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A DECOMPOSITION ALGORITHM FOR FUNCTIONS OF BOUNDED VARIATION

Ibraheem Alolyan
College of Science
King Saud University
P.O. Box 2455, Riyadh, 11451, KINGDOM OF SAUDI ARABIA
e-mail: alolyan@math.colostate.edu

Abstract: The well known Jordan Decomposition Theorem gives the useful characterization that any function of bounded variation can be written as the difference of two increasing functions. Functions which can be expressed in this way can be used to formulate an exclusion test for the recent cellular exclusion algorithms for numerically computing all zero points or the global minima of functions in a given cellular domain. In this paper we give an algorithm to approximate such increasing functions when only the values of the function of bounded variation can be computed. For this purpose, we are led to introduce the idea of ϵ-increasing functions, i.e., functions f such that $f(x) \leq f(y) + \epsilon$ for all $x < y$ in the domain of the function. It is shown that for any Lipschitz continuous function that has finite number of oscillation points, we can find two ϵ-increasing functions such that the Lipschitz function can be written as the difference of these functions.

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1. Introduction

The notion of functions of bounded variation plays a very significant and important role in the theory of real functions [1, 5], numerical analysis [3, 4] and

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optimization [7]. In the literature, several properties of these functions have been discussed (see for example [1, 6, 7, 9]); nevertheless, we focus our attention to one of these properties known as *Jordan Decomposition Theorem* (JDT).

Decomposable functions, which result from JDC, play an important role in optimization [2, 7]. For example, the cellular exclusion algorithm uses decomposable functions as a test function for the minimization condition [2, 8].

Let $f : [a,b] \to \mathbb{R}$ be a function and $\mathcal{P} := \{x_i \in [a,b] : a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$. We recall that the *variation* of f over $\mathcal{P}$ is the nonnegative real number

$$V_{\mathcal{P}}[f;a,b] = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

The function f is a function of bounded variation if there exists a number M such that for every partition $\mathcal{P}$ of $[a,b]$, we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M.$$

The total variation of f on $[a,b]$ is defined to be the number

$$V_f[a,b] := \sup_{\mathcal{P}\,|[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \tag{1}$$

where $\mathcal{P}\,|[a,b]$ means "$\mathcal{P}$ is a partition of $[a,b]$", see [6]. For simplicity, we will write $\sup_{\mathcal{P}}$ instead of $\sup_{\mathcal{P}\,|[a,b]}$. The set of all functions of bounded variation on $[a,b]$ is denoted by $\mathbb{BV}[a,b]$ and we have the following proposition which follows immediately from the definition of functions of bounded variation.

Proposition 1.1. (see [6]) *If* $f : [a,b] \to \mathbb{R}$ *is a* Lipschitz function *on* $[a,b]$, *i.e., there is a constant* C *such that*

$$|f(x) - f(y)| \leq C|x-y| \quad \forall x, y \in [a,b], \tag{2}$$

then $f \in \mathbb{BV}[a,b]$.

If we view the sum in (1) as a sum of positive and negative parts of the differences $f(x_i) - f(x_{i-1})$, then we can define $P_f[a,b]$ to be the summation of the positive parts of $f(x_i) - f(x_{i-1})$ and $N_f[a,b]$ to be the summation of the negative parts, i.e.,

$$\begin{aligned} P_f[a,b] &:= \sup_{\mathcal{P}} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+, \\ N_f[a,b] &:= \sup_{\mathcal{P}} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-, \end{aligned} \tag{3}$$

where $x^+ := \max\{0, x\}$ and $x^- := \max\{0, -x\}$, then we have

$$\begin{aligned} V_f[a,b] &= P_f[a,b] + N_f[a,b], \\ f(b) - f(a) &= P_f[a,b] - N_f[a,b]. \end{aligned}$$

Varying b in (3) we get two functions $P_f[a,.], N_f[a,.] : [a,b] \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\begin{aligned} P_f[a,x] &:= \sup_{\mathcal{P}\,|[a,x]} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+, \\ N_f[a,x] &:= \sup_{\mathcal{P}\,|[a,x]} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-. \end{aligned} \tag{4}$$

It was proven that these two functions are increasing on the interval $[a,b]$. If we take $p_J(x) = P_f[a,x] + f(a)$ and $n_J(x) = N_f[a,x]$ as Jordan functions then we have the following theorem.

Theorem 1.1. (Jordan Decomposition, see [6]) *If f is a function of bounded variation on $[a,b]$ then f can be written as the difference of two increasing functions*

$$f(x) = p_J(x) - n_J(x).$$

This theorem states that we can write f as the difference of two increasing functions where each function can be computed by finding the supremum among all partitions. However, the supremum sum over all partitions cannot be computed numerically. Therefore, we approximate the functions p_J and n_J by considering the uniform partition $\mathcal{P} := \{a + i(x-a)/(m+1)\}_{i=0}^{m+1}$ of the interval $[a,x]$, where $m \in \mathbb{N}$ and then study the consequences of this approximation.

