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VANISHING CARLESON MEASURE ON THE UNIT BALL IN C^n

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Abstract: Suppose that μ is a finite positive Borel measure on the unit ball $B \subset C^n$. In this paper, we will show that if the operator $T(f) = P[f]$ is compact as a mapping from $H^2(\sigma)$ into $L^2(B, d\mu)$, then μ is vanishing Carleson measure.

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1. Introduction

Throughout this paper, C^n will be the Cartesian product of n copies of C. For $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$ in C^n, the inner product is defined by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $|z|^2 = \langle z, z\rangle$.

Let B denote the unit ball in $C^n, n \geq 1$, and the boundary of B is the unit sphere $S = \{z \in C^n : |z| = 1\}$. σ is the normalized surface measure on its boundary S. For $1 \leq p \leq \infty$, $L^2(\sigma)$ denote the Lebesgue space of S induced by σ.

$A(B)$ is the class of all $f : B \to C$ that are continuous on the closed ball $\overline{B}$ and that holomorphic in its interior B. Let $H^2(\sigma)$ be the $L^2(\sigma)$ closure of the elements of $A(B)$ restricted to S. Then $H^2(\sigma)$ is a Hilbert subspace of $L^2(\sigma)$.

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For $f \in L^1(\sigma)$, $P[f]$ denotes the Poisson-Szegö integral defined for $z \in B$ by

$$P[f](z) = \int_S P(z,\zeta)f(\zeta)d\sigma(\zeta)\,,$$

where

$$P(z,\zeta) = \left(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\right)^n$$

is the Poisson-Szegö kernel for B. Each element f of $H^2(\sigma)$ has a holomorphic extension to B given by $P[f]$ (see [7, 1.5]).

A positive Borel measure μ on B is called a Carleson measure if

$$\| \mu \|_{\mathfrak{C}} = \sup\{\frac{\mu(B_\delta(\zeta))}{\delta^n} : \zeta \in S, \delta > 0\} < \infty\,,$$

where $B_\delta(\zeta) = \{z \in B : |1-\langle z,\zeta\rangle| < \delta\}$ (see [4, 8]).

In Section 2, we will show that if

$$\sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(w) < \infty,$$

then a positive Borel measure μ on B is Carleson measure on B.

A positive Borel measure μ on B is called a vanishing Carleson measure if

$$\lim_{\delta\to 0}\frac{\mu(B_\delta(\zeta))}{\delta^n} = 0\,,$$

uniformly in $\zeta \in S$.

Let X and Y be Banach spaces. We will denote the set of all bounded operators form X to Y by $\mathfrak{L}(X,Y)$. An operator $T \in \mathfrak{L}(X,Y)$ is called compact if and only if for every bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a subsequence convergent in Y.

In Section 3, we will show that if the operator $T(f) = P[f]$ is compact as a mapping from $H^2(\sigma)$ into $L^2(B,d\mu)$, then μ is vanishing Carleson measure.

2. Carleson Measure

Theorem 2.1. *If $f \in A(B)$, then $f(z) = P[f](z)$ for all $z \in B$.*

Proof. See [10, Theorem 3.2.4]. □

Theorem 2.2. *If $f \in A(B)$ and $z \in B$, Then*

$$f(z) = \int_S \frac{f(\zeta)}{(1 - \langle z, \zeta \rangle)^n} d\sigma(\zeta).$$

Proof. See [10, Theorem 3.2.4]. □

Theorem 2.3. *If $0 \leq r < 1$, $\zeta \in S$ and $\eta \in S$, then*

$$P(r\eta, \zeta) = P(r\zeta, \eta).$$

Also,

$$\int_S P(r\eta, \zeta) d\sigma(\zeta) = 1 = \int_S P(r\zeta, \eta) d\sigma(\eta).$$

Proof. See [10, Proposition 3.3.3]. □

Theorem 2.4. *The measures ν and σ are related by the formula*

$$\int_{C^n} f d\nu = 2n \int_0^\infty r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr.$$

In particular,

$$\int_B f d\nu = 2n \int_0^1 r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr.$$

Proof. See [10, Proposition 1.4.3]. □

Theorem 2.5. *If $\int_B P[f](z) d\mu(z) < \infty$ for all $f \in A(B)$, then*

$$\sup_{w \in B} \int_B \frac{(1 - |w|^2)^n}{|1 - \langle z, w \rangle|^{2n}} d\mu(z) < \infty.$$

Proof. For the following function f such that

$$f(\zeta) = \frac{(1 - |w|^2)^n}{(1 - \langle \zeta, w \rangle)^{2n}},$$

$$\begin{aligned} P[f](z) &= \int_S P(z, \zeta) \frac{(1 - |w|^2)^n}{(1 - \langle \zeta, w \rangle)^{2n}} d\sigma(\zeta) \\ &= (1 - |w|^2)^n (1 - |z|^2)^n \int_S \frac{1}{(1 - \langle z, \zeta \rangle)^n (1 - \langle \zeta, z \rangle)^n} \frac{1}{(1 - \langle \zeta, w \rangle)^{2n}} d\sigma(\zeta) \end{aligned}$$

$$= \frac{(1-|w|^2)^n}{(1-\langle z,w\rangle)^{2n}}\,,$$

by Theorem 2.

