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MINIMAL STRUCTURES AND SEPARATIONS PROPERTIES

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Abstract: In this work the notion of m-operator for an m-structure m_X on a set X is introduced. Also several separation forms for points of a set X are described and characterized, in a not necessarily topological context. We also study different relationships between these separation properties, and we establish conditions in regards to the operator which determine the equivalence between these separation forms.

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1. Introduction

After the works of Levine [5] and Kasahara [4], various mathematician turned their attention in introducing and studying diverse classes of sets related to the notion of operator associated to a topology on a set. Each one of these classes of sets is, in turn, used to obtain different separation properties and new forms of continuity. It is as well as they arise, among others: semi-open, pre-open, β-open, α-semi-open, θ-closed, semi-θ-open, (α,β)-semi-open, $\gamma-(\alpha,\beta)$-semi-open and the different axioms or formulated separation properties respectively, in terms of each of these classes of sets. The description, the properties and

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also in the study of situations referred previously, used by Maki [6], in abstract form by means of m-structure or minimal structure notions on a set. In this work, we introduce the notion of m-operator on m-structure and we show that they can be also described, studying the separation properties on a set, without necessarily to have a topology on it. We also find, that the obtained results constitute a generalization of many of the classic results and in particular, those obtained by Caldas et al in [1]. The results that are obtained also provide a conceptual frame that summarizes many relative separation forms in generalized sets derived in a topological space via operators existent in the literature.

2. Minimal Structures

In this section, we introduce the m-structure and the m-operator notions. Also, we define some important subsets associated to these concepts.

Definition 2.1. Let X be a nonempty set and let $m_X \subseteq P(X)$, where $P(X)$ denote the set of power of X. We say that m_X is an m-structure (or a minimal structure) on X, if $\emptyset$ and X belong to m_X.

The members of the minimal structure m_X are called m_X-open sets, and the pair (X, m_X) is called an m-space. The complement of an m_X-open set is said to be an m_X-closed set. An m-structure m_X on a nonempty set X, is said to have the property (B) of Maki, if the union of any family of elements of m_X belongs to m_X. Observe that any collection $\emptyset \neq \mathcal{J} \subseteq P(X)$, always is contained in an m-structure that have the property (B), as we know, $m_X(\mathcal{J}) = \{\emptyset, X\} \cup \{\bigcup_{M \in J} M : \emptyset \neq J \subseteq \mathcal{J}\}$. In particular, when $\mathcal{J} = m_X$, we denote by $m_X^{'} = m_X(\mathcal{J})$. Clearly $m_X = m_X^{'}$, if m_X have the property (B) of Maki. Note that if m_X is an m-structure and $Y \subseteq X$, then $\{M \cap Y : M \in m_X\}$ is an m-structure on Y, and is denoted by $m_{X|Y}$, and the pair $(Y, m_{X|Y})$ is called an m-subspace of (X, m_X).

It is important to observe that the m-structure notion, uses in abstract form the properties of many important collections of generalized sets without the necessity of a topological structure, some of them are illustrated in the following situations:

1. Given a topological space (X, τ), the collections: τ, τ_θ, $SO(X)$, $PO(X)$, $\beta(X)$ are m-structures on X, and all satisfy the property (B). Also, the collection of closed sets in X is an m-structure and satisfy the property (B) of Maki, if (X, τ) is an Alexandroff space.

2. If α is an operator associated with the topology τ on X in the sense of Carpintero et al [2] and [3], then the collections Γ_α and α-$SO(X, \tau)$ are m-

structures. Γ_α also has the property (B) and α-$SO(X,\tau)$ has the property (B), if α is a monotone operator.

3. If α, β are operators associated with a topology τ on X, the collection $(\alpha,\beta)-SO(X,\tau)$, introduced by Rosas et al in [9], also is an m-structure and satisfy the property (B).

4. If α, β and γ are operators associated with τ on X, the collection $\gamma-(\alpha,\beta)$-$SO(X,\tau)$, defined by Rosas et al in [11], is also an m-structure, and satisfy the property (B), when the operator γ is expansive on the class $(\alpha,\beta)-SO(X,\tau)$.

Definition 2.2. Let m_X be an m-structure on a set $X\neq\emptyset$. An m-operator on m_X, is an application $\alpha : P(X)\to P(X)$ that is expansive on m_X (that is, $U\subseteq\alpha(U)$, for all $U\in m_X$).

A particular case of the previous definition is when $m_X=\tau$, in which the m-operator notion is exactly the notion of operator associated with the topology introduced by Carpintero et al [2]. Also if α is an m-operator on m_X and $Y\subseteq X$, the restriction $\alpha\mid_{P(Y)}$ given by $\alpha\mid_{P(Y)}(M\cap Y)=\alpha(M)\cap Y$, for all $M\subseteq X$, is an m-operator on $m_{X|Y}$.

Definition 2.3. Given two m_X-operators $\alpha,\beta : P(X)\to P(X)$ on m_X. We say that $\alpha\preceq\beta$ if $\alpha(U)\subseteq\beta(U)$, for all $U\in m_X$.

Note that $\preceq$ defined previously, is an order on the class $\{\alpha : \alpha \text{ is an } m\text{-operator on } m_X\}$.

