{\frac{4}{n}-1}$$

(by (3.13)), which by (3.7) is exactly equation (2.7). The proof of Theorem 1 is therefore complete, where we have assumed the existence of ψ^{-1} in (3.3).

Remarks. 1. The case $P(x) = 0$. In equation (2.7) we generally take a non-zero potential $P(x)$. If $P(x) = 0$ then $\psi(x)$ is a constant function by (3.1) and therefore its inverse function $\psi^{-1}(x)$ in definition (3.3) does *not* exist, which means that the expression for V there has no meaning. However, if $P(x) = 0$ we can instead define $V(x)$ to be a constant function and we also define $\phi(t)$ to be a constant function, because of (3.2). First note that equation (3.4) still holds (with the left hand side there being 0 of course) since by (3.12), $0 = -\frac{n}{2}\dot{H} - \frac{n^2K^2}{12}Da^{-n} + \frac{nk}{2}a^{-2}$ (again as $D \overset{def.}{=} -\frac{12E}{n^2K^2}$), which multiplied by $\frac{4}{nK^2}$ gives

$$0 = -\frac{2}{K^2}\dot{H} - \frac{nD}{3}a^{-n} + \frac{2k}{K^2}a^{-2}, \tag{3.14}$$

as claimed. Next note that the right hand side of (3.5) indeed is a constant function of t (for $(P(x) = 0)$). Namely, differentiate (3.14) to obtain

$$\ddot{H} = \left(-\frac{2k}{a^2} + \frac{K^2n^2D}{6a^n}\right)H\,, \tag{3.15}$$

and then compute that

$$\frac{d}{dt}\left\{\frac{3}{K^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \frac{(n-6)}{6a^n}D\right\}$$
$$= \frac{3}{K^2}\left[2H\dot{H} + \frac{\ddot{H}}{3} - \frac{4kH}{3a^2}\right] - \frac{(n-6)nDH}{6a^n}$$
$$= \frac{3}{K^2}\left[2H\dot{H} + \left(\frac{-2k}{3a^2} + \frac{K^2n^2D}{18a^n}\right)H - \frac{4kH}{3a^2}\right] - \frac{(n-6)nDH}{6a^n} \quad \text{(by (3.15))}$$
$$= 3\left[\frac{2\dot{H}}{K^2} - \frac{2k}{K^2a^2}\right]H + \frac{n^2D}{6a^n}H - \frac{n^2DH}{6a^n} + \frac{nDH}{a^n}$$
$$= 3\left(\frac{-nD}{3}\right)a^{-n}H + \frac{nDH}{a^n} \quad \text{(by (3.14))}$$

$= 0$

$\Rightarrow$ (as claimed) that for some constant V_0 one has

$$\frac{3}{K^2}\left[H(t)^2 + \frac{\dot{H}(t)}{3} + \frac{2k}{3a(t)^2}\right] + \frac{(n-6)D}{6a(t)^n} = V_0. \tag{3.16}$$

In summary, in case $P(x) = 0$ in (2.7) we define $\phi(t) =$ any constant and $V(x) = V_0$ in (3.16) (as definition (3.3) no longer has a meaning). Then equations (3.14), (3.16) (the proper versions of equations (3.4), (3.5)) hold.

It follows that, again with $a(t) \stackrel{def.}{=} u(\sigma(t))^{-\frac{2}{n}}$, where $\dot{\sigma}(t) = u(\sigma(t))$, one does arrive at a solution (a, ϕ, V) of (i), (ii), in (2.1). Namely, in (2.2), (2.3), (2.4), $\rho_\phi(t) = V_0$, $p_\phi(t) = -V_0$,

$$\begin{aligned}
\Rightarrow \rho_T &= V_0 + Da^{-n} \\
&= \frac{3}{K^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \frac{nD}{6a^n} \quad (3.16)) \\
&= \frac{3}{K^2}\left[H^2 + \frac{K^2}{6}\left(-\frac{nDa^{-n}}{3} + \frac{2k}{K^2}a^{-2}\right) + \frac{2k}{3a^2}\right] + \frac{nD}{6a^n} \\
&\quad \text{(by (3.14))} \\
&= \frac{3}{K^2}\left[H^2 + \frac{k}{a^2}\right],
\end{aligned}$$

which is (i), and

$$\begin{aligned}
p_T &= -V_0 + \frac{(n-3)D}{3}a^{-n} \\
\Rightarrow \rho_T + 3p_T &= -2V_0 + (n-2)Da^{-n} \\
&= -\frac{6}{K^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] - \frac{(n-6)D}{3a^n} + (n-2)Da^{-n} \\
&\quad \text{(by (3.16))} \\
&= -\frac{6}{K^2}\left[H^2 + \frac{\dot{H}}{3}\right] - \frac{4k}{K^2a^2} + \frac{2Dn}{3a^n} \\
&= -\frac{6}{K^2}\left[H^2 + \frac{\dot{H}}{3}\right] + 2\left(-\frac{2}{K^2}\dot{H}\right) \quad \text{(by (3.14))} \\
&= -\frac{6}{K^2}\left[H^2 + \dot{H}\right]
\end{aligned}$$

$$= \quad -\frac{6}{K^2}\frac{\ddot{a}}{a},$$

which is (ii).

