International Journal of Pure and Applied Mathematics

Volume 35 No. 2 2007, 235-242

TOTAL K-DOMINATION NUMBER IN GRAPHS

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Abstract: Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$. A set $D \subseteq V$ is a *total k-dominating set* of G if every vertex in V is adjacent to at least k vertices of D. The *total k-domination number* $\gamma_t^k(G)$ of G is the minimum cardinality of a total k-dominating set of G. In this paper we establish some sharp bounds on $\gamma_t^k(G)$ of a graph G. Moreover, we give the exact value of total 2-domination number for a class of grid graphs.

AMS Subject Classification: 05C69
Key Words: k-domination, total k-domination, bounds, grid graph

1. Introduction

All graphs considered here are simple and finite. For standard graph theory terminology not given here, we refer the reader to [2].

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *open neighborhood* of a vertex v of G is denoted by $N(v) = \{u \in V \mid (u, v) \in E\}$. The degree of a vertex v of G is denoted by $d(v) = |N(v)|$. The maximum degree of vertices of G is denoted by Δ, and the minimum degree of vertices of

Received: December 19, 2006 © 2007, Academic Publications Ltd.

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G is denoted by δ. For any $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S.

A set $D \subseteq V$ is a *dominating set* of G if $|N(v) \cap D| \geq 1$ for all $v \in V \setminus D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Similarly, a set $D \subseteq V$ is a *total dominating set* of G if $|N(v) \cap D| \geq 1$ for all $v \in V$. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G. The literature on two parameters has been surveyed and detailed in two books by Haynes et al [5,6].

Fink et al [3] and Kulli [7] generalized the domination and total domination to k-domination and total k-domination, respectively. Let $k \in \mathbb{N}$, a set D of vertices of G is a *k-dominating set* if $|N(v) \cap D| \geq k$ for all $v \in V \setminus D$. The *k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a k-dominating set of G. The study of $\gamma_k(G)$ was continued by Rautenbach and Volkmann [8]. Similarly, a set D of vertices of G is a *total k-dominating set* of G if $|N(v) \cap D| \geq k$ for all $v \in V$. The *total k-domination number* $\gamma_t^k(G)$ of G is the minimum cardinality of a total k-dominating set of G. It is obvious that $\gamma_1(G) = \gamma(G)$ and $\gamma_t^1(G) = \gamma_t(G)$.

The *Cartesian product* of two graphs G and H is the graph, denoted by $G \Box H$, with $V(G \Box H) = V(G) \times V(H)$ (where $\times$ denotes the Cartesian product of sets) and $((u, u'), (v, v')) \in E(G \Box H)$ if and only if $u = v$ and $(u', v') \in E(H)$ or $u' = v'$ and $(u, v) \in E(G)$. We call that $P_i \Box P_j$ is a *grid graph*, $P_i \Box C_j$ for $j \geq 3$ is a *cylinder graph* and $C_i \Box C_j$ for $i \geq 3$ and $j \geq 3$ is a *torus graph*.

In [7], Kulli gave the values of total k-domination number for several special graphs such as the complete graphs, complete bipartite graphs, and proved that $\gamma_t^k(G) \geq \gamma_t(G) + \delta(G) - 1$ for a graph G. In this paper we continue to study the total k-domination number of a graph. We establish some sharp bounds on $\gamma_t^k(G)$ of a graph G and give the exact value of total 2-domination number of $P_2 \Box P_n$.

2. General Graphs

In this section we present some sharp bounds on total k-domination number of a graph.

Proposition 1. *If $K_{n_1,n_2,\ldots,n_m}$ is a complete m-partite graph, then $\gamma_t^k(K_{n_1,n_2,\ldots,n_m}) = k + 1$ for $m \geq k + 1$.*

Proof. Let $K_{n_1,n_2,\cdots,n_m} = (V_{n_1}, V_{n_2}, \cdots, V_{n_m}; E)$. Clearly,

$$\gamma_t^k(K_{n_1,n_2,\cdots,n_m}) \geq k + 1\,,$$

by the definition of total k-dominating set. We choose a vertex $v_i \in V_{n_i}$ for each $i = 1, 2, \cdots, k+1$, and let D be a set consisting of all v_i's. It is easily checked that S is a total k-dominating set of $K_{n_1,n_2,\ldots,n_m}$, and thus $\gamma_t^k(K_{n_1,n_2,\cdots,n_m}) \leq |D| = k+1$ as $m \geq k+1$. So $\gamma_t^k(K_{n_1,n_2,\cdots,n_m}) = k+1$. □

