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NEIGHBORHOOD CONDITIONS FOR BALANCED BIPARTITE GRAPHS TO BE HAMILTONIAN

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Abstract: Let G be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$. If for every balanced independent set S of four vertices $|N(S)| > n$, then G is Hamiltonian.

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1. Introduction

We use Behzad et al [2] for terminology and notation not defined here. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a simple, finite and undirected graph G. According to the (arbitrarily) orientation of a cycle C of G, the successor and predecessor of a vertex z of C are denoted by z^+ and z^-, respectively. Let G be a balanced bipartite simple graph of order $2n$, i.e. a graph with a bipartition into two independent vertex sets of the same cardinality. $N(S)$ is the neighborhood union of a balanced independent set S of four vertices, i.e. an independent set containing two vertices from each side of the bipartition. G is Hamiltonian if it has a cycle containing all the vertices of G.

The investigation of certain extremal problems involving neighborhood union conditions for balanced independent sets of cardinality four was initiated by

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Amar et al [1]. They posed the following question: Let H_{14} denote the class of graphs obtained from the graph depicted in Figure 1, where some (or all) of the four possible edges joining the top to the bottom might be present as well.

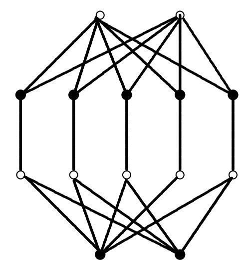

Figure 1: The minimal graph of H_{14}

Conjecture 1. (see Amar et al [1]) *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $\mid S \mid= 4$, we have $\mid N(S) \mid> n$, then either G is hamiltonian or $G \in H_{14}$.*

They also proved the following theorem.

Theorem 2. (see Amar et al [1]) *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $\mid S \mid= 4$, we have $\mid N(S) \mid> n$, then either G is Hamiltonian or G contains a spanning subgraph consisting of a cycle and an isolated edge.*

In this paper we shall prove the following theorem.

Theorem 3. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geq 4$. If for every balanced independent set S with $\mid S \mid= 4$, we have $\mid N(S) \mid> n$, then G is hamiltonian.*

The condition on the neighborhood union of G in Theorem 3 is sharp. To see this, take two complete balanced bipartite graphs, choose from each of them one side of the bipartition and join these sides completely. The resulting graph satisfies $\mid N(S) \mid= n$ for every balanced independent set S, with $\mid S \mid= 4$, but it is not Hamiltonian. H_{14} (see Figure 1) shows that also the minimum degree condition in Theorem 3 cannot be reduced. This is a non-Hamiltonian graph with minimum degree three, which satisfies the neighborhood union condition.

2. Previous Lemmas

Lemma 4. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 2$. If for every balanced independent set S with $\mid S \mid = 4$, we have $\mid N(S) \mid > n$, then G is connected.*

Proof. Let $\{A, B\}$ be a balanced bipartition of $V(G)$ and suppose that G is not connected; then there exist distinct components R and T of G. As $\delta(G) \geq 2$, each component has at least two vertices in A and two vertices in B. We may assume that $\mid V(R) \cap A \mid \geq \mid V(T) \cap B \mid$. Then every set S containing two vertices of $V(R) \cap A$ and two vertices of $V(T) \cap B$ is a balanced independent set with $\mid N(S) \mid \leq n$, contradicting the hypothesis. □

Lemma 5. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 1$. If for every balanced independent set S with $\mid S \mid = 4$, we have $\mid N(S) \mid \geq n$, then G has a perfect matching.*

Proof. Let $\{A, B\}$ be a balanced bipartition of $V(G)$. By the Koning-Hall Theorem (see Swamy and Thulasiraman [3]) it suffices to show that for every subset $R \subset A$ we have $\mid N(R) \mid \geq \mid R \mid$. Indeed, if $\mid N(R) \mid < \mid R \mid$ then $2 \leq \mid R \mid \leq n - 1$ as $\delta(G) \geq 1$. So every set S containing two vertices of R and two vertices of $B - N(R)$ is a balanced independent set of four vertices with $\mid N(S) \mid < n$, a contradiction. □

3. Proof of Theorem 3

Proof. Let $\{A, B\}$ be a balanced bipartition of $V(G)$. From Lemma 4, G is connected. Suppose G is not hamiltonian; then by Theorem 2 there exists a cycle C of length $2n-2$ such that $G-C$ is one edge. Without loss of generality let $C = a_2b_2...a_ib_i...a_nb_na_2$ and $a_1b_1 \in E(G-C)$. Since G has minimum degree $\delta(G) \geq 4$, then a_1 and b_1 have both at least three neighbors on C. Let

$$T = \{b_i^+ \in V(C) : b_i \in N_C(a_1)\} \cup \{a_j^+ \in V(C) : a_j \in N_C(b_1)\}.$$

It is obvious that T is a balanced independent set. Let $S \subset T$ be a balanced independent set such that $\mid S \mid = 4$. By Lemma 5, G has a perfect matching M and since $\mid N(S) \mid > n$, there is an edge of M, both ends of which are adjacent to S, thus we obtain a contradiction to the supposition. Therefore $\mid N(S) \mid \leq n$, which contradicts the hypothesis of the theorem. This proves the theorem. □

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