Definition 2.4. Let $\alpha : P(X)\to P(X)$ be an m-operator on m_X and $A\subseteq X$. A is called an α-m_X-open set, if for each $x\in A$ there exists an m_X-open set U such that $x\in U$ and $\alpha(U)\subseteq A$. The complement of an α-m_X-open set is an α-m_X-closed set.

We denote the collections of all α-m_X-open sets of X by $O(X,m_X,\alpha)$. Observe that the collection $O(X,m_X,\alpha)$ is stable under the union of sets and if m_X has the property (B), then we obtain that $O(X,m_X,\alpha)\subseteq m_X$.

Also, we note that:

1. If $\alpha=i_{P(X)}$ and m_X is any m-structure satisfying the property (B), then the α-m_X-open sets are elements of m_X. In particular, if $m_X=\tau$, where τ is a topology on X, and $\alpha=i_{P(X)}$, the α-m_X-open sets are open sets.

2. If $m_X=\tau$, where τ is a topology on X, and α is an operator associated with τ, the α-m_X-open sets are the α-open sets, described in [8].

3. If $m_X=\beta$-$SO(X,\tau)$, α and β operators associated with τ, where α is expansive on the class β-$SO(X,\tau)$, the α-m_X-open sets are the (α,β)-semi-open sets, described in [9].

4. If $m_X = \gamma$-(α, β)-$SO(X, \tau)$, α,β and γ are operators associated with τ, where γ is expansive on the class (α, β)-$SO(X, \tau)$, the α-m_X-open sets are the γ-(α, β)-semi-open sets, described in [11].

Definition 2.5. Let $\alpha : P(X) \to P(X)$ be an m-operator on m_X and $A \subseteq X$. A is called an α-m_X-semi-open set, if there exist an m_X-open set $U \subseteq X$ such that $U \subseteq A \subseteq \alpha(U)$. The complement of an α-m_X-semi-open set, is called an α-m_X-semi-closed set.

We denote by $SO(X, m_X, \alpha)$ the collection of all α-m_X-semi-open sets of X. Observe that $m_X \subseteq SO(X, m_X, \alpha)$. Also, if m_X has the property (B) of Maki, we obtain that:

$$O(X, m_X, \alpha) \subseteq m_X \subseteq SO(X, m_X, \alpha).$$

Note that the α-semi-open sets, introduced in [2] by Carpintero et al, generalize an extense class of sets in terms of which many generalized separation axioms were described. They constitute a particular case of the previous definition, when $m_X = \tau$ and α is an operator associated with a topology τ.

In general the α-m_X-open sets and the α-m_X-semi-open sets are not stable for the union. Nevertheless, for certain m-operators, the class of α-m_X-semi open sets are stable under union of sets, like it is demonstrated in the following lemma.

Lemma 2.1. *Let m_X be an m-structure which satisfy the property (B) of Maki and let $\alpha : P(X) \to P(X)$ be an m-monotone operator on m_X. If $A_i \in SO(X, m_X, \alpha)$ for each $i \in I$, then $\bigcup_{i\in I} A_i \in SO(X, m_X, \alpha)$.*

Proof. Suppose that m_X has the property (B), α is an m-monotone operator and $A_i \in SO(X, m_X, \alpha)$ for each $i \in I$. For each $i \in I$, there exists a set $U_i \in m_X$ such that $U_i \subseteq A_i \subseteq \alpha(U_i)$, in consequence, $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} \alpha(U_i)$. Since α is a monotone operator, then $\bigcup_{i\in I} \alpha(U_i) \subseteq \alpha(\bigcup_{i\in I} U_i)$; and $\bigcup_{i\in I} U_i \in m_X$, because m_X has the property (B). In consequence, $\bigcup_{i\in I} U_i \in m_X$ and $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq \alpha(\bigcup_{i\in I} U_i)$, therefore $\bigcup_{i\in I} A_i \in SO(X, m_X, \alpha)$. □

Definition 2.6. Let m_X be an m-structure on X and let α be an m-operator on m_X. We define the $\alpha - m_X$-closure and the $\alpha - m_X$-semi-closure of a set A of X, respectively, as follows:

i) $m_X - \mathrm{sCl}_\alpha(A) = \bigcap\{F : A \subseteq F, X \setminus F \in O(X, m_X, \alpha)\}$,
ii) $m_X - \mathrm{SCl}_\alpha(A) = \bigcap\{F : A \subseteq F, X \setminus F \in SO(X, m_X, \alpha)\}$.

Observe that if $\alpha = i_{P(X)}$ and m_X satisfy the property (B), then the above definition is justly the definition of m-closure, described in [1] and [7], that is;

$$m_X - \mathrm{SCl}_\alpha(A) = m_X - \mathrm{Cl}(A),$$

and

$$m_X - \mathrm{sCl}_\alpha(A) = m_X - \mathrm{SCl}_\alpha(A) = m_X - \mathrm{Cl}(A).$$

Also, we can observe that, for any $A \subseteq X$, the $m_X - \mathrm{sCl}_\alpha(A)$ is an $\alpha - m_X$-closed. But the $m_X - \mathrm{SCl}_\alpha(A)$ is not necessarily an $\alpha - m_X$-semi closed set, but according with Lemma 2.1 and the above definition, the $m_X - \mathrm{SCl}_\alpha(A)$ is an $\alpha - m_X$-semi-closed set when m_X has the property (B) and α is an m-monotone operator.