In Section 2 we explain the need for defining ϵ-increasing functions. In Section 3 we write our algorithm to approximate p_J and n_J; furthermore, we we state and prove Theorem 3.1 for the functions p and n resulting from Algorithm 3.1.

2. Need of ϵ-Increasing Definition

In order to approximate the functions p_J and n_J in Theorem 1.1, we use the uniform partition $\mathcal{P} := \{a+i(x-a)/(m+1)\}_{i=0}^{m+1}$ of $[a,x]$, where $m \in \mathbb{N}$. Then we define $P(x)$ and $N(x)$ to be

$$\begin{aligned} P(x) &:= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+, \\ N(x) &:= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-. \end{aligned} \tag{5}$$

These two functions approximate the functions $P_f[a,x]$ and $N_f[a,x]$, respectively. Moreover, P and N approach $P_f[a,x]$ and $N_f[a,x]$ as $m \to \infty$. Unfortunately, P and N are not guaranteed to be increasing if the function f is not monotone. We prove this in the following proposition.

Proposition 2.1. *If the function $f : [a,b] \to \mathbb{R}$ is not monotone on $[a,b]$, then no m can be chosen so that the functions P and N in (5) are increasing with respect to the uniform partition $\mathcal{P} := \{a + i(x-a)/(m+1)\}_{i=0}^{m+1}$.*

Proof. Suppose first that f is increasing on the interval $[a,c]$ and decreasing on the interval $[c,d]$ for $c,d \in (a,b]$ and $c < d$. In this case, the function P is increasing on $[a,c]$; nevertheless, if we let $e = \min\{c+(c-a)/m, d\}$ and consider the interval $\mathcal{I} = (c,e)$, then $P(x)$ is less than $P(c)$ for all $x \in \mathcal{I}$. In order to prove that, let x be any point in (c,e), and $\mathcal{P}$ be the uniform partition of $[a,x]$, then $f(x) < f(c)$ and $x_m = a + m(x-a)/(m+1) \in (a + m(c-a)/(m+1), c)$ and we have

$$\begin{aligned} P(x) = \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+ &= f(x_m) - f(x_0) + (f(x) - f(x_m))^+ \\ &< f(x_m) - f(x_0) + f(c) - f(x_m) = f(c) - f(x_0) = P(c). \end{aligned}$$

Therefore, the function P is not increasing on (c,e). If the function f is decreasing on the interval $[a,j]$ and increasing on the interval $[j,k]$ for $j,k \in (a,b]$ and $j < k$ then the function N is not increasing on the interval (j,k) by the same argument that was done for P in the first case. □

We define next the concept of *oscillation of a function.*

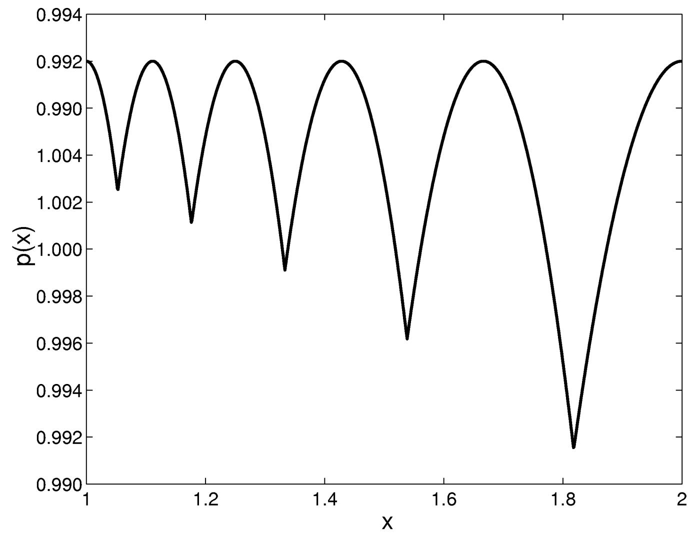
Definition 2.1. (Oscillation) We say that a continuous function $f : [a, b] \to \mathbb{R}$ *oscillates* k times on the interval $[a, b]$ if there are exactly k points $s_1, s_2, \dots, s_k \in (a, b)$ such that for all $i = 1, \dots, k$, the value $f(s_i)$ is either a strict local maximum or a strict local minimum. If the function f has an infinite number of maximum and minimum points on (a, b), we say that f oscillates infinitely often.

For example, the function $f : [-4, 4] \to \mathbb{R}$, defined by $f(x) = x(x-1)(x-2)(x-3)$, oscillates 3 times and the function $g : (0, 1) \to \mathbb{R}$, defined by $g(x) = x \sin(1/x)$, oscillates infinitely often. In the following example, the function f is smooth and oscillates only one time; nevertheless, the function P in this case is not increasing for any choice of m.

