International Journal of Pure and Applied Mathematics
Volume 34 No. 4 2007, 513-517

REMARKS ON THE QUASIDISKS AND THE HARDY-LITTLEWOOD PROPERTY

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Abstract: Suppose that D is a Jordan domain in the finit plane R^2, in this paper, the authors prove that D is a quasidisk if and only if $f \in \mathrm{Lip}_{1,J}(D)$, whenever f is analytic in D with $|f'(z)| \le d(z,\partial D)^{-1}$. Here $J(z_1,z_2) = \frac{1}{2}\log(1+\frac{|z_1-z_2|}{d(z_1,\partial D)})(1+\frac{|z_1-z_2|}{d(z_2,\partial D)})$ for $z_1, z_2 \in D$.

AMS Subject Classification: 30C62
Key Words: quasidisk, Hardy-Littlewood property, distance function

1. Introduction

In this paper, we shall adopt the notation and terminology as in paper, see [1], R^2 denotes the finite plane, $\bar{R}^2 = R^2 \cup \{\infty\}$. For a Jordan domain $D \subseteq R^2$, denotes $D^* = \bar{R}^2 \setminus \bar{D}$ the exterior of D. For $x \in R^2$ and $0 < r < \infty$, let $B^2(x,r) = \{z \in R^2 : |z-x| < r\}$, $\bar{B}^2(x,r)$ be the closure of $B^2(x,r)$, $B^2(r) = B^2(0,r)$ and $B^2 = B^2(1)$.

Suppose that D is a domain in the finite plane R^2 and that f is a real or complex valued function defined in D. We say that f is in $\mathrm{Lip}_k(D)$, $0 < k \le 1$, if there exists a constant m such that

$$|f(z_1) - f(z_2)| \le m|z_1 - z_2|^k \tag{1.1}$$

for all z_1 and z_2 in D, and we let $\|f\|_k$ denote the infimum of the numbers m for which (1.1) holds.

Received: November 13, 2006

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We say that a proper subdomain D of R^2 has the Hardy-Littlewood property of order k, $0 < k \leq 1$, if there exists a constant c such that f is in $\mathrm{Lip}_k(D)$ with $\|f\|_k \leq \frac{cm}{k}$ whenever f is analytic in D with

$$|f'(z)| \leq md(z, \partial D)^{k-1}. \tag{1.2}$$

Hardy and Littlewood in [2] proved the following theorem.

Theorem A. *If D is the open disk $B^2(x, r)$, then for any $0 < k \leq 1$, D has the Hardy-Littlewood property of order k.*

Suppose that D is a Jordan domain in R^2. We say that D is a quasidisk if there exists a $K-$quasiconformal mapping ($K \geq 1$) $f : \bar{R}^2 \to \bar{R}^2$ such that D is the image of the unit disk B^2 under f.

It is well-known that quasidisks play a very important role in quasiconformal mappings [3], complex dynamics [4], Fuchsian groups [5] and Teichmüller space theory [6], etc. A lot of interesting geometric and analytic properties for quasidisks are obtained by many authors.

Gehring and Martio in [7] proved the following theorem.

Theorem B. *If D is a quasidisk, then for any $0 < k \leq 1$, D has the Hardy-Littlewood property of order k.*

Theorem C. *Suppose that $D \subseteq R^2$ is a Jordan domain with $\infty \in \partial D$. If there exist constants $k, k^* \in (0, 1]$ such that D and D^* has the Hardy-Littlewood property of order k and k^* respectively, then D is a quasidisk.*

In this paper, we shall generalize the $\mathrm{Lip}_k(D)$ to $\mathrm{Lip}_{k,h}(D)$ as follows.

Suppose that D is a domain in R^2, f is a real or complex value function defined in D, h is a distance function defined in D. We say that f is in $\mathrm{Lip}_{k,h}(D)$, $0 < k \leq 1$, if there exists a constant m such that

$$|f(z_1) - f(z_2)| \leq mh(z_1, z_2)^k \tag{1.3}$$

for all z_1 and z_2 in D.