Note that the condition that α is an m-monotone operator, is not an artificial condition, because, we can find many operators $\alpha : P(X) \to P(X)$ that satisfying such conditions.

Lemma 2.2. *Let m_X be an m-structure on X and let α be an m-operator on m_X. For any subsets A and B of X, the following statements hold:*

1. if $A \subseteq B$, then $m_X - sCl_\alpha(A) \subseteq m_X - sCl_\alpha(B)$.

2. $x \in m_X - sCl_\alpha(A)$ if and only if $U \cap A \neq \emptyset$ for all $\alpha - m_X$-open set U such that $x \in U$;

3. A is an $\alpha - m_X$-closed set if and only if $A = m_X - sCl_\alpha(A)$;

4. $m_X - sCl_\alpha(m_X - sCl_\alpha(A)) = m_X - sCl_\alpha(A)$;

Lemma 2.3. *Let m_X be an m-structure on X and let α be an m-operator on m_X. For any subsets A, B of X, the following statements hold:*

1. If $A \subseteq B$, then $m_X - SCl_\alpha(A) \subseteq m_X - SCl_\alpha(B)$.

2. $x \in m_X - SCl_\alpha(A)$ if and only if $U \cap A \neq \emptyset$ for all $\alpha - m_X$-semi open set U such that $x \in U$.

3. $m_X - SCl_\alpha(m_X - SCl_\alpha(A)) = m_X - SCl_\alpha(A)$;

Also if m_X satisfies the property (B) and α is monotone, then

4. A is an $\alpha - m_X$-semi-closed set if and only if $A = m_X - SCl_\alpha(A)$;

Definition 2.7. Let m_X be an m-structure on X and let α be an m-operator on m_X. A point $x \in X$, is said to be an $\alpha - m_X$-adherent point of a set $A \subseteq X$ if and only if $\alpha(U) \cap A \neq \emptyset$ for all $U \in m_X$ such that $x \in U$.

The set of all $\alpha - m_X$-adherent points of A is denoted by $m_X - Cl_\alpha(A)$. A set A is called (m_X, α)-closed if $m_X - Cl_\alpha(A) = A$. The complement of an (m_X, α)-closed set is an (m_X, α)-open set.

Lemma 2.4. *Let m_X be an m-structure on X and let α be an m-operator on m_X; then*

$$m_X - SCl_\alpha(A) \subseteq m_X - Cl(A) \subseteq m_X - Cl_\alpha(A) \subseteq m_X - sCl_\alpha(A).$$

From the last result, follows that:

$$\alpha - m_X\text{-closed set} \Rightarrow (m_X, \alpha)\text{-closed set} \Rightarrow m_X - \text{closed set}$$

$$\Rightarrow \alpha - m_X\text{-semi-closed set},$$

or equivalently,

$$\alpha - m_X\text{-open set} \Rightarrow (m_X, \alpha)\text{-open set} \Rightarrow m_X - \text{open set}$$

$$\Rightarrow \alpha - m_X\text{-semi-open set},$$

when m_X satisfying the property (B) and α is a monotone operator.

The following definition, generalize the notions of D-sets introduced by Tong in [12].

Definition 2.8. Let m_X be an m-structure on X. A subset $A \subseteq X$, is called an m_X-Difference set (more precisely an m_X-D-set) if there exist subsets U, V in m_X such that $U \neq X$ and $A = U \setminus V$.

Observe that, any m_X-open set $U \neq X$, is an $m_X - D-$set, because trivially $U = U \setminus \emptyset$.

Definition 2.9. Let m_X be an m-structure on X and let α be an m-operator on m_X. A subset $A \subseteq X$, is called an $\alpha - m_X$-generalized closed set (abbreviated by $\alpha - m_X$-sg-closed) if $m_X - \mathrm{sCl}_\alpha(A) \subseteq U$, whenever $A \subseteq U$ and U is an $\alpha - m_X$-open set.

Definition 2.10. Let m_X be an m-structure on X and let α be an m-operator on m_X. A subset $A \subseteq X$, is said to be an $\alpha - m_X$-semi generalized closed set (abbreviated by $\alpha - m_X$-sg-semi-closed) if $m_X - \mathrm{SCl}_\alpha(A) \subseteq U$, whenever $A \subseteq U$ and U is an $\alpha - m_X$-semi-open set.

The followings theorems, characterize the $\alpha - m_X$-generalized closed sets and the $\alpha - m_X$-semi generalized closed sets.