Example 2.1. Let $f : [0, 2] \to \mathbb{R}$ be defined by $f(x) := 1 - (1-x)^2$. In theory, this function can be written as the difference of two increasing functions, e.g., $p(x) = 2x$ and $n(x) = x^2$. In order to compute p and n numerically, we notice that this function is increasing on the interval $[0, 1]$; therefore, the value of $P(x)$ will increase on this interval for any choice of m. Nevertheless, on the interval $(1, (m+1)/m)$; P will take values less than $P(1) = 1$. To explain this, let x be any point in $(1, (m+1)/m)$, and $\mathcal{P} := \{ix/(m+1)\}_{i=0}^{m+1}$ be the uniform partition of $[0, x]$, then $f(x) < f(1)$ and $mx/(m+1) \in (m/(m+1), 1)$ and we have

$$\begin{aligned} P(x) = \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+ \\ = f(mx/(m+1)) - f(0) + (f(x) - f(mx/(m+1)))^+ \\ < f(mx/(m+1)) - f(0) + (f(1) - f(mx/(m+1))) = f(1) - f(0) = 1 = P(1). \end{aligned}$$

Therefore, the function P is not increasing on $(1, (m+1)/m)$. In fact, $P(x)$ equals the value $1 = P(1)$ if there is i, $1 \le i \le m+1$, such that $ix/(m+1) = 1$, i.e., the point 1 is one of mesh points. Solving for x, we get $x = (m+1)/i$; therefore, p equals one if $x = (m+1)/(m+1) = 1$, $x = (m+1)/m$, $x = (m+1)/(m-1)$, $\dots$ or $x = m+1$. Figure 1 plots the function p on the interval $[1, 2]$ for $m = 10$ and 100. We can see from the plots that for $m = 10$, we have $P(x) = 1$ on the interval $[1, 2]$ for the values $x = 1, 10/9, 10/8, 10/7, 10/6$, and $10/5$; however, for all other values; P is strictly less than one. For $m = 100$, the function P oscillates more because it equals one at $1, 100/99, 100/98, \dots, 100/50$ and it is strictly less than one for all other values on $[1, 2]$. Although the functions in Figure 1 increase in oscillations, the amplitude decreases as m increases.



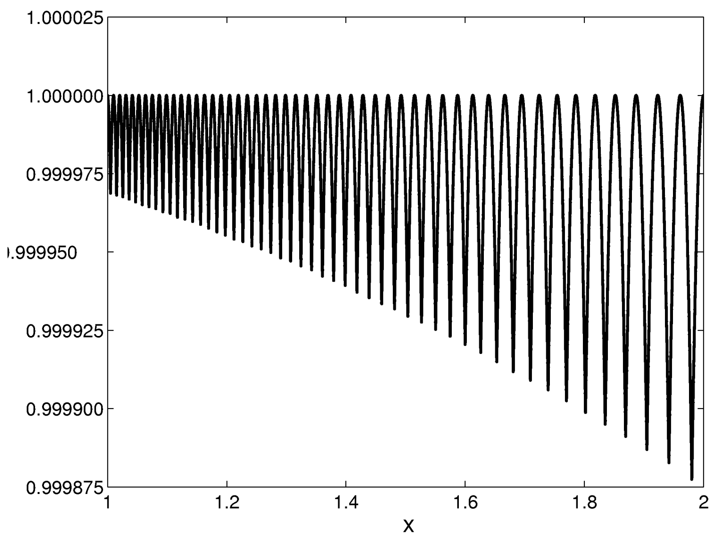


Figure 1: The function P in Example 2.1

Proposition 2.1 and Example 2.1 show that m cannot be chosen in such a way that P and N are always increasing. In fact, the functions P and N will oscillate in a small ϵ-neighborhood of $f(c)$. As m increases, the number ϵ becomes smaller and smaller. This idea led us to introduce a new definition that we call ϵ-increasing and we will use it in the decomposition algorithm.

Definition 2.2. (ϵ-Increasing) We say that the function $f : [a, b] \to \mathbb{R}$ is ϵ-increasing, where ϵ is a positive number, if $f(x) \leq f(y) + \epsilon$ for all $x < y$ and $x, y \in [a, b]$.

We now give the definition of *oscillation points* where the function oscillates infinite times in any neighborhood of this point.

Definition 2.3. (Oscillation Point) A point $x \in [a, b]$ is said to be an oscillation point of the function f, if for all $\delta > 0$, the function f has an infinite number of oscillations on $[x - \delta, x + \delta] \bigcap [a, b]$.

