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MULTIHOMOGENEOUS POLYNOMIAL INTERPOLATION OVER $\mathbb{Z}$

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Abstract: Here we prove a results on interpolation over the integers for multivariate multihomogeneous polynomials. The node are at integer points and we give a sharp result concerning the prime integers which must divide the determinant of the integral $k \times k$ interpolation matrix.

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1. Integral Polynomial Interpolation

Fix positive integers s, n_i and d_i, $1 \le i \le s$. Let $M(s; n_1, \dots, n_s; d_1, \dots, d_s; R)$ denote the R-module of all polynomials in the variables $x_{i,j}$, $1 \le i \le s$, $1 \le j \le d_i$, with coefficients in the commutative ring R and which have at most degree d_i with respect to the variables $x_{i,j}$, $1 \le j \le d_i$. Thus $M(s; n_1, \dots, n_s; d_1, \dots, d_s; R)$ is a free R-module with rank $\prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$. Set $N := n_1 + \cdots + n_s$. Here we prove the following result.

Theorem 1. *Fix positive integers s, n_i and d_i, $1 \le i \le s$. Set $N := n_1 + \cdots + n_s$, $\delta := \max\{d_1, \dots, d_s\}$ and $k := \prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$. Let $\wp_\delta$ denote the set of all prime integers p such that $2 \le p \le \delta$. For any $E \subset \mathbb{Z}^N$ such that $\sharp(E) = k$ let A_E denote the $k \times k$ interpolation matrix obtained evaluating ar*

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E the monomial basis of $M(s; n_1, \dots, n_s; d_1, \dots, d_s; \mathbb{Z})$ at E.

(i) For any $E \subset \mathbb{Z}^N$ such that $\sharp(E) = k$ the integer $\det(A_E)$ is divisble by all primes in $\wp_\delta$.

(ii) There exists $S \subset \mathbb{Z}^N$ such that $\sharp(E) = k$ and no prime $p \notin \wp_\delta$ divides $\det(A_S)$.

Proof. For any prime p and any $E \subset \mathbb{Z}^N$ such that $\sharp(E) = k$ let $A_E(p)$ denote the reduction modulo p of the matrix A_E. Thus $\det(A_E)$ is divisible by p if and only if $\det(A_E(p)) = 0$, i.e. if the reduction modulo p of E does not give a basis of the dual of the $\mathbb{F}_p$-vector space $M(s; n_1, \dots, n_s; d_1, \dots, d_s; \mathbb{F}_p)$. Similary, write E_p for the reduction modulo p of the set E. Thus $E_p \subseteq \mathbb{F}_p^N$, $\sharp(E_p) \le k$, if $\sharp(E_p) < k$, then $\det(A_E(p)) = 0$, while $\det(A_E(p)) = \det(A_{E_p})$ if $\sharp(E_p) = k$.

(a) Fix E and a prime $p \in \wp_\delta$. If $\sharp(E_p) < k$, then $\det(A_E) \equiv 0 \pmod{p}$ and hence we are done. Now assume $\sharp(E_p) = k$. Let i be an integer such that $d_i = \delta$. The polynomial $t_{i,1}^p - t_{i,1}$ vanishes at all points of $\mathbb{F}_p^N$ and hence $\det(A_{E_p}) = 0$.

(b) To prove part (ii) it is sufficient to find S such that $\det(A_S(p)) \neq 0$ for all $p \notin S$. To construct a set with this property we will construct a set $S \subset \mathbb{Z}^N$ such that $\sharp(S) = k$, every $P \in S$ has as coordinates non-negative integers in the interval $[0, \delta]$ (and hence $\sharp(S_p) = k$ for all $p \notin \wp_\delta$, and $\det(A_{S_p}) \neq 0$ (as an element of $\mathbb{F}_p$) if $p \notin \wp_\delta$. First assume $N = 1$. Hence $s = 1$, $n_1 = 1$, $\delta = d_1$ and $k = \delta + 1$. Set $S := \{0, 1, \dots, \delta\}$. Hence $\det(A_S)$ is the product of all integers $j - i$, with $0 \le j < i \le \delta$ (Vandermonde determinant). Hence $\sharp(S_p) = k$ and $\det(A_{S_p}) \neq 0$ for all primes $p \notin \wp_\delta$, proving the case $N = 1$. Now assume $N \ge 2$ and that the result is true for the integer $N - 1$ and all other data. First assume $n_s = 1$ and hence $s \ge 2$. Set $s' = s - 1$, $n'_i = n_i$ and $d'_i = d_i$ for all $1 \le i \le s' = s - 1$. We see $\mathbb{Z}^{N-1}$ as the linear subspace $\mathbb{Z}^{N-1} \times \mathbb{Z}$. Take a solution $S' \subset \mathbb{Z}^{N-1}$ with respect to the data s', n'_i, d'_i and set $S := \bigcup_{P \in S'} \bigcup_{0 \le j \le d_s} (P, j)$. Applying d_s times the so-called Horace Lemma we conclude. Now assume $n_s \ge 2$. Set $s' := s$, $n'_j = n_j$ and $d'_j = d_j$ for $1 \le j \le s - 1$ and $n'_s = n_s - 1$. For each integer a such that $0 \le a \le d_s$ set $d'_{s,a} := d'_s - a = d_s - a$. For every integer a such that $0 \le j \le a$ let $S_a \subset \mathbb{Z}^{N-1}$ be a solution for the data s, n'_i, d'_i for $1 \le i \le s-1$, $n'_s = n_s - 1$ and $d'_s := d_s - a$. Set $S := \bigcup_{a=0}^{d_s} \bigcup_{P \in S_a} (P, a)$ and apply again d_s times Horace Lemma. □

Part (b) of the proof of Theorem 1 gives the following result.

Theorem 2. *Fix a finite field extension K of $\mathbb{Q}$ and a Dedeking domain D with K has its field of fractions. Fix positive integers s, n_i and d_i, $1 \le i \le s$.*

Set $N := n_1 + \cdots + n_s$, $\delta := \max\{d_1, \ldots, d_s\}$ *and* $k := \prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$. *Let* $\wp_\delta$ *denote the set of all maximal ideals* M *of* D *such that* $\sharp(D/M) \geq \delta$. *For any* $E \subset D^N$ *such that* $\sharp(E) = k$ *let* A_E *denote the* $k \times k$ *interpolation matrix obtained evaluating ar* E *the monomial basis of* $M(s; n_1, \ldots, n_s; d_1, \ldots, d_s; D)$ *at* E. *Then there exists* E *as above such that* $\det(A_E) \notin M$ *for all maximal ideals* M *of* D *such that* $M \notin \wp_\delta$.

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