Theorem 2.1. *Let m_X be an m-structure on X that satisfies the property (B) and let α be an m-monotone operator on m_X. $A \subseteq X$ is an $\alpha - m_X$-sg-semi-closed set if and only if there are not exist $\alpha - m_X$-semi-closed set F such that $F \neq \emptyset$ and $F \subseteq m_X - SCl_\alpha(A) \setminus A$.*

Proof. Suppose that A is an $\alpha - m_X$-sg-semi-closed and let $F \subseteq X$ be an $\alpha - m_X$-semi-closed set such that $F \subseteq m_X - \mathrm{SCl}_\alpha(A) \setminus A$. It follows that, $A \subseteq X \setminus F$ and $X \setminus F$ is an $\alpha - m_X$-semi open set, since A is an $\alpha - m_X$-sg-semi-closed, we have that $m_X - \mathrm{SCl}_\alpha(A) \subseteq X \setminus F$ and $F \subseteq X \setminus m_X - \mathrm{SCl}_\alpha(A)$. It follows that

$$F \subseteq m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus m_X - \mathrm{SCl}_\alpha(A)) = \emptyset,$$

implying that $F = \emptyset$. Reciprocally, if $A \subseteq U$ and U is an $\alpha - m_X$-semi-open set, then $m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus U) \subseteq m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus A) = m_X - \mathrm{SCl}_\alpha(A) \setminus A$. Since $m_X - \mathrm{SCl}_\alpha(A) \setminus A$ does not contain subsets $\alpha - m_X$-semi-closed different from the empty set, we obtain that $m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus U) = \emptyset$, and this implies that $m_X - \mathrm{SCl}_\alpha(A) \subseteq U$ in consequence A is an $\alpha - m_X$-sg-closed. □

In a similar form, we can prove the following characterization.

Theorem 2.2. *Let m_X be an m-structure on X and let α be an m-operator on m_X. $A \subseteq X$ is an $\alpha - m_X$-sg-closed if and only if there are not exist $\alpha - m_X$-closed set F such that $F \neq \emptyset$ and $F \subseteq m_X - sCl_\alpha(A) \setminus A$.*

3. Separation Properties on m-Structures

In this section, we introduce and study different separation properties on a set X with an m-structure. We also look for the existent relation between the different set defined before.

Definition 3.1. Let m_X be an m-structure on X. We say that:

1. X is an m_X-T_0 if for each pair of distinct points $x, y \in X$, there exists an m_X-open sets U of X, such that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.
2. X is an m_X-T_1 if for each pair of distinct points $x, y \in X$, there exists an m_X-open set of X containing x but not y and an m_X-open set of X containing y but not x.
3. X is an m_X-T_2 if for each pair of distinct points $x, y \in X$, there exist disjoint m_X-open sets U and V such that $x \in U$ and $y \in V$.

We can see that the collections $O(X, m_X, \alpha)$ and $SO(X, m_X, \alpha)$ are m-structures in the sense of the Definition 2.1. If we take m_X as $O(X, m_X, \alpha)$ (respectively $SO(X, m_X, \alpha)$), in the Definition 3.1, we obtain separation properties denoted by (m_X, α)-sT_i (respectively (m_X, α)-ST_i), for $i = 0, 1, 2$. From Definition 3.1, it is immediate that:

$$m_X - T_i \Rightarrow m_X - T_{i-1}; \quad (m_X, \alpha) - sT_i \Rightarrow (m_X, \alpha) - sT_{i-1} \quad \text{and}$$

$$(m_X,\alpha)-ST_i \Rightarrow (m_X,\alpha)-ST_{i-1}, \text{ for } i=1,2.$$

Definition 3.2. Let m_X be an m-structure on X and let α be an m-operator on m_X. We say that:

1. X is an (m_X,α)-T_0 if for each pair of distinct points $x,y\in X$, there exists an m_X-open set U of X, such that $x\in U$ and $y\notin\alpha(U)$, or $y\in U$ and $x\notin\alpha(U)$.
2. X is an (m_X,α)-T_1 if for each pair of distinct points $x,y\in X$, there exist m_X-open sets U and V of X containing x and y, respectively, such that $y\notin\alpha(U)$ and $x\notin\alpha(V)$.
3. X is an (m_X,α)-T_2 if for each pair of distinct points $x,y\in X$, there exist m_X-open sets U and V of X, such that $x\in U$, $y\in V$ and $\alpha(U)\cap\alpha(V)=\emptyset$.

From Definition 3.2, follows that,

$$(m_X,\alpha)-T_i \Rightarrow (m_X,\alpha)-T_{i-1}, \quad i=1,2.$$

Also

$$(m_X,\alpha)-sT_i \Rightarrow (m_X,\alpha)-T_i \Rightarrow m_X-T_i \Rightarrow (m_X,\alpha)-ST_i,$$

for $i=0,1,2$.

It is important to observe that the above definition generalize many of the well known separation axioms seen in the literature. As we specify.

1. For $\alpha=i_{P(X)}$ and m_X any m-structure, the properties of the (m_X,α)-T_i are the separation properties described in the Definitions 3.1, for $i=0,1,2$. 2. Let m_X be any m-structure, ρ an m-operator on m_X and $\alpha=m_XCl_\rho$. The properties of the (m_X,α)-T_i are the m-Uryshon axioms introduced in [7].

3. For $m_X=\tau$, α an operator associated with τ, the notions of separations described in the above definition are the α-T_i notions introduced in [8].

4. If $m_X=(\alpha,\beta)-SO(X,\tau)$ and the m-operator α is taken as the operator γ, then the above definition is just the separation axioms $\gamma-(\alpha,\beta)$-T_i introduced in [11].

Now we characterize some properties of the m-spaces described above.