3. Decomposition Algorithm

In this section we we introduce the *decomposition algorithm* to compute the functions p and n that approximate p_J and n_J, respectively. These functions are computed with respect to the uniform partition $\pi := \{a + i(x - a)/(m + 1)\}_{i=0}^{m+1}$ of the interval $[a, x]$ where m is the number of points between a and x. Consequently, the functions p and n will not be always increasing. We showed in Example 2.1 that p and n cannot be increasing on $[a, b]$ even for smooth functions and for any value of m. However, we prove in Theorem 3.1 that the resulting functions, p and n, are ϵ-increasing under some assumptions. In this algorithm we define p and n to be

$$\begin{aligned} p(x) &:= f(a) + \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+, \\ n(x) &:= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-. \end{aligned} \tag{6}$$

Algorithm 3.1. INPUT: a, f and x.
OUTPUT: p and n evaluated at x.
choose m (number of points between a and x).

$d := (x - a)/(m + 1)$.
for $i = 0 : m + 1$
    $x_i := a + id$.
$p = f(a)$.
$n = 0$.
for $i = 1 : m + 1$
    if $s = f(x_i) - f(x_{i-1}) \geq 0$

$p = p + s.$
else
$n = n + s.$

Note that we can choose $m+1 = 2^k$ to decrease the number of computations when we increase m (i.e., if we choose $m + 1 = 2^{k+1}$ then we already have computed the the values $f(x_i)$ for i is even). The main question about this algorithm is how to compute m such that p and n are increasing. Unfortunately, m cannot always be computed even for smooth functions. We give the following example to illustrate this point.

We have discussed in Proposition 1.1 that each Lipschitz function on $[a, b]$ is a function of bounded variation in this interval. In our next discussion, we choose the space of Lipschitz functions because in this space, the value $|f(x_i) - f(x_{i-1})|$ can be controlled by $|x_i - x_{i-1}|$. We come now to the main theorem in this paper which shows that p and n in Algorithm 3.1 have, in fact, a special property that can be exploited numerically.

Theorem 3.1. *If $f : [a, b] \to \mathbb{R}$ is a Lipschitz function with a Lipschitz constant C, and the number of oscillation points is finite (can be 0) then for all $\epsilon > 0$ there exists $m := m_\epsilon \in \mathbb{N}$ such that p and n in Algorithm 3.1 are ϵ-increasing.*

In order to prove this theorem, we need to prove the following propositions. We will always use the partitions $\{x_i\}_{i=0}^{m+1}$ and $\{y_i\}_{i=0}^{m+1}$ to be the uniform partitions of $[a, x]$ and $[a, y]$, respectively.

Lemma 3.1. *1. If f is increasing on $[a, b]$ then p is increasing and we have $p(x) = f(x)$ and $n(x) = 0$ for all $x \in [a, b]$.*

2. If f is decreasing on $[a, b]$ then n is increasing and we have $p(x) = f(a)$ and $n(x) = f(a) - f(x)$ for all $x \in [a, b]$.

Proof. This follows immediately from the way p and n are constructed in Algorithm 3.1 □

In the following discussion, we will consider the function p and prove that this function is ϵ-increasing. A similar argument can be made to prove that the function n is also ϵ-increasing.

Proposition 3.1. *If $f : [a, b] \to \mathbb{R}$ is a Lipschitz function with a Lipschitz constant C and there is a point $c \in [a, b]$ such that f is increasing on $[a, c]$ and decreasing on $[c, b]$, then for all $\epsilon > 0$, there exists an integer $m_\epsilon \in \mathbb{N}$ such that if we choose $m = m_\epsilon$ in Algorithm 3.1, then the resulting function p is*

ϵ-increasing on $[a, b]$, *i.e.*,

$$p(x) \leq p(c) \leq p(x) + \epsilon \quad \forall x > c.$$

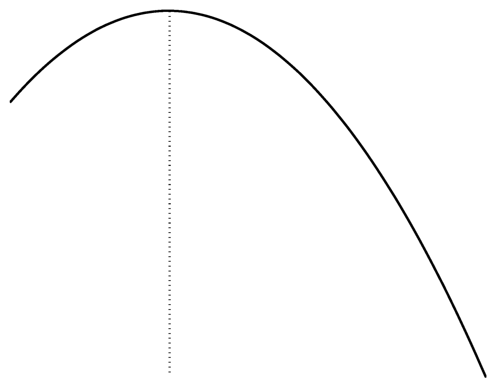


Figure 2: The shape of the function $f(x)$ in Proposition 3.1

Proof. Let $\epsilon > 0$ be given, then choose $m := m_\epsilon$ such that $1/(m+1) < \epsilon/C(b-a)$. If $x \in (c, b]$ and $\{x_i = a + i(x-a)/(m+1)\}_{i=0}^{m+1}$ is the uniform partition of $[a, x]$, then $|f(x_i) - f(x_{i-1})| \leq C(x_i - x_{i-1}) \leq C(b-a)/(m+1) < \epsilon$. Since $x > c$, there must be an index k between 0 and m such that $x_k \leq c < x_{k+1}$. We now compute $p(x)$ and assume that $m < k$ (if $m = k$, then the last summation $\sum_{i=k+2}^{m+1}$ will not appear).

