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CONTROLLABILITY WITH TARGET STATE ON THE BOUNDARY

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Abstract: Various aspects of the boundary regional controllability (when the target state to be reached is given in a subregion of the boundary) introduced in [11] have been discussed in [1] for the heat equation, in [3] for parabolic systems and in [18] for hyperbolic systems. In this paper we consider the system:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \Delta y(x,t) + f(x,t)\,, & \Omega \times [0,T]\,, \\ \dfrac{\partial y}{\partial n}(\xi,t) = 0\,, & \partial\Omega \times [0,T]\,, \\ y(x,0) = y_0(x)\,, & \Omega\,. \end{cases}$$

An approach to the general problem of this kind is given and an approximation method to calculate the optimal solution is proposed. This approach uses a result relating to the null controllability from [11] and consequently according to [12] and establishes at the same time the time-invariance of reachable states set.

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1. Introduction

Many real problems in the control of distributed systems can be reformulated as problem of analysis for infinite-dimensional systems. Among the most important is controllability which has been widely developed (see Curtain and Zwart [6] and the references therein). Some problems relating to regional analysis have been studied when the target region is inside the considered domain [11], More precisely, the regional controllability problem is to find a control which steers a system (S), at time T, to a prescribed function defined on subregion ω. In the real world, we are often led to consider the case where the target region is situated on the boundary of the domain. For example in the thermic isolation problem it may be that the control is only required to vanish the temperature flux before crossing the wall (see Figure 1).

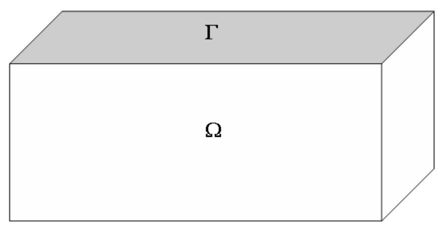

Figure 1: The system domain Ω and the subregion target Γ

The study of this notion is motivated by many real applications. This is the case for example of the problem of safe ground boundary determination of the radiation field from multiple radiation sources, or also the problem of the nontoxic reservoir water surface boundary determination and zone control due to a toxic diffusion source. The diffusion boundary determination and zone control, are reformulated as model-based distributed control tasks (see [5]). On the theoretical side, some recently new concepts and numerical results have been given (see [3]). This kinds of problems are also quoted in [2, 5] and [17].

The subject of this paper is to study this problem with respect to the input function, and is structured as follows: In Section 2, we give definition of regional controllability, in Section 3, we give result ensuring the existence and the uniqueness of the control which achieve such a subject. In Section 4, an approach to the general problem of this kind is given and an approximation method to calculate the optimal solution is proposed.

2. Definitions and Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^n$, T a positive number, $\partial\Omega$ the boundary of Ω, Γ a nonempty subregion of $\partial\Omega$. We denote $Q = \Omega\times]0,T[$ and $\Sigma = \partial\Omega\times[0,T]$ then we consider the system:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \Delta y(x,t) + f(x,t)\,, & Q\,, \\ \dfrac{\partial y}{\partial n}(\xi,t) = 0\,, & \Sigma\,, \\ y(x,0) = 0\,, & \Omega\,. \end{cases} \tag{2.1}$$

For η_d a given state in $L^2(\Gamma)$, the problem consists in finding a control function in an appropriate space that steer the system to the state η_d in a time T and minimize a given cost function.

— γ_0: $H^1(\Omega) \longrightarrow L^2(\partial\Omega)$, denotes the trace operator of order zero. It is linear continuous. γ_0^* denotes the adjoint operator.

— If $\Gamma \subset \partial\Omega$, we consider $\chi_\Gamma : L^2(\partial\Omega) \longrightarrow L^2(\Gamma)$ the restriction to Γ while χ_Γ^* is considered for its adjoint.

— If $\omega \subset \Omega$, we consider also $\chi_\omega : H^1(\Omega) \longrightarrow H^1(\omega)$ the restriction to ω.