Theorem 3.1. *Let m_X be an m-structure on X and let α be an m-operator on m_X. X is an (m_X,α)-ST_0 if and only if for any pair of distinct points x, $y\in X$, we have that $m_X-SCl_\alpha(\{x\})\neq m_X-SCl_\alpha(\{y\})$.*

Proof. Suppose that X is an (m_X,α)-ST_0, then for any pair of distinct points x, y of X there exists an $\alpha-m_X$-semi-open set U such that $x\in U$ and $y\notin U$ or $y\in U$ and $x\notin U$. It follows from Lemma 2.3, that $m_X-\mathrm{SCl}_\alpha(\{x\})\neq$

$m_X - \mathrm{SCl}_{\alpha}(\{y\})$. Reciprocally, if $m_X - \mathrm{SCl}_{\alpha}(\{x\}) \neq m_X - \mathrm{SCl}_{\alpha}(\{y\})$, there exists a point $z \in m_X - \mathrm{SCl}_{\alpha}(\{x\})$ and $z \notin m_X - \mathrm{SCl}_{\alpha}(\{y\})$, but this implies that, there exists an $\alpha - m_X$-semi-open set U_z such that $x \in U_z$ and $y \notin U_z$. Therefore, X is an (m_X, α)-ST_0. □

Theorem 3.2. *Let m_X be an m-structure on X that satisfies the property (B) and let α be an m-monotone operator on m_X. The following properties are equivalent:*

1. X is an (m_X, α)-ST_1.

2. For any $x \in X$, the unitary set $\{x\}$ is an $\alpha - m_X$-semi-closed set.

3. Each subset A of X is the intersection of all $\alpha - m_X$-semi-open sets of X containing A.

Proof. (1)⇒(2). If X is an (m_X, α)-ST_1 and $x \in X$, then for each $y \in X \setminus \{x\}$, there exists an $\alpha - m_X$-semi-open set U_y such that $y \in U_y$ and $x \notin U_y$, follows that $U_y \cap \{x\} = \emptyset$, therefore $y \in U_y \subseteq X \setminus \{x\}$. In consequence $X \setminus \{x\}$ is an $\alpha - m_X$-semi-open, but this implies that $\{x\}$ is an $\alpha - m_X$-semi-closed set.

(2)⇒(3). Observe that for any $A \subseteq X$, $A = \bigcap_{x \notin A} X \setminus \{x\}$. By hypothesis each unitary set $\{x\}$ is an $\alpha - m_X$-semi-closed set, then each set $X \setminus \{x\}$, with $x \notin A$, is an $\alpha - m_X$-semi-open set.

(3)⇒(1). By hypothesis each unitary set $\{x\}$ is the intersection of all $\alpha - m_X$-semi open sets containing $\{x\}$. In consequence, for each $y \neq x$, there exists an $\alpha - m_X$-semi-open set containing x but not y, follows that X is an (m_X, α)-ST_1. □

In a similar form as the classical case, but not in a necessarily topological context, we have the following separation forms.

Definition 3.3. Let m_X be an m-structure on X and let α be an m-operator on m_X. X is said to be an (m_X, α)-$T_{1/2}$ if each $\alpha - m_X$-sg-closed is an $\alpha - m_X$-semi-closed set.

The following theorem, characterizes the m-spaces satisfying the property (m_X, α)-$ST_{1/2}$.

Theorem 3.3. *Let m_X be an m-structure on X that satisfies the property (B) and let α be an m-monotone operator on m_X. Then X is an (m_X, α)-$ST_{1/2}$ if and only if each unitary set $\{x\}$ in X is an $\alpha - m_X$-semi-open set or an $\alpha - m_X$-semi-closed set.*

Proof. Sufficiency. Suppose that X is an (m_X,α)-$ST_{1/2}$. Then for any $x \in X$, the unitary set $\{x\}$ can be $\alpha - m_X$-semi-closed set or not. In the case that $\{x\}$ is an $\alpha - m_X$-semi-closed set, the result follows. In the other case, $X \setminus \{x\}$ is an $\alpha - m_X$-sg-closed in m_X. Now using hypothesis, we obtain that $X \setminus \{x\}$ is an $\alpha - m_X$-semi-closed set and therefore, $\{x\}$ is an $\alpha - m_X$-semi-open.

Necessity. Let A be an $\alpha - m_X$-sg-closed in m_X and $x \in m_X - \mathrm{SCl}_\alpha(A)$. If $\{x\}$ is an $\alpha - m_X$-semi-open set, then $\{x\} \cap A \neq \emptyset$ and therefore, $x \in A$. In the case that $\{x\}$ is an $\alpha - m_X$-semi-closed set, then we have that $x \in A$, because if $x \notin A$, then $\{x\} \subseteq m_X - \mathrm{sCl}_\alpha(A) \setminus A$. but this is impossible by Theorem 2.1. □

As a consequence of the last theorem, we have the following corollary.