$$\begin{aligned}
p(x) &= f(a) + \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+ \\
&= f(a) + \underbrace{\sum_{i=1}^{k} (f(x_i) - f(x_{i-1}))^+}_{=f(x_k)-f(a)} + \underbrace{(f(x_{k+1}) - f(x_k))^+}_{\geq 0} \\
&\quad + \underbrace{\sum_{i=k+2}^{m+1} (f(x_i) - f(x_{i-1}))^+}_{=0} \\
&\geq f(a) + f(x_k) - f(a) \\
&\geq f(c) - \epsilon \qquad \text{because} f(c) - f(x_k) \leq \epsilon \\
&= p(c) - \epsilon.
\end{aligned}$$

To prove the left inequality, we note that $(f(x_{k+1}) - f(x_k))^+ \leq f(c) - f(x_k)$ because $f(x_{k+1}) < f(c)$. Then it follows that $p(x) \leq f(c) = p(c)$. □

Corollary 3.1. *If $f[a,b] \to \mathbb{R}$ satisfies the conditions in Proposition 3.1 then p is ϵ-increasing on $[a,b]$.*

Proof. Let $x, y \in [a,b]$ and $x < y$ then we have three cases:
1. If $x < y < c$, then $p(x) \leq p(y)$ because p is increasing on $[a,c]$.
2. If $x < c < y$, then $p(x) \leq p(c) \leq p(y) + \epsilon$ (from Proposition 3.1).
3. If $c < x < y$, then $p(x) \leq p(c)$ and $p(c) \leq p(y) + \epsilon$; therefore, $p(x) \leq p(y) + \epsilon$. □

Proposition 3.2. *If $f : [a,b] \to \mathbb{R}$ is a Lipschitz function with a Lipschitz constant C and there is a point $c \in [a,b]$ such that f is decreasing on $[a,c]$ and increasing on $[c,b]$ then there exists an $m_\epsilon \in \mathbb{N}$ such that if we choose $m = m_\epsilon$ in Algorithm 3.1, then the resulting function p is ϵ-increasing on $[a,b]$.*

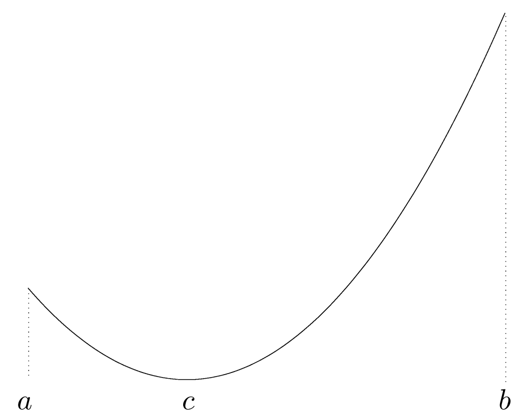


Figure 3: The shape of the function $f(x)$ in Proposition 3.2

Proof. Let $x, y \in [a,b]$ be such that $x < y$. Choose $m := m_\epsilon$ such that the following inequality holds $C(b-a)/(m+1) < \epsilon/3$ and consider the two uniform partitions $\{x_i\}_{i=0}^{m+1}$ and $\{y_i\}_{i=0}^{m+1}$ for $[a,x]$ and $[a,y]$, respectively. If $x \leq c$, then $f(x) = f(a)$ and $f(y) \geq f(a)$ because $p(y) - f(a) = \sum_{i=1}^{m+1}(f(y_i) - f(y_{i-1}))^+ \geq 0$. So, we assume that $c < x < y$. Since $\{y_i\}$ is a partition of $[a,y]$, there is k between 0 and m such that $y_k \leq c < y_{k+1}$. If $y_k = c$, then $p(y) = f(a) + f(y_{m+1}) - f(y_k)$ which is obviously greater than $p(x)$. If $y_k < c < y_{k+1}$, then there are some points x_i such that $y_k \leq x_i \leq y_{k+1}$ for $i = r+1, r+2, \ldots, r+s$. Now either $r+s = m+1$ or $r+s < m+1$. If

$r+s=m+1$ then

$$\begin{aligned}\sum_{i=r+1}^{r+s}(f(x_i)-f(x_{i-1}))^+ &= (f(x_{r+1})-f(x_r))^+ + \sum_{i=r+2}^{r+s}(f(x_i)-f(x_{i-1}))^+\\ &\le C(x_{r+1}-x_r) + C\sum_{i=r+1}^{r+s}(x_i-x_{i-1})\\ &\le C(x_{r+1}-x_r)+C(y_k-y_{k-1})\\ &< 2C\frac{b-a}{m+1} < \frac{2\epsilon}{3}.\end{aligned}$$

