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NUMBER OF INTERSECTIONS OF A FIXED PLANE CURVE WITH PLANE CONICS

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Abstract: Let $C \subset \mathbf{P}^2$ be an integral degree d plane curve. Here we study the integers $\sharp(A \cap C)$ (and related integers when A contains a singular point of C) when A varies among the set Σ all plane conics or a suitable subset of Σ (e.g. the set of all smooth conics or the set of all singular conics).

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1. Number of Intersections of a Fixed Plane Curve with Plane Conics

We work over an algebraically closed field $\mathbb{K}$ and then adapt our definitions (but do not prove anything) to the case of a finite field. Let $A, C \subset \mathbf{P}^2$ be plane curves defined over $\mathbb{K}$. We will alway assume that C is irreducible and that it is not an irreducible component of A. Hence the set $A \cap C$ is finite and we will set $[A \cdot C] := \sharp(A \cap C)$. Hence $1 \le [A \cdot C] \le \deg(A) \cdot \deg(C)$. Now assume that A varies in a set Φ of plane conics. We want to study the set of integers $\{[A \cdot C]\}_{A \in \Phi}$ in the following cases:

(i) Φ is the set of all smooth conics;

(ii) Φ is the set of all reduced conics; hence here we allow smooth conics and unions of two distinct lines, but no double lines;

(iii) Φ is the set of all conics; hence here we allow all smooth conics, all pairs of distinct lines and all double lines.

(iv) Φ is the set of all singular conics;

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(v) Φ is the set of all reduced, but singular conics, i.e. the set of all unions of two distinct lines;

(vi) Φ is the set of all double lines.

If C is singular and $A \cap C = \{P_1, \dots, P_s\}$ with $P_i \neq P_j$ for all $i \neq j$, set $\bullet(A, C) := \sum_{i=1}^{s} r_i$, where r_i is the number of branches of C at P_i. Hence $[A \cdot C] \leq \bullet(A, C) \leq \deg(A) \cdot \deg(C)$. We may consider the sets $\{\bullet(A, C)\}_{A \in \Phi}$ when Φ is as in (i), (ii), (iii), (iv), (v) or (vi). We prove the following result.

Theorem 1. *Fix an integer $d \geq 3$ and a degree d integral plane curve C. Let Φ_1 (resp. Φ_2, resp. Φ_3, resp Φ_4, resp. Φ_5, resp. Φ_6) denote the set of all smooth conics, resp. all reduced conics, all conics, resp. all singular conics, resp all reduced singular conics, resp. all double lines).*

(a) *If either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > 2d$, *then* $\{2d, 2d-1, 2d-2, 2d-3\} \subseteq \{[A \cdot C\}_{A \in \Phi_i} \cap \{\bullet(A, C)\}_{A \in \Phi_i}$ *for* $i = 1, 2, 3$.

(b) *If either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > d$, *then* $\{d, d-1\} \subsetneqq \{[A \cdot C\}_{A \in \Phi_i} \cap \{\bullet(A, C)\}_{A \in \Phi_6}$.

(c) *If either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > d(d-1)$, *then* $\{2d, 2d-1, 2d-2, 2d-3\} \subseteq \{[A \cdot C\}_{A \in \Phi_i} \cap \{\bullet(A, C)\}_{A \in \Phi_i}$ *for* $i = 4, 5$.

Now we fix a prime power q and assume $\mathbb{K} = \bar{\mathbb{F}}_q$. We assume that A and C are defined over $\mathbb{F}_q$. Set $[A \cdot C]_q := \sharp(A(\mathbb{F}_q) \cap C(\mathbb{F}_q))$. We also define $\bullet(A, C)$ counting only the branches of C defined over $\mathbb{F}_q$. We may consider the set of integers when A varies in the family $\Phi(q)$ obtained from a family of plane conic as in (i), (ii), (iii), (iv), (v) and (vi) taking only conics defined over $\mathbb{F}_q$. In case (ii), (iii), (iv), (v) and (vi) we write (ii;q), (iii;q), (iv;q), (v;q) and (vi;q) if we allow only union of two lines such that each of them is defined over $\mathbb{F}_q$. Now we consider a more general problem. We fix an integer $e \geq 1$ and look at the sets of integers $\{[A \cdot C]_{q^e}\}_{A \in \Phi(q)}$ and $\{\bullet(A, C)_{q^e}\}_{A \in \Phi(q)}$. Roughly speaking, we count the solutions over $\mathbb{F}_{q^e}$, but we allow only conics defined over $\mathbb{F}_q$. It is easy to adapt the proofs in [1] to get for suitable $q \gg d$ the existence of certain degree d curves such that the sets of integers appearing as $\bullet(A, C)$ or as $[A, \cdot C]$ is large. However, we do not know if for every integral degree d curve C these sets must contain a suitable prescribed integer, e.g. the integer $2d$.