Corollary 3.1. *Let m_X be an m-structure on X that satisfy the property (B) and let α be an m-monotone operator on m_X. Then X is an (m_X,α)-$ST_{1/2}$ if and only if each subset A of X is the intersection of $\alpha - m_X$-semi-open set and $\alpha - m_X$-semi-closed set that contain A.*

Proof. Sufficiency. Suppose that X is an (m_X,α)-$ST_{1/2}$, since each subset $A \subseteq X$ can be written as $A = \bigcap_{x \notin A} X \setminus \{x\}$. Using Theorem 3.3, we obtain that each $A \subseteq X$ is the intersection of sets that are $\alpha - m_X$-semi-open set or $\alpha - m_X$-semi-closed set that contain A.

Necessity. For each $x \in X$, the set $X \setminus \{x\}$ can be written as the intersections of $\alpha - m_X$-semi-open set and $\alpha - m_X$-semi-closed set that contain $X \setminus \{x\}$, then $\{x\} = \bigcup_{i \in I} S_i$. Here each $S_i \subseteq \{x\}$ and S_i is $\alpha - m_X$-semi-open set or $\alpha - m_X$-semi-closed set. In consequence, for some $j \in I$, we obtain that $\{x\} = S_j$. It follows that $\{x\}$ is an $\alpha - m_X$-semi-open set or an $\alpha - m_X$-semi-closed set . □

Corollary 3.2. *Under the hypothesis of Theorem 3.3, The following statements hold:*

1. $(m_X,\alpha) - ST_{1/2} \Rightarrow (m_X,\alpha) - ST_0$.
2. $(m_X,\alpha) - ST_1 \Rightarrow (m_X,\alpha) - ST_{1/2}$.

In analogous form, under flexible conditions on the m-structure and the m-operator α, we characterize the m-spaces that satisfying the property (m_X,α)-$sT_{1/2}$, as follows.

Theorem 3.4. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then X is an (m_X,α)-$sT_{1/2}$ if and only if each unitary set $\{x\}$ in X is an $\alpha - m_X$-open set or an $\alpha - m_X$-closed set.*

Corollary 3.3. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then X is an (m_X, α)-$sT_{1/2}$ if and only if each subset A of X is the intersection of $\alpha - m_X$-open set and $\alpha - m_X$-closed set that contain A.*

Definition 3.4. Let m_X be an m-structure on X. We say that:

1. X is an m_X-D_0 if for each pair of distinct points $x, y \in X$, there exists an m_X-D-set W of X, such that $x \in W$ and $y \notin W$, or $y \in W$ and $x \notin W$.
2. X is an m_X-D_1 if for each pair of distinct points $x, y \in X$, there exist m_X-D-sets, W and Z in X containing x and y, respectively, such that $y \notin W$ and $x \notin Z$.
3. X is an m_X-D_2 if for each pair of distinct points $x, y \in X$, there exist m_X-D-sets, W and Z of X, such that $x \in W$, $y \in Z$ and $W \cap Z = \emptyset$.

If we take m_X as $O(X, m, \alpha)$ (respectively $SO(X, m, \alpha)$), in the above definition, we obtain the following separation properties, denoted by $(m_X, \alpha) - sD_i$ (respectively $(m_X, \alpha) - SD_i$) for $i = 0, 1, 2$. It is immediate that:

$$m_X - D_i \Rightarrow m_X - D_{i-1}; \quad (m_X, \alpha) - sD_i \Rightarrow (m_X, \alpha) - sD_{i-1}\,,$$

$$(m_X, \alpha) - SD_i \Rightarrow (m_X, \alpha) - SD_{i-1}, \text{ for } i = 1, 2.$$

Also,

$$m_X - T_i \Rightarrow m_X - D_i, \text{ for } i = 0, 1, 2.$$

Theorem 3.5. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then:*

(i) X is an $m_X - D_0$ if and only if X is an $m_X - T_0$.

Also, if m_X satisfies the property (B),

(ii) X is an $m_X - D_1$ if and only if X is an $m_X - D_2$.

Proof. (i) *Necessity.* If X is an $m_X - D_0$ and $x \neq y$, there exist $U, V \in m_X$, $U \neq X$, such that: $x \in U \setminus V$ and $y \notin U \setminus V$ or $y \in U \setminus V$ and $x \notin U \setminus V$. If the case is $x \in U \setminus V$ and $y \notin U \setminus V$, then $x \in U$ and $x \notin V$. Since $y \notin U \setminus V$, can happen that $y \notin U$ or $y \in U$ and $y \in V$. If the case is $y \notin U$ then $x \in U$ and $y \notin U$. In the case that $y \in U$ and $y \in V$, we have that $y \in V$ and $x \notin V$. A similar result is obtaining if $y \in U \setminus V$ and $x \notin U \setminus V$. From the above, we conclude that X is an $m_X - T_0$.

Sufficiency. It is immediate, because all m_X-open set different from X, are m_X-D-sets.