Therefore,

$$p(x) \le f(a) + \frac{2\epsilon}{3} < p(y)+\epsilon.$$

If $r+s<m+1$ then we have $y_k \le x_i \le y_{k+1} < x_{r+s+1}$ and the value of $p(x)$ is

$$\begin{aligned}p(x) =& f(a) + \underbrace{\sum_{i=1}^{r}(f(x_i)-f(x_{i-1}))^+}_{=0} + \underbrace{\sum_{i=r+1}^{r+s}(f(x_i)-f(x_{i-1}))^+}_{\le 2\epsilon/3}\\ &+ \underbrace{(f(x_{r+s+1})-f(x_{r+s}))^+}_{\le\epsilon/3} + \sum_{i=r+s+2}^{m+1}(f(x_i)-f(x_{i-1}))^+.\end{aligned}$$

For the last part of the expression of $p(x)$ we have $f(y_{k+1}) < f(x_{r+s+1})$ and $f(y_{m+1}) > f(x_{m+1})$, this implies

$$\begin{aligned}\sum_{i=r+s+2}^{m+1}(f(x_i)-f(x_{i-1}))^+ =& f(x_{m+1})-f(x_{r+s+1})\\ <& f(y_{m+1})-f(y_{k+1})\\ =& \sum_{i=k+2}^{m+1}(f(y_i)-f(y_{i-1}))^+.\end{aligned}$$

Now we compute $p(y)$,

$$p(y) = f(a) + \underbrace{\sum_{i=1}^{k}(f(y_i)-f(y_{i-1}))^+}_{=0} + \underbrace{(f(y_{k+1})-f(y_k))^+}_{\ge 0}$$

$$+\sum_{k+2}^{m+1}(f(y_i)-f(y_{i-1}))^+$$
$$\geq p(x)-\epsilon. \qquad \square$$

We treated the case where f oscillates one time. In the following proposition we will prove that if p is ϵ-increasing on the interval $[a,c]$ where the function f is increasing (decreasing), then p will remain ϵ-increasing in the interval $[c,b]$ where f is decreasing (increasing).

Proposition 3.3. *If $f:[a,b]\to\mathbb{R}$ is a Lipschitz function and there are $c,d,e\in(a,b)$ with $a<c<d<e<b$ such that f is increasing on $[a,c]$ and $[d,e]$ and decreasing on $[c,d]$ and $[e,b]$ then for all $\epsilon>0$, there is an $m_\epsilon\in\mathbb{N}$ such that if we choose $m=m_\epsilon$ in Algorithm 3.1, then the resulting function p is ϵ-increasing on $[a,b]$.*

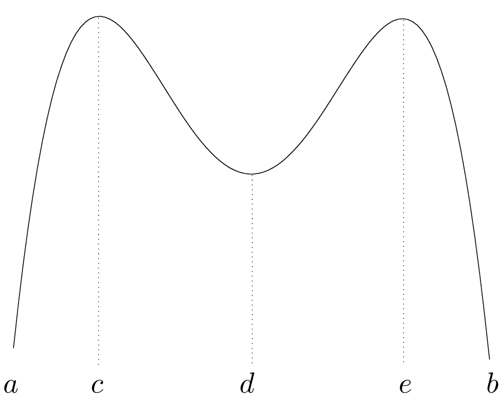


Figure 4: The shape of the function $f(x)$ in Proposition 3.3

Proof. Let $\epsilon>0$ be given, choose $m:=m_\epsilon$ such that $C(b-a)/(m+1)<\epsilon/5$. We will consider three cases

1. On the interval $[a,d]$; it follows from Proposition 3.1 that the function p is ϵ-increasing. Furthermore, for all $x\in[a,d]$, we have $p(x)\leq p(c)$.

2. On the interval $[d,e]$; let $y\in[d,e]$ and $\{y_i\}_{i=0}^{m+1}$ be the uniform partition of $[a,y]$ then we claim that $p(c)\leq p(y)-\epsilon/5$. To show this, we notice the existence of k such that $y_k\leq d<y_{k+1}$ and

$$p(y)=\underbrace{f(a)+\sum_{i=1}^{k}(f(y_i)-f(y_{i-1}))^+}_{=p(y_k)}+\underbrace{(f(y_{k+1})-f(y_k))^+}_{\geq 0}$$