— $H : U = L^2(0,T) \longrightarrow H^1(\Omega)$, the controllability operator defined by

$$Hu = \int_0^T S(T-\tau)Bu(\tau)d\tau.$$

We recall some definitions related to regional controllability described by El Jai et al [9].

Let $\omega \subset \Omega$ be of positive Lebesgue measure and $\mathcal{G}$ be any subset of $H^1(\Omega)$.

The system (2.1) is said to be $\omega-$exactly (resp. ω-approximately) controllable to $\mathcal{G}$ if for all $z_d \in \mathcal{G}$ (given $\varepsilon > 0$) there exists a control $u \in U$ such that $\chi_\omega y_u(T) = z_d$ (resp. $\|\chi_\omega y_u(T) - z_d\|_{H^1(\omega)} \leq \varepsilon$).

Let E be any subset of $H^{\frac{1}{2}}(\Gamma)$, then we have the following definition.

Definition 2.1. 1. The system (2.1) is said to be exactly regionally boundary controllable to E on Γ if

$$E \subset \chi_\Gamma\gamma_0 S(T)y_0 + \mathrm{Im}\,(\chi_\Gamma\gamma_0 H).$$

2. The system (2.1) is said to be approximately regionally boundary controllable to E on Γ if

$$E \subset \chi_\Gamma \gamma_0 S(T) y_0 + \overline{\mathrm{Im}\,(\chi_\Gamma \gamma_0 H)}.$$

In what follows, we shall say that a system is $\mathcal{B}$-controllable on Γ ($\mathcal{B}$ for the boundary).

It is clear that:

— The above definitions mean that we are only interested by the transfer of the system (2.1) to a desired state on the subregion $\Gamma \subset \partial\Omega$ and the control u implicitly depends on Γ.

— The above definitions do not allow for pointwise or boundary controls since, for such systems the operator B is not bounded. However the extension can be carried out in similar manner if one takes regular controls such that $y_u(T) \in H^1(\Omega)$ as in (El Jai and Pritchard [10]).

Problems of this type, often to be found in real world, are of importance in a large field of applied science. Since they deal with subregions of the considered domain, they allow to solve problems that have no solution in the whole domain (see [5]).

3. Existence

For a function $f \in L^2(Q)$, it is well known that the system (2.1) has a solution $y \in L^2(Q)$ and one can consider the trace of the terminal state on the subregion Γ:

$$\eta = \eta(.) = y(T,.)|_\Gamma = \chi_\Gamma \gamma_0 (y(T,.)) = \gamma(y(T,.))\,, \tag{3.1}$$

where γ is the restriction of the trace operator to Γ.

Now, let X be a closed subspace of $L^2(Q)$ and define the operator C from X into $L^2(\Gamma)$ by

$$f \to C(f) = \eta\,. \tag{3.2}$$

The operator C is continuous and compact.

Let C be the operator defined by $C = \gamma_0 C_0$, where

$$C_0 : X \to H^{\frac{3}{4}}(\Omega)\,, \qquad f \mapsto C_0(f) = y(T,\cdot)\,. \tag{3.3}$$

The state $y(t,.)$ is the solution of (2.1) corresponding to f.

Consider γ: $H^{\frac{3}{4}}(\Omega) \to L^2(\Gamma)$ the trace operator, then we have the following lemma.

Lemma 3.1. *The map γ has a dense range.*

Proof. Let $\eta \in L^2(\Gamma)$ and $\bar{\eta}$ be a smooth function $\in L^2(\Gamma)$ that approximate η we conclude the result by choosing for example $y \in H^{\frac{3}{4}}(\Omega)$ that satisfies the system:

$$\begin{cases} \Delta y & = 0\,, \quad \Omega\,, \\ y|_{\partial\Omega} & = \bar{\eta}\,, \quad \partial\Omega\,, \end{cases} \tag{3.4}$$

which has a unique solution. □

Lemma 3.2. *The map C_0 has a dense range.*

Proof. From [13], we know that one has null controllability from any state in $L^2(\Gamma)$ when he takes controls in any closed subset of $L^2(Q)$, then we can approximate any $y \in H^{\frac{3}{4}}(\Omega)$ from X. □