It is obvious, if we let $h(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in D$, then $\mathrm{Lip}_{k,h}(D) = \mathrm{Lip}_k(D)$. We introduce the following important distant function J in D:

$$J(z_1, z_2) = \frac{1}{2}\log(1 + \frac{|z_1 - z_2|}{d(z_1, \partial D)})(1 + \frac{|z_1 - z_2|}{d(z_2, \partial D)}) \tag{1.4}$$

for $z_1, z_2 \in D$.

The distance function J has been researeched by Gehring, Martio, Palka, Osgood and Hag in [8-10].

The main purpose of this paper is to use $\mathrm{Lip}_{1,J}(D)$ to depict the analytic properties of quasidisks, we obtain the following result.

Theorem. *If D is a Jordan domain in the finite plane R^2, then D is a quasidisk if and only if $f \in \mathrm{Lip}_{1,J}(D)$ whenever f is analytic in D with $|f'(z)| \leq d(z, \partial D)^{-1}$.*

2. Two Lemmas and The Proof of Theorem

We shall first introduce the following two lemmas, they are the key of the proof of the theorem.

Lemma 1. (see [1], [9], [14]) *A Jordan domain D in R^2 is a quasidisk if and only if there exists a constant c such that*

$$K(z_1, z_2) \leq cJ(z_1, z_2) \tag{2.1}$$

for all $z_1, z_2 \in D$. Here

$$K(z_1, z_2) = \inf_{\gamma} \int_{\gamma} d(z, \partial D)^{-1} ds \tag{2.2}$$

and the infimum in (2.2) is taken over all rectifiable arcs γ joining z_1 and z_2 in D.

Lemma 2. (see [7]) *If D is a Jordan domain in R^2, then*

$$K(z_1, z_2) \leq b\delta(z_1, z_2) \tag{2.3}$$

for all $z_1, z_2 \in D$. In here $b > 0$ is an absolute constant, and $\delta(z_1, z_2)$ is a distance function in D which defined by

$$\delta(z_1, z_2) = \sup |f(z_1) - f(z_2)|, \tag{2.4}$$

where the supremum in (2.4) is taken over all analytic functions f on D with

$$|f'(z)| \leq d(z, \partial D)^{-1}. \tag{2.5}$$

Remark. Lemma 2 is a generalization of a result by Kaufman and Wu [14], the proof is essentially identical, so we omit it here.

Theorem. *If D is a Jordan domain in the finite plane R^2, then D is a quasidisk if and only if $f \in \mathrm{Lip}_{1,J}(D)$ whenever f is analytic in D with $|f'(z)| \leq d(z, \partial D)^{-1}$.*

Proof. ($\Rightarrow$) If D is a quasidisk, then by Lemma 1

$$K(z_1, z_2) \leq cJ(z_1, z_2) \tag{2.6}$$

for all $z_1, z_2 \in D$, where c is a constant which depends only on D.

Next for any $z_1, z_2 \in D$, by [9] we know that there exists quasihyperbolic geodesics $\gamma \subseteq D$ which joining z_1 and z_2 such that

$$K(z_1, z_2) = \int_\gamma d(z, \partial D)^{-1} ds. \tag{2.7}$$

If f is analytic and satisfies(2.5) in D, then (2.5), (2.6) and (2.7) imply

$$|f(z_1) - f(z_2)| = |\int_\gamma f'(z)dz| \leq \int_\gamma |f'(z)|ds$$
$$\leq \int_\gamma d(z, \partial D)^{-1} ds = K(z_1, z_2) \leq cJ(z_1, z_2).$$

($\Leftarrow$) Now suppose that every f is analytic and satisfies (2.5) in D also satisfies

$$|f(z_1) - f(z_2)| \leq mJ(z_1, z_2), \tag{2.8}$$

where m is a constant which depends only on D.

Then by Lemma 2 and (2.8) we have

$$K(z_1, z_2) \leq b\delta(z_1, z_2) \leq bmJ(z_1, z_2) \tag{2.9}$$

for all $z_1, z_2 \in D$.

(2.9) and Lemma 1 imply that D is a quasidisk. □

3. Acknowledgments

This research is supported by NNSF of P.R. China(10471039) and Foundation of the Educational Committee of Zhejiang Province Under Grant (20060306).

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