Since a positive Borel measure μ satisfies $\int_B P[f](z)d\mu(z) < \infty$ for all $f \in A(B)$,

$$\sup_{w\in B}\int_B \frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\mu(z) < \infty. \qquad \square$$

Theorem 2.6. *If a positive Borel measure μ satisfies*

$$\sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(w) < \infty,$$

then measure μ on B is Carleson measure on B.

Proof. For $z = 0$,

$$\int_B \frac{(1-|z|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(w) = \mu(B).$$

This implies that if $\delta \geq 1/4$, then

$$\mu(B_\delta(\zeta)) \leq \mu(B) \leq \sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(w)$$
$$\leq 4^n\delta^n \sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(w)\,,$$

for all $\zeta \in \partial B$.

Suppose $\delta < 1/4$ and $\zeta \in S$. Put

$$z_0 = (1-\frac{\delta}{2})\zeta.$$

For $z \in B$:

$$|1-\langle z,z_0\rangle| = |1-\langle z-z_0+z_0,z_0\rangle|$$
$$= |1-\langle z-z_0,z_0\rangle - \langle z_0,z_0\rangle| \leq (1-|z_0|^2) + |\langle z-z_0,z_0\rangle|.$$

If $z \in B_\delta(\zeta)$, then

$$|\langle z-z_0,z_0\rangle| = |\langle z,z_0\rangle - \langle z_0,z_0\rangle| = |\langle z,\frac{z_0}{|z_0|}\rangle|z_0| - \langle\frac{z_0}{|z_0|},\frac{z_0}{|z_0|}\rangle|z_0|^2|$$

$$= |z_0||\langle z,\zeta\rangle - \langle \zeta,\zeta\rangle |z_0|| = |z_0||\langle z,\zeta\rangle - |z_0|| \le \frac{3}{2}\delta|z_0|.$$

This implies that, for $z \in B_\delta(\zeta)$,

$$\begin{aligned}|1-\langle z,z_0\rangle| \le (1-|z_0|^2) + \frac{3}{2}\delta|z_0| &= (1-|z_0|^2) + 3(1-|z_0|)|z_0| \\ &= (1-|z_0|^2) + 3(1-|z_0|)(1+|z_0|) \le 4(1-|z_0|^2).\end{aligned}$$

Above result shows that

$$\frac{(1-|z_0|^2)^n}{|1-\langle z,z_0\rangle|^{2n}} \ge \frac{(1-|z_0|^2)^n}{|4(1-|z_0|^2)|^{2n}} = \frac{1}{4^{2n}}\frac{(1-|z_0|^2)^n}{(1-|z_0|^2)^{2n}} = \frac{1}{4^{2n}}\frac{1}{(1-|z_0|^2)^n},$$

for all $z \in B_\delta(\zeta)$. This implies that

$$\begin{aligned}\int_B \frac{(1-|z_0|^2)^n}{|1-\langle z,z_0\rangle|^{2n}} d\mu(z) &\ge \int_{B_\delta(\zeta)} \frac{(1-|z_0|^2)^n}{|1-\langle z,z_0\rangle|^{2n}} d\mu(z) \\ &\ge \int_{B_\delta(\zeta)} \frac{1}{4^{2n}}\frac{1}{(1-|z_0|^2)^n} d\mu(z) \ge \int_{B_\delta(\zeta)} \frac{1}{4^{2n}}\frac{1}{2^n(1-|z_0|)^n} d\mu(z) \\ &\ge \frac{1}{4^{2n}}\frac{\mu(B_\delta(\zeta))}{\delta^n},\end{aligned}$$

for all $z \in B_\delta(\zeta)$. Since

$$\frac{1}{4^{2n}}\frac{\mu(B_\delta(\zeta))}{\delta^n} \le \int_B \frac{(1-|z_0|^2)^n}{|1-\langle z,z_0\rangle|^{2n}} d\mu(z),$$

$$\sup\{\frac{1}{4^{2n}}\frac{\mu(B_\delta(\zeta))}{\delta^n} : \delta > 0, \zeta \in S\} \le \sup_{w\in B}\int_B \frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}} d\mu(z) < \infty.$$