Theorem 3.3. *The range of the map C is dense.*

Proof. Let $\eta \in L^2(\Gamma)$, then for each $\epsilon > 0$ we have from Lemma 3.1, $\exists y \in H^{\frac{3}{4}}(\Omega)$:

$$\|\gamma(y) - \eta\| \leq \frac{\varepsilon}{2}\,. \tag{3.5}$$

For this y, from Lemma 3.2, $\exists f \in X$ such that:

$$\|C_0(f) - y\| \leq \frac{\varepsilon}{2\,\|\gamma\|}\,, \tag{3.6}$$

and then:

$$\|C(f) - \eta\| = \|\gamma(C_0(f)) - \gamma(y) + \gamma(y) - \eta\| \leq \varepsilon\,. \tag{3.7}$$

Now, let us define the cost function J_0 and J_λ for $\lambda > 0$ by:

$$J_0(f) = \frac{1}{2}\,\|f\|^2_{L^2(Q)}\,, \tag{3.8}$$

$$J_\lambda(f) = J_0(f) + \frac{\lambda}{2}\,\|C(f) - \eta_d\|^2_{L^2(\Gamma)}\,, \tag{3.9}$$

and for $\varepsilon \geq 0$, let X_0 and X_ε be defined by:

$$X_0 = \{f \in X : C(f) = \eta_d\}\,, \tag{3.10}$$

$$X_\varepsilon = \left\{f \in X : \|C(f) - \eta_d\|_{L^2(\Gamma)} \leq \varepsilon\right\}\,. \tag{3.11}$$

It is clear that X_0 and X_ε are convex and $X_0 \subset X_\varepsilon \subset X$. □

Theorem 3.4. *There exists a unique solution $\bar{f}_\lambda$ that minimizes $J_\lambda(f)$ in X.*

Proof. For each $\lambda > 0$, $J_\lambda(f)$ is bounded by bellow. Let $j_\lambda = \inf_{f \in X} J_\lambda(f)$ and $(f_n)_{n \geq 0}$ a minimizing sequence in X, that is:

$$\lim_{n \to \infty} J_\lambda(f_n) = j_\lambda \,. \tag{3.12}$$

$(J_\lambda(f_n))_{n \geq 0}$ must be bounded and then $(\|f_n\|)_{n \geq 0}$ is bounded.

As X is a Hilbert space, there is a subsequence - denoted also by (f_n) - that converges weakly to $\bar{f}_\lambda$. Since X is closed $\bar{f}_\lambda \in X$, and while C is continuous and compact $C(f_n)$ converges strongly to $C(\bar{f}_{\lambda)} = \bar{\eta}_\lambda$. Besides we have:

$$\left\|\bar{f}_\lambda\right\| \leq \lim_{n \to \infty} \inf \|f_n\| \tag{3.13}$$

(as $\|.\|$ is lower semi-continuous in a weak topology), and

$$\|\bar{\eta}_\lambda - \eta_d\| = \lim_{n \to \infty} \|C(f_n) - \eta_d\| \;\; \text{(strong convergence)} \,. \tag{3.14}$$

So,

$$J_\lambda(\bar{f}_\lambda) \leq \lim_{n \to \infty} \inf J_\lambda(f_n) = j_\lambda \,. \tag{3.15}$$

The result follows. □

The uniqueness stems from the convexity.

