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BAD POSTULATION OF GENERAL SURFILINEAR FATTENING OF GENERAL CONNECTED CURVILINEAR SUBSCHEMES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be an integral projective surface. For all integers t, m such that $t \geq 2$ and $m \geq 2$ be the set of all zero-dimensional degree tm connected schemes supported by a smooth point P of X and complete intersections of multiples of order t and m of germs of curves intersecting transversally at P. Here we check for many t, m and many $M \in \mathrm{Pic}(X)$ that a general $Z \in X(t, m)$ has bad postulation with respect to the complete linear system $|M|$.

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1. Bad Postulation of Surfilinear Fattenings

Let X be an integral projective variety, $Z \subset X$ a zero-dimensional scheme, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, L)$ a linear subspace. We will say that Z has bad postulation with respect to V or to the linear system $|V|$ if the restriction map $\rho_{V,Z} : V \to H^0(Z, L|Z)$ has not maximal rank, i.e. it is neither injective nor surjective. Now assume $n := \dim(X) \geq 2$. Fix positive integers m, t. A connected multi-curvilinear subscheme Z of type (t, m) of X is a zero-dimensional scheme Z such that Z_{red} is a smooth point P of X and there are germs at P of

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a smooth surface Y and of smooth curves C, D contained in Y and such that $Z = tC \cap mD$, where we see tC and mD as effective Cartier divisors of X. Hence $\deg(Z) = tm$. $Z = \{P\}$ if $t = m = 1$. Z is curvilinear if either $t = 1$ or $m = 1$. If $t \geq 2$ and $m \geq 2$, then Z has 2-dimensional Zariski tangent space at P. We will say that $\emptyset$ is a connected multi-curvilinear subscheme of X of type $(0, 0)$. Let $X(t, m)$ denote the set of all connected multi-curvilinear subscheme Z of type (t, m) of X. Since each $Z \in X(t, m)$ is locally a complete intersection an contained in a smooth curve, it is easy to check that $X(t, m)$ is an integral quasi-projective variety. A general connected curvilinear scheme has good postulation with respect to every linear system. We will see that this is not true in the case $m \geq 2$, $t \gg 0$, and that counterexamples are very common. More precisely, we will prove the following result.

Theorem 1. *Fix an integer $m \geq 2$, an integeral projective surface X such that $n := \dim(X) \geq 2$ and an ample line bundle L on X. Then there exists a positive integer $a_0(X, L, m)$ such that for all integers $a \geq a_0(X, L, m)$ there is an integer $t \geq 2$ such that a general $Z \in X(t, m)$ has bad postulation with respect to the complete linear system $|L^{\otimes a}$. Furthermore, there exists an integer $t_0(X, L, m)$ such that for all $t \geq t_0(X, L, m)$ a general $Z \in X(t, m)$ has bad postulation with respect to some complete linear system $|L^{\otimes z}|$.*

Proof. For any any integer t and any zero-dimensional scheme $Z \subset X$ let $\rho_{Z,t} : H^0(X, L^{\otimes a}) \to H^0(Z, L^{\otimes a})$ denote the restriction map. Set $b(z) := h^0(X, L^{\otimes z})$ and $c := L^2$. Since L is ample, $c > 0$, $h^i(X, L^{\otimes z}) = 0$, $i = 1, 2$, for $z \gg 0$, say for $z \geq a_1$, and $L^{\otimes z}$ is very ample for $z \gg 0$, say for $z \geq a_2 \geq a_1$. Furthermore, the function $b(z)$ is strictly increasing for $z \geq a_2$. For any integer $y \geq b(a_2)$ let $\alpha(y)$ be the minimal integer z such that $b(z) > y$. By Riemann-Roch we have $b(z) = cz^2/2 + O(z)$ for $z \gg 0$. Thus $\alpha(y) = \sqrt{2y/c} + o(\sqrt{y})$. Hence $\lim_{y \to +\infty} \alpha(my)/\alpha(y) = \sqrt{m}$. Fix $z \geq m \cdot a_2$. There is an integer $a_3 \geq ma_2$ such that $\alpha((b(z)+1)/m)) \leq \lfloor z/m \rfloor$ for all integers $z \geq a_3$. Fix the integer $z \geq a_3$ and set $t := \alpha((b(z)+1)/m))$ and take a general $Z \in X(t, m)$. There is a curvilinear scheme $W \subset Z$ such that Z is an m-fattening of W. Since $\alpha(t) \leq \lfloor z/m \rfloor$, there is $C \in |L^{\otimes \lfloor z/m \rfloor}|$ such that $W \subset C$. Hence $Z \subset mC$. Since $tm > b(z) \geq b(m\lfloor z/m \rfloor)$, Z has bad postulation with respect to $|L^{\otimes z}|$. To check the last asertion it is sufficient to check that for $z \gg 0$ the previous proof works taking as t any integer $\alpha(e/m)$ with $b(z) + 1 \leq e \leq b(z+1)$. $\square$

In many cases we may give explicit bounds for the integer $a_0(X, L, m)$ appearing in the statement of Theorem 1. As an example we will give the case $X = \mathbf{P}^2$.

Proposition 1. *Fix integers d, m, t such that $m \geq 2$ and $t \geq 2$. Let x be the first positive integer such that $x^2 + 3x \leq 2t$. Assume $x \leq \lfloor d/m \rfloor$ (i.e. $2t \leq \lfloor d/m \rfloor^2 + 3\lfloor d/m \rfloor$) and either $(d+2)(d+1)/2 \leq tm$ or $(d+2)(d+1)/2 > tm$ and $(d+2)(d+1)/2 - tm < (x^2+3x)/2 - t$ or $xm < d$, $(d+2)(d+1)/2 > tm$, and $(d^2+3d) - 2tm < (d-mx)^2 + 3(d-mx)$. Then the general multi-curvilinear subscheme of type (t, m) of $\mathbf{P}^2$ has bad postulation with respect to the complete linear system $|\mathcal{O}_{\mathbf{P}^2}(d)|$.*

Proof. We fix a general $Z \in \mathbf{P}^2(t, m)$. There is a curvilinear scheme $W \subset Z$ such that Z is an m-fattening of W. Hence $t = \text{length}(W)$. Since $t \leq (x^2 + 3x)/2$, there is a degree x curve $C \subset \mathbf{P}^2$ such that $W \subset C$. Hence $Z \subset mC$. Since $\deg(mC) = mx \leq d$ by the assumption on x, Z has bad postulation if $(d+2)(d+1)/2 \leq tm$. Now assume $(d+2)(d+1)/2 > tm$, but $(d+2)(d+1)/2 - tm < (x^2+3x)/2 - t$. Fix a general curve E of degree $d - m\lfloor d/m \rfloor$, with the convention $E := \emptyset$ if $d \equiv 0 \pmod{m}$. Varying C in the linear system $|\mathcal{I}_W(x)|$, we see that Z is contained in many curves $mC + E$, so many that Z must have bad postulation with respect to the complete linear system $|\mathcal{O}_{\mathbf{P}^2}(d)|$. Now assume $xm < d$, $(d+2)(d+1)/2 > tm$, and $(d^2+3d) - 2tm < (d-mx)^2 + 3(d-mx)$. Fix C as in the first part. The curves $mC + F$ with $F \in |\mathcal{O}_{\mathbf{P}^2}(d - xm)|$ show that Z has bad postulation with respect to the complete linear system $|\mathcal{O}_{\mathbf{P}^2}(d)|$. □

When $m = 1$ the opposite picture holds, at least in characteristic zero (see [1]).

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References

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