Theorem 2. *Let $k \in \mathbb{N}$ and suppose $G = (V, E)$ is a graph with minimum degree $\delta \geq k$. Then a total k-dominating set D of G is minimal if and only if for every vertex $v \in D$ there exists a vertex $u \in N(v)$ such that $|N(u) \cap D| = k$.*

Proof. Let D be a minimal total k-dominating set in G. Suppose there exists a vertex $v \in D$ such that $|N(u) \cap D| \geq k+1$ for every vertex $u \in N(v)$. Set $D' = D - \{v\}$. Since every vertex $w \in V$ is adjacent to at least k vertices in D', it follows that D' is a total k-dominating set of G, contradicting the minimality of D.

Now assume that D is a total k-dominating set satisfying that for every vertex $v \in D$ there exists a vertex $u \in N(v)$ such that $|N(u) \cap D| = k$. Consider the set $D' = D - \{v\}$ for any vertex $v \in D$. According to our assumption, there exists a vertex $u \in N(v)$ such that $|N(u) \cap D| = k$. So D' would not total k-dominate u, and hence it would not be a total k-dominating set of G. Thus D is a minimal total k-dominating set of G. □

Theorem 3. *Let $G = (V, E)$ be a graph of order n and minimum degree $\delta \geq k$, $k \in \mathbb{N}$. Then $\gamma_t^k(G) = n$ if and only if for every vertex $v \in V$ there exists a vertex $u \in N(v)$ such that $|N(u)| = k$.*

Proof. Suppose $\gamma_t^k(G) = n$, then the condition holds. Otherwise there exists a vertex $v \in V$ such that $|N(u)| \geq k+1$ for every $u \in N(v)$. Then the set $D = V - \{v\}$ is a total k-dominating set of G, and hence $\gamma_t^k(G) \leq |D| = n-1$, a contradiction.

Conversely, suppose that for every vertex $v \in V$, there exists a vertex $u \in N(v)$ such that $|N(u)| = k$, we claim that $\gamma_t^k(G) = n$. Otherwise let D be a minimum total k-dominating set satisfying $|D| = \gamma_t^k(G) < n$. Then $V \backslash D \neq \emptyset$. Take $v \in V \backslash D$. According to the assumption, there exists a vertex $u \in N(v)$ such that $|N(u)| = k$ but $|N(u) \cap D| < k$, thus D is not a total k-dominating set, contracting the choice of D. □

As an immediate consequence of Theorem 3, we have the following result.

Corollary 4. (see [7]) *If $G = (V, E)$ is a k-regular graph of order n and $k \in \mathbb{N}$, then $\gamma_t^k(G) = n$.*

Next we will give lower and upper bounds on $\gamma_t^k(G)$ in terms of order, size, maximum and minimum degrees of a graph G.

Theorem 5. *Let $k \in \mathbb{N}$ and suppose $G = (V, E)$ be a graph with minimum degree $\delta \geq k$. Then $\gamma_t^k(G) \geq kn/\Delta$ and this bound is sharp.*

Proof. Let D be a minimum total k-dominating set of G, and $S = V \backslash D$. We distinguish the following two cases.

Case 1. If $S \neq \emptyset$, let t denote the number of edges between D and S, since the degree of each vertex in D is at most $(\Delta - k)$, thus $t \leq (\Delta - k)\gamma_t^k(G)$. But since each vertex in S is adjacent with at least k vertices in D, we know $t \geq k(n - \gamma_t^k(G))$. Combining these two inequalities we have $\gamma_t^k(G) \geq kn/\Delta$.

Case 2. If $S = \emptyset$, then $V = D$. Since $\Delta \geq \delta \geq k$, we know $(k/\Delta) \leq 1$. Thus $\gamma_t^k(G) = |V| = n \geq kn/\Delta$.

Corollary 4 implies that the sharpness of the lower bound. □

Corollary 6. *If $G = (V, E)$ is an l-regular graph of order n and $l \geq k$, $k \in \mathbb{N}$, then $\gamma_t^k(G) \geq kn/l$.*

Using the approach developed by Harary and Haynes in [4], we obtain the lower bound on $\gamma_t^k(G)$ in terms of the order and size of a graph.