Theorem 3.5. *J_0 has a unique minimum in X_ε.*

Proof. Since the range of C is dense, $X_\varepsilon \neq \emptyset$. Let $(f_n)_n$ be a minimizing sequence in X_ε, that is:

$$\lim_{n \to \infty} J_0(f_n) = j_\varepsilon \,, \tag{3.16}$$

where $j_\varepsilon = \inf_{f \in X_\varepsilon} J_0(f)$. $(f_n)_n$ is bounded and converges weakly to $\bar{f}_\varepsilon$,

$$\left\|C(\bar{f}_\varepsilon) - \eta_d\right\| = \lim_{n \to \infty} \|C(f_n) - \eta_d\| \leq \varepsilon \,. \tag{3.17}$$

Since $f_n \in X_\varepsilon$, then $\bar{f}_\varepsilon \in X_\varepsilon$ Thus

$$\frac{1}{2} \left\|\bar{f}_\varepsilon\right\|^2 \leq \lim_{n \to \infty} \inf J_0(f_n) = j_\varepsilon \tag{3.18}$$

the uniqueness stems from the convexity. □

Theorem 3.6. *If η_d is reachable then J_0 has a unique minimum in X_0.*

Proof. Analogous to the proof of Theorem 3.5. □

We can establish a relation between $\bar{f}_0$, $\bar{f}_\varepsilon$ and $\bar{f}_\lambda$ by the following result.

Corollary 3.7. *If η_d is reachable then $\bar{f}_\lambda$ converges strongly to $\bar{f}_0$ as $\lambda \to \infty$.*

Proof. Since η_d is reachable, from Theorem 3.4 $\bar{f}_0$ exists and $j_\lambda \le j_0$ since $X_0 \subset X$ then

$$\lambda \left\|C(\bar{f}_\lambda) - \eta_d\right\|^2 \le \left\|\bar{f}_0\right\|^2 . \tag{3.19}$$

It follows that

$$\lim_{\lambda\to\infty} \left\|C(\bar{f}_\lambda) - \eta_d\right\| = 0 . \tag{3.20}$$

Also

$$\left\|\bar{f}_\lambda\right\| \le \left\|\bar{f}_0\right\| \text{ and then } \lim_{\lambda\to\infty} \bar{f}_\lambda \to \hat{f} . \tag{3.21}$$

So

$$\left\|C(\hat{f}) - \eta_d\right\| = \lim_{\lambda\to\infty} \left\|C(\bar{f}_\lambda) - \eta_d\right\| = 0 . \tag{3.22}$$

It follows that $\hat{f} \in X_0$ and $\left\|\bar{f}_0\right\| \le \left\|\hat{f}\right\|$, but

$$\left\|\hat{f}\right\| \le \lim_{\lambda\to\infty} \inf \left\|\bar{f}_\lambda\right\| \le \left\|\bar{f}_{,0}\right\| , \tag{3.23}$$

whence $\left\|\bar{f}_0\right\| = \left\|\hat{f}\right\|$. □

Corollary 3.8. *If η_d is reachable then $\bar{f}_\varepsilon$ converges strongly to $\bar{f}_0$ as $\varepsilon \to 0$.*

Proof. We proceed as in the proof of the Corollary 3.7. □

Moreover the control $\bar{f}_\lambda$ associated to the cost function J_λ can be characterized by the following result.

Theorem 3.9. *$\bar{f}_\lambda$ satisfy the equation: $(1 + \lambda C^*C)\bar{f}_\lambda = \lambda C^*\eta_d$.*

Proof. Since the range of C is dense and $L^2(\Gamma)$ may be identified to its dual the above equation is defined. For any $\sigma \ge 0$ and any $g \in X$ we have:

$$\begin{aligned} &J_\lambda(\bar{f}_\lambda + \sigma g) \\ &\qquad = J_\lambda(\bar{f}_\lambda) + \sigma^2 J_\lambda(g) + \sigma[\langle \bar{f}_\lambda, g\rangle + \lambda\langle C(\bar{f}_\lambda) - \eta_d, C(g)\rangle] , \end{aligned} \tag{3.24}$$

and since $J_\lambda(\sigma)$ reaches its minimum at $\sigma = 0$, we obtain

$$\begin{aligned} \langle \bar{f}_\lambda, g\rangle + \lambda\langle C(\bar{f}_\lambda) - \eta_d, C(g)\rangle &= \langle \bar{f}_\lambda, g\rangle + \lambda\langle C^*(C(\bar{f}_\lambda) - \eta_d), g\rangle \\ &= \langle \bar{f}_\lambda + \lambda C^*(C(\bar{f}_\lambda) - \eta_d), g\rangle = 0 , \end{aligned} \tag{3.25}$$

$\forall g \in X$, whence the result. □

4. Numerical Approximation

In order to approximate the problem by finite elements method, we proceed the following way.