(ii) *Necessity.* Suppose that X is an $m_X - D_1$ and $x \neq y$, then there exist $m_X - D-$ sets $U \setminus V$ and $U' \setminus V'$ such that: $x \in U \setminus V$, $y \notin U \setminus V$, $x \notin U' \setminus V'$ and

$y \in U^{'} \setminus V^{'}$. Since $x \notin U^{'} \setminus V^{'}$ then, it can happen some of the following cases: $x \notin U^{'}$ or $x \in U^{'} \cap V^{'}$. If $x \notin U^{'}$, since $y \notin U \setminus V$, we have that: $x \notin U^{'}$, $y \notin U$ or $x \notin U^{'}$, $y \in U \cap V$. In the first case, $x \in U \setminus (U^{'} \cup V)$, because $x \in U \setminus V$, and $y \in U^{'} \setminus (U \cup V^{'})$, but $y \in U^{'} \setminus V^{'}$, also $(U \setminus (U^{'} \cup V)) \cap (U^{'} \setminus (U \cup V^{'})) = \emptyset$. In the second case, we have $x \in U \setminus V$, $y \in V$ and $(U \setminus V) \cap V = \emptyset$. Finally if $x \in U^{'} \cap V^{'}$, then $y \in U^{'} \setminus V^{'}$, $x \in V^{'}$ and $(U^{'} \setminus V^{'}) \cap V^{'} = \emptyset$. Therefore in any case, x and y, can be separated by disjoint $m_X - D-$sets, that is, X is an $m_X - D_2$.

Sufficiency. It is immediate. □

Theorem 3.6. *Let m_X be an m-structure on X and let $\alpha : P(X) \to P(X)$ be an application such that $\alpha(U) \cap V = U \cap \alpha(V) = \emptyset$, for any pair of m_X-open sets U and V, $U \cap V = \emptyset$. Then:*

$$m_X\text{-}T_2 \quad \Rightarrow \quad (m_X, \alpha)\text{-}sT_1 \quad \Rightarrow (m_X, \alpha)\text{-}sT_0$$

Proof. Suppose that X satisfies the property m_X-T_2, and x, y two distinct points in X. It follows that, for each point $z \in X \setminus \{y\}$, there exist m_X sets U_z and U_y, such that $z \in U_z$, $y \in U_y$ and $U_z \cap U_y = \emptyset$. Using the property of α, we have that $\alpha(U_z) \cap U_y = \emptyset$, and we obtain the following inclusions, $\alpha(U_z) \subseteq X \setminus U_y \subseteq X \setminus \{y\}$, but this implies that $X \setminus \{y\}$ is an $\alpha - m_X$-open set that contain x but not y. Proceeding in a similar form, we conclude that $X \setminus \{x\}$ is an $\alpha - m_X$-open set that contains y, but not x, we conclude that X is an (m_X, α)-sT_1. Now, using Theorem 3.4, it follows that (m_X, α)-sT_1 implies (m_X, α)-sT_0. □

We can observe, as follows, that there are many situations under which the hypothesis of Theorem 3.6 are satisfied.

1. All the generalized forms of closure (Definition 2.5 and Definition 2.6) on an m-structure are m-operators (in the sense of Definition 2.2) satisfying the conditions of Theorem 3.6.

2. Let $\emptyset \neq Y \subseteq X$ and define an m structure $m_X = \{A \subseteq X : A \cap Y = \emptyset\} \cup X$, the m-operator $\alpha(A) = A \cup Y$, α is an m-operator that satisfies the hypothesis of Theorem 3.6 and $\alpha \neq m_X$-Cl, because $\alpha(\emptyset) = Y$ and $m_X - \mathrm{Cl}\,(\emptyset) = \emptyset$.

3. Clearly if α satisfies the hypothesis of Theorem 3.6, any operator β with $\beta \preceq \alpha$ also satisfies it. Even more, if α and β satisfying the conditions of Theorem 3.6, then $\rho(A) = \alpha(A) \cup \beta(A)$ and $\rho(A) = \alpha(A) \cap \beta(A)$ are also m-operators satisfying the conditions of Theorem 3.6.

In general, the property (m_X, α)-sT_0 does not imply the property m_X-T_2. But, under certain conditions on the application $\alpha : P(X) \to P(X)$ that acts on an m-structure on X, the reverse implication is valid, as we can see in the following theorem.

Theorem 3.7. *Let m_X be an m-structure on X that satisfies the property B and let $\alpha : P(X) \to P(X)$ be an application that satisfies the following condition $m_X Cl \preceq \alpha$, then: (m_X, α)-sT_0 implies m_X-T_2.*

Proof. Suppose that X is an (m_X, α)-sT_0. For each pair of points x, y in X, such that $x \neq y$, can happen the following cases:

$$\text{(a)}\ \ x \in U,\, y \notin U\,,\ \text{ or }\ \text{(b)}\ \ y \in U,\, x \notin U;$$

for some $\alpha - m_X$-open set U in X.

In the case (a), there exists a set $U_x \in m_X$ such that $x \in U_x$ and $\alpha(U_x) \subseteq U$. By hypothesis $m_X Cl \prec \alpha$, we have that $x \in U_x \subseteq m_X \mathrm{Cl}\,(U_x) \subseteq \alpha(U_x) \subseteq U$. Follows that, $m_X \mathrm{Cl}\,(U_x) \subseteq U$, and since $y \notin U$ we obtain that $y \in X \setminus U \subseteq X \setminus m_X \mathrm{Cl}\,(U_x)$. Therefore, there exist m_X-open sets, U_x and $X \setminus m_X \mathrm{Cl}\,(U_x)$ containing x and y, respectively, such that $U_x \cap (X \setminus m_X \mathrm{Cl}\,(U_x)) = \emptyset$.