Notation 1. Fix plane curves A, C and $P \in A_{reg} \cap C_{reg}$. Let $o_P(A, C)$ denote the contact order of A and C at P.

Remark 1. Fix an integral plane curve $C \subset \mathbf{P}^2$ such that $d := \deg(C) \geq 3$. Let Ψ denote the family of all lines, Θ the family of all double lines and Ξ the family of all reducible conics which are union of two distinct lines. By Bertini's

Theorem there are several lines L intersecting transversally C. Thus there are $A \in \Theta$ and $B \in \Xi$ such that $[A \cdot C] = \bullet(A, C) = d$ and $[B \cdot C] = \bullet(B, C) = d$. By Bertini's Theorem there is a smooth conic B such that $[B \cdot C] = \bullet(B, C) = 2d$. Obviously, $x = [L \cdot C]$ (resp. $x = \bullet(L, C)$) for some line L if and only if $x = [2L \cdot C]$ (resp. $x = \bullet(2L, C)$) for some double line $2L$.

Remark 2. Here we assume $\mathrm{char}(\mathbb{K}) = 0$ and take the set-up of (i), i.e. Φ is the set of all smooth conics. Fix C and a general $P \in C$. Let $u : X \to C$ be the normalization map. Set $d := \deg(C)$.

First Claim. *There are smooth conics A_i, $1 \leq i \leq 5$, such that $P \in A_i$ and $o_P(A_i, C) = i$ for all i. Furthermore, A_5 is unique.*

Proof of First Claim. The linear system $|\mathcal{O}_{\mathbf{P}^2}(2)|$ of all plane conics induces a morphism $u_2 : X \to \mathbf{P}^5$. Since $\mathrm{char}(\mathbb{K}) = 0$, the contact order of u_2 at a general point $u^{-1}(P)$ of X is the classical one ([2], Theorem 15). This is equivalent to the First Claim, except that a priori some of the conics A_i may be singular. By the generality of P the tangent line $T_C P$ of C at P has contact order 2 with C at P. This implies that A_5 must be smooth and easily shows (using a dimensional count) that we may take as A_i, $1 \leq i \leq 4$, a smooth conic.

Second Claim. *There are smooth conics A_i, $1 \leq i \leq 5$, such that $P \in A_i$, $o_P(A_i, C) = i$ and $A_i \cap \mathrm{Sing}(C) = \emptyset$ for all i.*

Proof of Second Claim. Fix $Q \in \mathrm{Sing}(C)$. The linear system $|\mathcal{I}_Q(2)|$ induces a morphism $u_Q : X \to \mathbf{P}^4$. The statement $Q \notin A_5$ for a general P is equivalent to the fact that u_Q has classical Hermite sequence (which is true by [2], Theorem 15). For sufficiently general A_i, $1 \leq i \leq 4$, this is true by the same quotation. Since $\mathrm{Sing}(C)$ is finite, we get the Second Claim.

Third Claim. *Fix a general $(P, P') \in C \times C$ and use P for the First and Second Claims. Let $B_i(P')$ (resp. $B_i'(P')$ denote the set of all conics (resp. smooth conics) smooth at P and containing P'. Let $D_i(P')$ (resp. $D_i'(P')$) denote the set of all $A \in B_i(P')$ (resp. $A \in B_i(P')$) which are either singular at P' or tangent to C at P'. Then:*

(a) $B_5(P') = B_5'(P') = D_5(P') = D_5'(P') = D_4(P') = D_4'(P') = \emptyset$.

(b) *If* $1 \leq i \leq 4$, *then* $\dim(B_i(P')) = \dim(B_i'(P')) = 5 - i - 1$.

(c) *If* $1 \leq i \leq 3$, *then* $\dim(D_i(P')) = \dim(D_i'(P')) = 5 - i - 2$.

Proof of the Third Claim. Let $\{2P', C\}$ denote the degree two effective divisor of C with P' as its support. We will see $\{2P', C\}$ as a length 2 zero-dimensional subscheme of $\mathbf{P}^2$. Since (P, P') is general in $C \times C$, we may fix P' and then (with fixed P'). Use that the linear system $|\mathcal{I}_P(2)|$ (i.e. the linear projection of $u_2(X)$ from $u_2(u^{-1}(P')))$ has classical Hermite sequence ([2], Theorem 15) to prove all the assertions concerning the sets $B_i(P')$ and hence on the sets $B_i'(P')$. Use that the linear system $|\mathcal{I}_{\{2P',C\}}(2)|$ to prove all the assertions concerning the sets $D_i(P')$ and hence the sets $D_i'(P')$.