Theorem 7. *Let $k \in \mathbb{N}$ and suppose $G = (V, E)$ be a graph of size m with minimum degree $\delta \geq k$. Then $\gamma_t^k(G) \geq 2(n - \frac{m}{k})$ and this bound is sharp.*

Proof. Let D be a minimum total k-dominating set of G. Then each vertex of $V \backslash D$ is adjacent to at least k vertices in D. Further, each vertex of D has at least k neighbors in D. Thus the number m of edges of G satisfies

$$m \geq k|V - D| + k\frac{|D|}{2} = k(n - \gamma_t^k(G)) + k\frac{\gamma_t^k(G)}{2}.$$

So $\gamma_t^k(G) \geq 2(n - \frac{m}{k})$.

Let G_1 be the complete bipartite graph $K_{k,k}$ with vertex classes X and Y and let $G_2 = K_1$. Let G be the graph obtained from disjoint union of G_1 and G_2 by joining the unique vertex of G_2 with all vertices of X. By construction, it is easy to see that G realize the sharp lower bound. □

Theorem 8. *Let $G = (V, E)$ be a graph of order n and minimum degree $\delta \geq k$, $k \in \mathbb{N}$. If $\frac{\delta}{\ln \delta} \geq 2k$, then*

$$\gamma_t^k(G) \leq \frac{n}{\delta}(k \ln \delta + \sum_{i=0}^{k-1} \frac{k-i}{i!\delta^{k-1-i}}).$$

Proof. Let $p = k\frac{\ln\delta}{\delta}$, then $0 \le p \le \frac{1}{2}$. We form a set D by picking every vertex v of G independently at random with $P[v \in D] = p$. For $0 \le i \le k-1$, let $D_i = \{v \in V || N(v) \cap D| = i\}$. For $v \in V$ and $0 \le i \le k-1$, we have

$$\begin{aligned} P[v \in D_i] &= \binom{d(v)}{i} p^i (1-p)^{(d(v)-i)} \le \binom{d(v)}{i} (1-p)^{(d(v))} \\ &\le \frac{1}{i!}(d(v))^i (1-p)^{d(v)} \le \frac{1}{i!}(d(v))^i e^{-pd(v)} \quad (1 - x \le e^{-x} \quad for \quad x \in \mathbb{R}) \\ &= \frac{1}{i!} e^{-pd(v)+i\ln d(v)}\,. \end{aligned}$$

Since $p \ge \frac{k-1}{\delta}$, we have

$$\frac{\partial}{\partial d(v)}(-pd(v) + i\ln d(v)) = -p + \frac{i}{d(v)} \le -p + \frac{k-1}{\delta} \le 0$$

and the exponent of e in the last expression is monotonically decreasing in $d(v)$ which implies

$$P[v \in D_i] \le \frac{1}{i!} e^{-p\delta + i\ln\delta} = \frac{1}{i!} e^{-k\ln\delta + i\ln\delta} = \frac{1}{i!}\frac{1}{\delta^{k-i}}\,.$$

Hence

$$E[|D_i|] \le \sum_{v\in V} P[v \in D_i] \le \frac{1}{i!}\frac{n}{\delta^{k-i}}\,.$$

Since $\delta \ge k$, for $0 \le i \le k-1$ there is a set $D'_i \subseteq V \setminus D$ such that $|D'_i| \le (k-i)|D_i|$ and $|N(v) \cap D'_i| \ge k-i$ for all $v \in D_i$ (such a set can be constructed for example as the union of the sets containing $(k-i)$ neighbors of $v \in D_i$ that do not lie in D). Since $D \cup (\cup_{i=0}^{k-1} D')$ is a total k-dominating set of G, the first moment method [1] implies

$$\begin{aligned} \gamma_t^k(G) &\le E(|D \cup (\bigcup_{i=0}^{k-1} D'_i)|) \le E(|D|) + \sum_{i=0}^{k-1} E(|D'_i|) \\ &\le E(|D|) + \sum_{i=0}^{k-1}(k-i)E(|D_i|) \le \frac{n}{\delta}(k\ln\delta + \sum_{i=0}^{k-1}\frac{k-i}{i!\delta^{k-1-i}})\,. \end{aligned}$$

This completes the proof. □

3. Grid Graphs

In this section we give the exact value of total 2-domination number of $P_2 \Box P_n$. We first give the following lemma.

Lemma 9. *Let $k \in \mathbb{N}$ and suppose $G = (V, E)$ be a graph with minimum degree $\delta \geq k$. If $d(v) = k$, then each vertex in $N(v)$ is in every total k-dominating set of G.*

Theorem 10. *For any integer $n \geq 2$, $\gamma_t^2(P_2 \Box P_n) = 2\lfloor \frac{n}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor + 2$.*

Proof. It is easy to verify that the assertion is true for small $n \leq 4$. Hence we may assume $n \geq 5$ in the following.