In a similar form, we can prove the case (b), that is, there exists an $U_y \in m_X$ such that $y \in U_y$, $x \in X \setminus m_x \mathrm{Cl}\,(U_y)$ also $U_y \cap (X \setminus m_x \mathrm{Cl}\,(U_y)) = \emptyset$, and we conclude that X is an m_X-T_2. □

Observe that Theorem 4.8 in [1], corresponds to the trivial case, because $m_X Cl \preceq m_X Cl$.

An immediate consequence of the last two theorems and Theorem 3.3, is the following corollary.

Corollary 3.4. *Under the hypothesis of Theorems 3.4 and 3.5. The following properties are equivalent:*

1. (X, m) is $(m_X, \alpha) - sD_2$; 2. (X, m) is $(m_X, \alpha) - sD_1$;
3. (X, m) is $(m_X, \alpha) - sD_0$; 4. (X, m) is (m_X, α)-sT_0;
5. (X, m) is (m_X, α)-sT_1; 6. (X, m) is m_X-T_2.

Observe that from the comments on Theorem 3.9, there are many m-operators different from $m_X - Cl$ and $m_X - \mathrm{SCl}_\alpha$ satisfying the hypothesis of Theorems 3.4 and 3.5.

Definition 3.5. Let m_X be an m-structure on X and let $\alpha : P(X) \to P(X)$ be an m-operator on m_X. We say that α is regular respect to m_X, if for each $x \in X$ and each $U \in m_X$ such that $x \in U$, there exists $V \in m_X$ such that $x \in V$ and $\alpha(V) \subseteq U$.

The following theorems characterizes the operators that are regular with respect to an m-structure m_X.

Theorem 3.8. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then:*

$$\alpha \text{ is regular with respect to an } m_X \iff m_X = \{A : A \text{ is } (m_X, \alpha)\text{-open}\}.$$

Proof. Sufficiency. Suppose that α is regular with respect to m_X and that there exists an m_X-closed subset F such that $m_X - Cl_\alpha(F) \nsubseteq F$. It follows that, there exists a point x such that $x \in m_X - Cl_\alpha(F)$ and $x \notin F$; but this imply that $x \in X \setminus F$, but $X \setminus F$ is an m_X-open set. Now using the hypothesis, there exists $V \in m_X$ such that $x \in V$ and $\alpha(V) \subseteq X \setminus F$, therefore $\alpha(V) \cap F \subseteq (X \setminus F) \cap F = \emptyset$, but this is impossible, because $x \in m_X - Cl_\alpha(F)$. In consequence, all m_X-closed set is an $(m_X, \alpha)^*$-closed, now using Lemma 2.4, it follows that all $(m_X, \alpha)^*$-closed set are m_X-closed.

Necessity. If $m_X = \{A : A$ is an (m_X, α)-open$\}$ and $x \in X$ with $x \in U$, where $U \in m_X$, then, we have that $x \notin X \setminus U = m_X - \mathrm{Cl}\,(X \setminus U)$, therefore, there exists $V \in m_X$ for which $x \in V$ and $\alpha(V) \cap (X \setminus U) = \emptyset$, it follows that $\alpha(V) \subseteq U$. □

Observe that the above theorem generalizes the characterizations of the regular spaces, in the case when is using the semi regular topology given in [7].

Theorem 3.9. *Let m_X be an m-structure on X and let $\alpha : P(X) \to P(X)$ be an m-operator on m_X. If α is regular with respect to m_X. Then the following properties hold:*

1. All m_X-open set are α-m_X-open set.

2. $m_X - Cl_\alpha(A) = m_X - Cl(A)$, for all $A \subseteq X$.

3. For all m-operator $\beta : P(X) \to P(X)$ on m_X such that $\beta \preceq \alpha$, we have:

$$\{A : A \text{ is an } (m_X, \beta)\text{-open set}\} = \{A : A \text{ is } (m_X, \alpha)\text{-open set}\} = m_X.$$

Proof. 1. If $U \in m_X$, then for all $x \in U$ there exists $V_x \in m_X$ such that $x \in V_x$ and $\alpha(V_x) \subseteq U$. This implies that U is an $\alpha - m_X$-open set.

2. Suppose that $x \notin m_X - \mathrm{Cl}\,(A)$, it follows that $x \in X \setminus m_X - \mathrm{Cl}\,(A)$ and $X \setminus m_X - \mathrm{Cl}\,(A) \in m_X$, but by hypothesis, there exists $V \in m_X$ such that $x \in V$ and $\alpha(V) \subseteq X \setminus m_X - \mathrm{Cl}\,(A)$, in consequence, $\alpha(V) \cap A \subseteq (X \setminus m_X - \mathrm{Cl}\,(A)) \cap A = \emptyset$, but it implies that $x \notin m_X - Cl_\alpha(A)$. Therefore, $m_X - Cl_\alpha(A) \subseteq m_X - \mathrm{Cl}\,(A)$. Now using Lemma 2.4, we obtain the other inclusion.

3. If α is a regular operator with respect to m_X and $\beta \preceq \alpha$, it follows that β is regular with respect to m_X and by Lemma 3.1 the result follows. □

We can observe, from the last result, that many separation axioms or separation properties described before are satisfied for a regular m-operator on m_X.

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