Let h and ε be two positive numbers such that $0 < \varepsilon \leq h$ and C_h a sequence of operators approximating C

$$X_h = \{f \in X : \|C_h(f) - \eta_d\| \leq h\}\,, \tag{4.1}$$

where $\bar{f}_h$ the minimizer of J_0 on X_h. We have $\bar{f}_0 \in X_h$ and we choose ε such that:

$$\left\|C_h(\bar{f}_0) - \eta_d\right\| \leq \varepsilon\,. \tag{4.2}$$

Theorem 4.1. *The sequence $C_h(\overline{f}_h)$ converges strongly to η_d as $h \to 0$.*

Proof. As $\overline{f_0} \in X_h, X \neq \emptyset$ and $\left\|\overline{f_h}\right\| \leq \left\|\overline{f_0}\right\|$ then $(\overline{f}_h)_{h>0}$ converges weakly to $\widehat{f}$ as $h \to 0$, also since C_h^* is compact, $C_h^*(\zeta)$ converges strongly to $C^*(\zeta)$ for each $\zeta \in X$.

Now we have

$$\begin{aligned}\langle C_h(\overline{f}_h) - C_h(\overline{f}_h), \zeta\rangle &= \langle C_h(\overline{f}_h), \zeta\rangle - \langle C(\widehat{f}), \zeta\rangle \\ &= \langle \overline{f}_h, C_h^*(\zeta)\rangle - \langle \widehat{f}, C^*(\zeta)\rangle = \langle \overline{f}_h - \widehat{f}, C_h^*(\zeta)\rangle + \langle \widehat{f}, C_h^*(\zeta) - C^*(\zeta)\rangle\,,\end{aligned} \tag{4.3}$$

so $C_h(\overline{f}_h)$ converges weakly to $C_h(\overline{f}_h)$ as $h \to 0$, but

$$\left\|\widehat{f}\right\| \leq \lim_{h\to 0}\inf \left\|\overline{f}_h\right\| \leq \left\|\overline{f_0}\right\|\,, \tag{4.4}$$

then

$$\begin{aligned}\left\|C(\widehat{f}) - \eta_d\right\| &= \lim_{h\to 0}\left\|C(\overline{f}_h) - \eta_d\right\| \\ &\leq \lim_{h\to 0}(\left\|C(\overline{f}_h) - C_h(\overline{f}_h)\right\| + \left\|C_h(\overline{f}_h) - \eta_d\right\|) = 0\,.\end{aligned} \tag{4.5}$$

It follows that $\widehat{f} \in X_0$ and $\left\|\overline{f}_0\right\| \leq \left\|\widehat{f}\right\|$, then $\left\|\overline{f}_0\right\| = \left\|\widehat{f}\right\|$,

$$\begin{aligned}\lim_{h\to 0}\left\|C_h(\overline{f}_h) - \eta_d\right\| &\leq \lim_{h\to 0}\left\|C_h(\overline{f}_h) - C(\overline{f}_h)\right\| \\ &\quad + \left\|C(\widehat{f}) - \eta_d\right\| + \left\|C(\overline{f}_h) - C(\widehat{f})\right\| = 0\,,\end{aligned} \tag{4.6}$$

and finally $\mathrm{C}_h(\overline{f}_h)$ converges strongly to η_d. □

5. Conclusion

The boundary controllability problem is considered and solved by means the study of the considered operators. The used method gives characterization of optimal control achieving such an objective. An open question may be the case when one consider the observability problem. Characterization of such notion by means the similar techniques and operators, is under consideration and the work will appear in a separate paper.

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