$$+\underbrace{\sum_{i=k+2}^{m+1}(f(y_i)-f(y_{i-1}))^+}_{\geq 0}.$$

Since $p(c)-p(y_k)\leq \epsilon/5$, we have $p(y)\geq p(c)-\epsilon/5$. Now, choose $x<y$, then either $x\in[a,d]$ and therefore, $p(x)\leq p(c)\leq p(y)+\epsilon/5$ or $x\in(d,e]$. If $x\in(d,e]$ then there are r and s such that $y_k\leq x_i\leq y_{k+1}$ for $i=r+1,\ldots,r+s$. If $r+s=m+1$ then we have

$$\begin{aligned}\sum_{i=r+1}^{r+s}(f(x_i)-f(x_{i-1}))^+ &= (f(x_{r+1})-f(x_r))^+ + \sum_{i=r+2}^{r+s}(f(x_i)-f(x_{i-1}))^+\\ &\leq C(x_{r+1}-x_r)+C\sum_{r+1}^{r+s}(x_i-x_{i-1})\\ &\leq C(x_{r+1}-x_r)+C(y_k-y_{k-1})\\ &< 2C\frac{b-a}{m+1}<\frac{2\epsilon}{5}.\end{aligned}$$

Therefore,

$$\begin{aligned}p(x) &= \underbrace{f(a)+\sum_{i=1}^{r}(f(x_i)-f(x_{i-1}))^+}_{=p(x_r)} + \underbrace{\sum_{i=r+1}^{r+s}(f(x_i)-f(x_{i-1}))^+}_{\leq 2\epsilon/5}\\ &\leq p(c)+\frac{2\epsilon}{5}\\ &< p(y)+3\epsilon/5.\end{aligned}$$

If $r+s<m+1$, then we have $y_k\leq x_i\leq y_{k+1}<x_{r+s+1}$ for $i=r+1,\ldots,r+s$. Moreover,

$$\sum_{i=r+s+2}^{m+1}(f(x_i)-f(x_{i-1}))^+\leq \sum_{i=k+2}^{m+1}(f(y_i)-f(y_{i-1}))^+.$$

We now compute $p(y)$ and $p(x)$ and compare them

$$p(y)=\underbrace{f(a)+\sum_{i=1}^{k}(f(y_i)-f(y_{i-1}))^+}_{=p(y_k)}+\underbrace{(f(y_{k+1})-f(y_k))^+}_{\geq 0}$$

$$+ \sum_{i=k+2}^{m+1} (f(y_i) - f(y_{i-1}))^+$$
$$\geq p(c) - \epsilon/5 + \sum_{i=r+s+2}^{m+1} (f(x_i) - f(x_{i-1}))^+,$$

and

$$p(x) = \underbrace{f(a) + \sum_{i=1}^{r}(f(x_i) - f(x_{i-1}))^+}_{=p(x_r)} + \underbrace{\sum_{i=r+1}^{r+s} (f(x_i) - f(x_{i-1}))^+}_{\leq 2\epsilon/5}$$
$$+ \underbrace{(f(x_{r+s+1}) - f(x_{r+s}))^+}_{\leq \epsilon/5} + \sum_{i=r+s+2}^{m+1} (f(x_i) - f(x_{i-1}))^+$$
$$\leq p(c) + 2\epsilon/5 + \epsilon/5 + p(y) - p(c) + \epsilon/5$$
$$= p(y) + 4/5\epsilon.$$

3. On the interval $[e, b]$; let $x \in [a, b]$ and $y \in (e, b]$ such that $x < y$. We first show that $p(e) \leq p(y) + \epsilon/5$. For the partition $\{y_i\}$, there is an index k such that $y_k \leq e < y_{k+1}$ and we have

$$p(y) = \underbrace{f(a) + \sum_{i=1}^{k}(f(y_i) - f(y_{i-1}))^+}_{=p(y_k)} + \underbrace{(f(y_{k+1}) - f(x_k))^+}_{\geq 0}$$
$$+ \underbrace{\sum_{i=k+2}^{m+1} (f(y_i) - f(y_{i-1}))^+}_{=0}$$
$$\geq p(e) - \epsilon/5.$$

If $x \in [a, e]$, then $p(x) \leq p(e) + 4\epsilon/5$ and $p(e) \leq p(y) + \epsilon/5$ which implies $p(x) \leq p(y) + \epsilon$. So we assume that $x \in (e, b]$. Then for the partition $\{x_i\}$, we have $y_k \leq x_i \leq y_{k+1}$ for $i = r + 1, \ldots, r + s$. If $r + s = m + 1$, we have

$$p(x) = \underbrace{f(a) + \sum_{i=1}^{r}(f(x_i) - f(x_{i-1}))^+}_{\leq p(e)+\epsilon/5} + \underbrace{(f(x_{r+1}) - f(x_r))^+}_{<\epsilon/5}$$

$$+\underbrace{\sum_{i=r+2}^{r+s}(f(x_i)-f(x_{i-1}))^+}_{<\epsilon/5}$$
$$\leq p(e)+3\epsilon/5.$$

If $r+s<m+1$, we will have two more terms added to the previous $p(x)$

$$\underbrace{(f(x_{r+s+1})-f(x_{r+s}))^+}_{<\epsilon/5}+\underbrace{\sum_{i=r+s+2}^{m+1}(f(x_i)-f(x_{i-1}))^+}_{=0}\leq \epsilon/5;$$

therefore, $p(y)\geq p(e)-\epsilon/5\geq p(x)-\epsilon$. □

After stating and proving the previous propositions, we now have the tools to prove Theorem 3.1.