2. Equations (3.4), (3.5) imply (i), (ii). The argument following the establishment of equations (3.4), (3.5) actually shows (independently of $u(x)$) that if $a(t)$ is given a priori, and if $\phi(t)$, $V(x)$ are functions satisfying equations (3.4), (3.5), then automatically (a, ϕ, V) solves equations (i), (ii) in (2.1).

4. Some Examples

As a simple illustration of the application of Theorem 1, choose $u(x) = \left(1 - A^4\omega^2x^2\right)/A^2$ for $A, \omega > 0$, which solves equation (2.7) for $E = 0$, $P(x) = 2A^2\left(1 - A^2\omega^2\right)\left(1 - A^4\omega^2x^2\right)^{-1} = K^2B^2A^4\left(1 - A^4\omega^2x^2\right)^{-1}$, $B^2 = 2(1 - A^2\omega^2)/K^2A^2$, $n = 4$, $k = 1$. In (3.1) we can take $\sigma(t) = \tanh(\omega t)/A^2\omega$, $\psi(x) = \psi_0 \pm \frac{B}{\omega}\arcsin(A^2\omega x) = \psi_0 \pm \frac{B}{\omega}\arctan(\frac{A^2\omega x}{\sqrt{1-A^4\omega^2x^2}})$ for $A^2\omega|x| < 1$ and obtain from (3.2), (3.3), $a(t) = A\cosh(\omega t)$, $\phi(t) = \psi_0 \pm \frac{B}{\omega}\arcsin(\tanh(\omega t)) = \psi_o \pm \frac{B}{\omega}\arctan(\sinh(\omega t)) = \psi_0' \pm \frac{2B}{\omega}\arctan(e^{\omega t})$, $V(x) = \frac{3\omega^2}{K^2} + B^2\cos^2(\frac{\omega}{B}(x - \psi_0))$, for constants ψ_0, ψ_0', which is the Ellis-Madsen solution in Section 4.3 of [2]. We note that the 2 in the expression $\sin\left(2\frac{\omega}{B}(\phi - \phi_0)\right)$ in equation (42) of [2] should not appear there. One can obtain in fact all of the solutions in [2] for a suitable choice of $u(x)$ and n.

Özer and Taha have considered in [6] two string-motivated solutions (a_j, ϕ_j, V_j), $j = 1, 2$, for $k = 1$, $D_j = 0$, where the potentials V_j were specified a priori. One can also take the point of view of specifying the scale factors $a_1(t) = (a_0^2 + t^2)^{\frac{1}{2}}$, $a_2(t) = a_0 + t^2/2a_0$, for $a_0 \neq 0$, and then applying Remark 2 to find (ϕ_j, V_j). For $a_2(t)$, for example, $H_2(t) = 2t/(2a_o^2 + t^2)$, $\dot{H}_2(t) = 2(2a_0^2 - t^2)/(2a_0^2 + t^2)^2$, and one obtains by (3.2) $\dot{\phi}_2(t)^2 = 4t^2/K^2(2a_0^2 + t^2)^2 \Rightarrow \phi_2(t) = \phi_0 + \frac{1}{K}\log\left(1 + \frac{t^2}{2a_0^2}\right)$, where we choose the positive square root. Also, $H_2(t)^2 + \dot{H}_2(t)^2/3 + 2/3a_2(t)^2 = (10t^2 + 12a_0^2)/3(2a_0^2 + t^2)^2 \Rightarrow V_2(\phi_2(t)) = (10t^2 + 12a_0^2)/K^2(2a_0^2 + t^2)^2$, by (3.5). For $x > \phi_0$, we can take $\phi_2^{-1}(x) = \sqrt{2}|a_0|\left[e^{K(x-\phi_0)} - 1\right]^{\frac{1}{2}}$ and compute that $V_2(x) = V_2(\phi_2(\phi_2^{-1}(x))) = \left[5e^{-K(x-\phi_0)} - 2e^{-2K(x-\phi_0)}\right]/K^2a_0^2$. Similarly, for the above scale factor $a_1(t)$ one can obtain via (3.4), (3.5) $\phi_1(t) = \phi_0' + \frac{1}{K}\log\left(1 + \frac{t^2}{a_0^2}\right)$, $V(x) = \left[4e^{-K(x-\phi_0')} - e^{-2K(x-\phi_0')}\right]/K^2a_0^2$, say for $x > \phi_0'$. These solutions (a_j, ϕ_j, V_j) can also be obtained by taking $u_j(x) = a_j(\sigma^{-1}(x))^{-\frac{n_j}{2}}$ (which is motivated by (3.2)) for the convenient choices $n_1 = 4$, $n_2 = 2$, and using the corresponding solutions $\sigma_1(t) = a_0^{-1}\arctan(a_0^{-1}t)$, $\psi_1(x) = \psi_0 - \frac{2}{K}\log(\cos(a_0 x))$, $\sigma_2(t) = \sqrt{2}\arctan((\sqrt{2}a_0)^{-1}t)$, $\psi_2(x) = \psi_0 - \frac{2}{K}\log(\cos(\frac{x}{\sqrt{2}}))$ of the equations in (3.1) for $P_1(x) = 4a_0^2\tan^2(a_0 x)$, $P_2(x) = \tan^2(\frac{x}{\sqrt{2}})$; $E_1 = E_2 = 0$. One can go beyond the assumption $D_1 = D_2 = 0$ (in [6]) and obtain solutions ϕ_j, $V_j \circ \phi_j$ via (3.4), (3.5). For example, for $D_1 \neq 0$ one can check, using *Mathematica* for example, that