This shows that measure μ on B is Carleson measure on B. □

3. Vanishing Carleson Measure

Lemma 3.1. *If μ is a positive Borel measure such that $\int_B P(z,\zeta)d\mu(z) < \infty$, then a linear operator T such that $T(f) = P[f]$ is mapping from $H^2(\sigma)$ into $L^2(B,d\mu)$.*

Proof. Since

$$\begin{aligned}|P[f](z)|^2 &= |\int_S f(\zeta)P(z,\zeta)d\sigma(\zeta)|^2\\ &\le \int_S |f(\zeta)|^2 P(z,\zeta)d\sigma(\zeta)\int_S P(z,\zeta)d\sigma(\zeta) = \int_S |f(\zeta)|^2 P(z,\zeta)d\sigma(\zeta),\end{aligned}$$

$$\begin{aligned}\int_B |P[f](z)|^2 d\mu(z) &\le \int_B\int_S |f(\zeta)|^2 P(z,\zeta)d\sigma(\zeta)d\mu(z)\\ &\le \int_S |f(\zeta)|^2 \int_B P(z,\zeta)d\mu(z)d\sigma(\zeta).\end{aligned}$$

This implies that

$$\int_B |P[f](z)|^2 d\mu(z) \le M\int_S |f(\zeta)|^2 d\sigma(\zeta)\,,$$

where $M = \int_B P(z,\zeta)d\mu(z)$. □

Theorem 3.2. *A linear operator T on Hilbert space H is compact if and only if $\| Tx_n \| \to 0$ whenever $x_n \to 0$ weakly in H.*

Proof. See [9, Theorem VI.11]. □

Lemma 3.3. *If a linear operator T such that $T(f) = P[f]$ is compact as mapping from $H^2(\sigma)$ into $L^2(B, d\mu)$, then*

$$\lim_{|w|\to 1^-}\int_B \frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\mu(z) = 0.$$

Proof. For the following function f such that

$$f(\zeta) = \frac{(1-|w|^2)^{n/2}}{(1-\langle\zeta,w\rangle)^n},$$

$$\begin{aligned}&\int_S \frac{(1-|w|^2)^{n/2}}{(1-\langle w,\zeta\rangle)^n}g(\zeta)d\sigma(\zeta)\\ &\qquad= \int_S \frac{(1-|w|^2)^n}{(1-\langle w,\zeta\rangle)^n(1-\langle\zeta,w\rangle)^n}\frac{(1-\langle\zeta,w\rangle)^n}{(1-|w|^2)^{n/2}}g(\zeta)d\sigma(\zeta)\\ &\qquad= \int_S \frac{(1-|w|^2)^n}{|1-\langle w,\zeta\rangle|^{2n}}\frac{(1-\langle\zeta,w\rangle)^n}{(1-|w|^2)^{n/2}}g(\zeta)d\sigma(\zeta) = \frac{(1-|w|^2)^n}{(1-|w|^2)^{n/2}}g(w)\end{aligned}$$

$$= (1 - |w|)^{n/2} g(w)\,,$$

for all $g \in H^2(\sigma)$. This implies that

$$\lim_{|w|\to 1^-} \int_S \frac{(1-|w|^2)^{n/2}}{|1-\langle w,\zeta\rangle|^n} g(\zeta)d\sigma(\zeta) = 0\,,$$

for all $g \in H^2(\sigma)$. By Theorem 8,

$$\lim_{|w|\to 1^-} \int_B \frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}} d\mu(z) = 0. \qquad \square$$

Corollary 3.4. *If the operator $T(f) = P[f]$ is compact as a mapping from $H^2(\sigma)$ into $L^2(B, d\mu)$, then μ is vanishing Carleson measure.*

Proof. For $\zeta \in S$, put

$$z_0 = (1 - \frac{\delta}{2})\zeta.$$

If $w \in B_\delta(\zeta)$,

$$\frac{1}{4^{2n}} \frac{\mu(B_\delta(\zeta))}{\delta^n} \le \int_B \frac{(1-|z_0|^2)^n}{|1-\langle z,z_0\rangle|^{2n}} d\mu(z)$$

by the proof of Theorem 6. As $\delta \to 0$, $|z_0| \to 1$. This implies that

$$\frac{1}{4^{2n}} \lim_{\delta\to 0} \frac{\mu(B_\delta(\zeta))}{\delta^n} \le \lim_{|z_0|\to 1} \int_B \frac{(1-|z_0|^2)^n}{|1-\langle z,z_0\rangle|^{2n}} d\mu(z) = 0.$$

This implies that μ is vanishing Carleson measure. $\square$

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