Proof of Theorem 3.1. Since $f\in Lip[a,b]$ and has a finite number of oscillation points, we have that either f is monotone, has finite number of oscillations or has finite number of oscillation points.

1. If f is monotone then p is increasing.

2. If f oscillates finitely many times then there is a partition $\Gamma:=\{\xi_i\}_{i=0}^{l}$ (needs not be uniform) such that f is increasing on some intervals and decreasing on others. We then use induction on Proposition 3.3 to prove that p is ϵ-increasing.

3. If f has finite number of oscillation points in $[a,b]$, say $s_1,\dots,s_k$, then for all $\delta>0$, the function f oscillates at most finitely many times on $[a,b]-\bigcup([s_i-\delta,s_i+\delta]\bigcap[a,b])$. Since this is true for any $\delta>0$, we can choose $\delta<\epsilon/4Ck$. For $i=1,2,\dots,k$, let $\{t_{ij}\}_{j=1}^{n_i}$ be a partition of the interval $[s_i-\delta,s_i+\delta]\bigcap[a,b]$ then we have

$$\sum_{j=1}^{n_i}(f(t_{ij})-f(t_{i(j-1)}))^+\leq C\sum_{j=1}^{n_i}(t_{ij}-t_{i(j-1)})\leq 2C\delta<\epsilon/2k.$$

If we take the summation over all the intervals around $s_1,\dots,s_k$, we get

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(f(t_{ij})-f(t_{i(j-1)}))^+<\epsilon/2.$$

Now we will consider the case where we have one oscillation point and then use induction. Suppose that s is the only oscillation point of f on $[a,b]$ and f

oscillates n_s times on $[a,b]-[s-\delta,s+\delta]$ where $2\delta<\epsilon/6$. We prove next that the function p is ϵ-increasing on $[a,b]$. Choose $m_\epsilon := m$ to be $C(b-a)/(m+1) < \epsilon/3(2n_s+4)$ and let $x,y\in[a,b]$ such that $x<y$. We will consider the case where $x>s+\delta$ (if $x\le s+\delta$ then this case can be proven with the same argument). Since f has finitely many oscillations on $[a,s-\delta]$ and $[s+\delta,b]$, the function p is $\epsilon/3$-increasing on these intervals. We consider the uniform partitions $\{x_i\}_{i=1}^{m+1}$ and $\{y_i\}_{i=1}^{m+1}$ of $[a,x]$ and $[a,y]$, respectively. Let y_k be such that $y_k\le s-\delta<y_{k+1}$ and $y_{k+r}\le s+\delta<y_{k+r+1}$ where r can take the value 0. For the partition $\{x_i\}$, let $x_e\le y_k<x_{e+1}$ and $y_{k+r}\le x_{e+f}<y_{k+r+1}$. Now p is $\epsilon/3$-increasing on $[a,s-\delta]$; therefore, $p(y_k)>p(x_e)-\epsilon/3$.

$$p(x_{e+f})-p(x_e)=\sum_{i=e+2}^{e+f}(f(x_i)-f(x_{i-1}))^+<\epsilon/3. \tag{7}$$

We have $y_{k+r}\le x_{e+f}$, and $y>x$; therefore,

$$p(y)-p(y_{k+r})>p(x)-p(x_{e+f})-\epsilon/3.$$

Finally, we compute $p(y)$

$$\begin{aligned} p(y) &= \underbrace{p(y)-p(y_{k+r})}_{\ge p(x)-p(x_{e+f})-\epsilon/3} + \underbrace{p(y_{k+r})-p(y_{k+1})}_{\ge 0} + \underbrace{p(y_{k+1})}_{\ge p(x_e)-\epsilon/3} \\ &> p(x)-p(x_{e+f})+p(x_e)-2\epsilon/3 \\ &> p(x)-p(x_{e+f})+p(x_e)+p(x_{e+f})-p(x_e)-\epsilon \qquad \text{(from (7))} \\ &= p(x)-\epsilon. \end{aligned}$$

□

If we choose ϵ to be the machine epsilon ϵ_{mach} of some computer, then the functions p and n will be increasing in this computer because if $x<y$ then $p(x)\le p(y)+\epsilon_{\text{mach}}=p(y)$ and $n(x)\le n(y)+\epsilon_{\text{mach}}=n(y)$.

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