Let D be a total 2-dominating set of $P_2 \Box P_n$ and let $x_{i,j}$ be the vertex of $P_2 \Box P_n$ in row i ($i = 1, 2$) and column j ($j = 1, 2, \cdots, n$). According to Lemma 9, we know that $S = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{1,n-1}, x_{1,n}, x_{2,n-1}, x_{2,n}\}$ must be in D. Let $f(l)$ be the number of vertices of D in the first l columns of $G' = G[V(P_2 \Box P_n) \backslash S]$. We claim that $f(l+3) \geq f(l)+4$ for any $1 \leq l \leq n-7$. Indeed, let C_k,C_{k+1} and C_{k+2} be three consecutive columns of G'. To 2-dominate the vertices $x_{1,k+1}$ and $x_{2,k+1}$, we need two vertices from $\{x_{1,k}, x_{1,k+2}, x_{2,k+1}\}$ and two more vertices from $\{x_{2,k}, x_{1,k+1}, x_{2,k+2}\}$. Since $f(1) \geq 0$, $f(2) \geq 2$ and $f(3) \geq 4$, we obtain

$$f(n-4) \geq \begin{cases} 4\lfloor \frac{n-4}{3} \rfloor & \text{if } n-4 \equiv 0, 1 \pmod 3, \\ 4\lfloor \frac{n-4}{3} \rfloor + 2 & \text{if } n-4 \equiv 2 \ (\text{mod}3) \end{cases}$$
$$= \begin{cases} 4\lfloor \frac{n-4}{3} \rfloor & \text{if } n \equiv 1, 2 \pmod 3, \\ 4\lfloor \frac{n-4}{3} \rfloor + 2 & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

Thus,

$$|D| = 8 + f(n-4) \geq \begin{cases} 4\lfloor \frac{n-4}{3} \rfloor + 8 & \text{if } n \equiv 1, 2 \pmod 3 \\ 4\lfloor \frac{n-4}{3} \rfloor + 10 & \text{if } n \equiv 0 \pmod 3 \end{cases}$$
$$= 2\lfloor \frac{n}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor + 2 \, .$$

Since D includes any minimum total 2-dominating set of G, we have $\gamma_t^2(P_2 \Box P_n) \geq 2\lfloor \frac{n}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor + 2$.

Now we construct a total 2-dominating set D of G as follows: If $n \equiv 0$ $(mod\ 3)$, let

$$D = (\bigcup_{k=0}^{\lfloor \frac{n-2}{3} \rfloor} \{x_{1,3k+1}, x_{1,3k+2}, x_{2,3k+1}, x_{2,3k+2}\}) \cup \{x_{1,n}, x_{2,n}\}.$$

If $n \equiv 1\ (mod\ 3)$, let

$$D = (\bigcup_{k=0}^{\lfloor \frac{n-2}{3} \rfloor} \{x_{1,3k+1}, x_{1,3k+2}, x_{2,3k+1}, x_{2,3k+2}\}) \cup \{x_{1,n-1}, x_{1,n}, x_{2,n-1}, x_{2,n}\}.$$

If $n \equiv 2\ (mod\ 3)$, let

$$D = \bigcup_{k=0}^{\lfloor \frac{n-2}{3} \rfloor} \{x_{1,3k+1}, x_{1,3k+2}, x_{2,3k+1}, x_{2,3k+2}\}.$$

Thus

$$\begin{aligned} |D| &= \begin{cases} 4\lfloor \frac{n}{3} \rfloor + 2 & \text{if } n \equiv 0\ (mod\ 3)\,, \\ 4\lfloor \frac{n}{3} \rfloor + 4 & \text{if } n \equiv 1\ (mod\ 3)\,, \\ 4\lfloor \frac{n}{3} \rfloor + 4 & \text{if } n \equiv 2\ (mod\ 3)\,, \end{cases} \\ &= 2\lfloor \frac{n}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor + 2 \end{aligned}$$

Thus $\gamma_t^2(P_2 \Box P_n) \leq |D| = 2\lfloor \frac{n}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor + 2$. Combining these two inequalities produces $\gamma_t^2(P_2 \Box P_n) = 2\lfloor \frac{n}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor + 2$. □

Using the above method, we also obtain the exact values of $\gamma_t^2(P_2 \Box C_n)$, here we give the result and omit its proof.

Proposition 11. *For any $n \geq 3$, $\gamma_t^2(P_2 \Box C_n) = 2\lfloor \frac{n-1}{3} \rfloor + 2\lfloor \frac{n+2}{3} \rfloor$.*

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