$$\phi_1(t) = \phi_0' + \frac{2}{\sqrt{3}K}\left\{\frac{-\sqrt{3a_0^2+DK^2}}{a_0}\operatorname{arctanh}\left(\frac{\sqrt{3a_0^2+DK^2}t}{a_0\sqrt{-DK^2+3t^2}}\right)\right. \tag{4.1}$$

$$\left. + \sqrt{3}\log\left(3t+\sqrt{-3DK^2+9t^2}\right)\right\} \tag{4.2}$$

and that

$$V_1(\phi_1(t)) = \frac{3(4t^2+3a_0^2) - DK^2}{3K^2(a_0^2+t^2)^2}. \tag{4.3}$$

As a fourth example, and final one, for a parameter $\lambda \in \mathbb{R}\setminus\{0\}$, let $u(x) = -\frac{3}{4}\sqrt{3}\cdot\tanh\left(\sqrt{\frac{27}{8}}\lambda x\right)$. Then $u(x)$ solves equation (2.7) for $E = A + \frac{27}{4}\lambda^2$, $A > 0$, $P(x) = A$, $n = 1$, $k = -8\lambda^2$. By an application of Theorem 1, one obtains the solutions $(a, \phi_\pm, V)$ given by

$$\begin{aligned} a(t) &= \frac{16}{27}\left[1 + e^{\frac{27\lambda}{4\sqrt{2}}(t-c)}\right], \\ \phi_\pm(t) &= \pm\frac{4}{3}\sqrt{\frac{2}{3}}\frac{\sqrt{A}}{\lambda K}\operatorname{arcsinh}\left[e^{-\frac{27\lambda}{8\sqrt{2}}(t-c)}\right], \\ V(x) &= \frac{3^7\lambda^2}{2^5K^2} + \frac{135A}{8K^2}\tanh^2\left[\frac{3}{4}\sqrt{\frac{3}{2}}\frac{\lambda K}{\sqrt{A}}(x-\phi_0)\right], \end{aligned} \tag{4.4}$$

for integration constants c, ϕ_0.

References

[1] T. Christodoulakis, T. Grammenos, C. Helias, P. Kevrekidis, G. Papadopoulos, F. Williams, *On 3+1-Dimensional Scalar Field Cosmologies, from Progress in General Relativity and Quantum Cosmology Research*, Nova Science Pub. (2005).

[2] G. Ellis, M. Madsen, Exact scalar field cosmologies, *Classical and Quantum Gravity,* **8** (1991), 667-676.

[3] R. Hawkins, J. Lidsey, The Ermakov-Pinney equation in scalar field cosmologies, *Physical Review D,* **66** (2002), 023523-023531.

[4] P. Kevrekidis, F. Williams, On 2+1-dimensional Friedmann-Robertson-Walker universes: an Ermakov-Pinney equation approach, *Classical and Quantum Gravity*, **20** (2003), L177-L184.

[5] J. Lidsey, Cosmic dynamics of Bose-Einstein Condensates, *Classical and Quantum Gravity*, **21** (2004), 777-785.

[6] M. Özer, M. Taha, Exact solutions in string-motivated scalar field cosmology, *Physical Review D*, **45** (1992), R997-R999.

[7] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley and Sons (1972).

[8] F. Williams, Einstein field equations: An alternate approach towards exact solutions for an FRW universe, In: *Proceedings of the Sixth Alexander Friedmann International Seminar on Gravitation and Cosmology*, Cargèse, France (2004); *International J. of Modern Physics A*, **20** (2005), 2481-2484.