Fourth Claim. *Let U_i, $1 \le i \le 4$, be the linear system on X which is the linear span of the set of all effective divisors on X induced by all smooth conics E_i such that $o_P(E_i, C) \ge i$. Then U_i has no base point.*

Proof of Fourth Claim. Since $U_4 \subseteq U_j$ for $j = 1, 2, 3$, it is sufficient to show that U_4 has no base points. The existence of A_4 gives that $\pi^{-1}(P)$ is not a base point of U_4. Fix $Q \in C_{reg} \backslash \{P\}$. If $Q \notin T_P C$, then the double line $2T_P C$ shows that $u^{-1}(Q)$ is not a base point of U_4. Now assume $Q \in T_P C \cap C$ and $Q \neq P$. There is a smooth conic $A_4 \in U_4$. Since $o_P(T_P C, A_4) = 2$ and $Q \in T_P C$, $Q \notin A_4$, by Bezout, proving the Fourth Claim.

Fifth Claim. *There are smooth conics A_i, $1 \le i \le 4$, such that $P \in A_i$, $o_P(A_i, C) = i$ and A_i intersects transversally C outside P, i.e. $[A_i \cdot C] = \bullet(A_i, C) = 2d + 1 - i$ for all i.*

Proof of Fifth Claim. By the Fourth Claim the associated linear systems have no base points. Hence the Fifth Claim is true by Bertini's Theorem.

Remark 3. Fix an integral plane curve $C \subset \mathbf{P}^2$ such that $d := \deg(C) \ge 3$. Assume $\mathrm{char}(\mathbb{K}) > 2d$. Then the First, Second, Third and Fourth Claims of Remark 2 are true under our assumption, because we again may quote [2], 15. To avoid the use of Bertini's Theorem in the Fifth Claim we use a dimensional count of tangent vectors as in a classical proof of Bertini's Theorem.

Proposition 1. *Fix an integer $d \ge 3$ and let C be a general degree d plane curve. Let Ψ denote either the set of all lines or the set of all double lines of $\mathbf{P}^2$. Then $\{[L \cdot C]\}_{L \in \Psi} = \{\bullet(L, C)\}_{L \in \Psi} = \{d, d-1, d-2\}$.*

Proof. Since C is smooth, $\bullet(L, C) = [L \cdot C]$ for every L. The proposition follows from the following assertions:

(a) $o_P(T_P C, C) = 2$ for a general $P \in C$;

(b) C has a non-empty set of flexes;

(c) each flex is an ordinary flex and every flex line is tangent to C at a unique point;

(d) no line is tangent to C at more than two points.

All the assertions are well-known and easy in arbitrary characteristic (e.g if $d > 3$ use a degeneration of C to a suitable reducible curve). To get (b) use the Brill-Segre formula ([2], Theorem 9) and the fact that by part (a) C has classical Hermite sequence, i.e. in the formula in [2] put $b_i = i$ for $i = 0, 1, 2$. □

Remark 4. Fix an integer $d \geq 3$ and assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > d(d-1)$. Let C be an integral degree d plane curve. By Bertini's Theorem there is a line L such that $\sharp(C \cap L) = d$ and $L \cap \mathrm{Sing}(X) = \emptyset$. Our assumptions on $\mathrm{char}(\mathbb{K})$ imply that a general tangent line D has two as order of contact. Hence $[D \cdot C] = \bullet(D, C) = d-1$. By the Brill-Segre formula applied to the normalization of C ([2], Theorem 9) we get the existence of a line R such that $\bullet(R, C) \leq d-2$. Hence $[R \cdot C] \leq d-2$.

Remark 5. Fix an integer $d \geq 3$ and assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > d$. Let C be an integral degree d plane curve. Fix a general $P \in C$. By [2], Theorem 14, we have $o_P(T_P C, C) = 2$. Furthermore, $T_P C \cap \mathrm{Sing}(C) = \emptyset$, because P is general and C cannot be strange by our assumption on $\mathrm{char}(\mathbb{K})$. Hence there is $Q \in C_{reg} \cap T_P C$ such that $Q \neq P$. The degree of the Gaussian map of the normalization of C is at most $d(d-1)$. Thus by our assumptions on $\mathrm{char}(\mathbb{K})$ and the generality of P we also get that $T_P C$ is not multisecant. Hence $\bullet(T_P C, C) = [T_P C \cdot C] = d-1$. For a general line D through Q we have $\bullet(T_P C \cup D, C) = [T_P C \cup D \cdot C] = 2d-2$.

Proof of Theorem 1. The statement concerning Φ_1 follows from Remarks 2 and 3. The statement concerning Φ_1 implies the statements concerning Φ_2 and Φ_3. The statement concerning Φ_6 is true by Remarks 1 and 4. The statements concerning Φ_4 and Φ_5 are true by Remark 5. □

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