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STABILITY THEOREMS OF WEITZENBÖCK'S AND ZHANG-YANG'S INEQUALITIES FOR SIMPLEX

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Abstract: A new deviation metric which is called the deviation regular metric of a simplex is introduced. Utilizing the deviation metric, stability theorems of Weitzenböck's and Zhang-Yang's inequalities for simplicity is established.

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1. Introduction

Stability of geometric inequalities which is called stability versions, first investigated by Minkowsksi and Bounesen [2], [10]. It is systematically studied until in the late half of last century, see [7], [6], [3], [4], [5]. The definition of the stability of geometric inequalities given by H. Groemer, and, he further established some classical stability versions of geometric inequalities (see [5]). However, there is no one investigated the stability of geometric inequalities involving simplex because of no appropriate support function or radial functions, and some reason of techniques. In this paper, we established theorems of stability versions of Weitzenböck's and Zhang-Yang's inequalities involving simplex. Our approach is to use a new deviation metric which is called the deviation regular metric of a simplex is introduced.

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We only consider this problem in the compact set in n-dimensional Euclidean space in this paper. Suppose $\mathbf{E} \subseteq R^n (n > 2)$, and $\mathbf{E}$ is compact, R^n is an n-dimensional linear vector space, $\Omega \subseteq \mathbf{E}$, is a simplex, $\tau = \{A_1, A_1, \ldots, A_{n+1}\}$ is the set of its top cluster, $a_{ij} (1 \leq i < j \leq n+1)$ is edge lengths, it is mean $a_{ij} = |A_i A_j|$, and we designate

$$\mu_{n,k} = \binom{n+1}{k+1} = \frac{(n+1)!}{(k+1)!(n-k)!} \ ,$$

$$\mu_{n,1} = \binom{n+1}{2} = \frac{n(n+1)}{2} \ .$$

Definition 1.1. Suppose a_i $(i = 1, 2, \cdots, \mu_{n,1})$ is simplex edge lengths, then partial positive metric of simplex is defined by

$$\delta(\Omega, \overline{\Omega}) = \sum_{i=1}^{\mu_{n,1}} (a_i - \overline{a})^2 \ , \tag{1.1}$$

where $\overline{a} = \frac{1}{\mu_{n,1}} \sum\limits_{i=1}^{\mu_{n,k}} a_i$, $\overline{\Omega}$ is a regular simplex with edge lengths $\overline{a}$.

Definition 1.2. Suppose Ω is a simplex, its partial positive metric is defined by

$$\delta_k(\Omega, \overline{\Omega}) = \sum_{i=1}^{\mu_{n,k}} (V_i(k) - \overline{V(k)})^2 \ , \tag{1.2}$$

where $V_i(k)$ $(1 \leq k \leq n-1)$ is volume of $k-$dimensional side of simplex Ω, $\overline{V}(k) = \frac{1}{\mu_{n,k}} \sum\limits_{i=1}^{\mu_{n,k}} V_i(k)$.

Obviously, partial positive metric is convenient than Hausdorff metric or radial metric when we deal it with computer, and it has significant geometrical meaningful also too.

Now we begin investigate the geometric inequalities of simplex. Suppose $V_i(k)$ $(i = 1, 2, \cdots, \mu_{n,k}, 1 \leq k \leq n-1)$ is the volume of $k-$dimensional side of simplex (expressly, $V_i(n-1) = V_i$, $V_i(1) = a_i$), then, the famous Weitzenböck inequality of simplex is

$$\sum_{i=1}^{\mu_{n,k}} V_i(k)^2 \geq \mu_{n,k} (\frac{n!}{\sqrt{n+1}})^{\frac{2k}{n}} \frac{k+1}{k!^2} V^{\frac{2k}{n}} \ , \tag{1.3}$$

if and only if Ω is regular, then it is equal. When $n = 2, k = 1$, (1.3) is the famous Weitzenböck inequality for triangle:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} S_\triangle \ .$$

By (1.3), we have the following result.

Theorem 1.1. *Suppose* $1 \le k \le n-1$ *and* $\delta_k(\Omega, \overline{\Omega})$ *is* $k-$ *patial positive metric of* Ω, $\forall\, \varepsilon \ge 0$, *then*

$$\sum_{i=1}^{\mu_{n,k}} V_i^2(k) - \mu_{n,k}[(\frac{k+1}{k!})^{\frac{1}{k}}(\frac{n!^2}{n+1})^{\frac{1}{n}}]^k V^{\frac{2k}{n}} \le \varepsilon \Rightarrow \delta_k(\Omega, \overline{\Omega}) \le \frac{\varepsilon}{\alpha_n} \ , \tag{1.4}$$

or stability versions of inequality (1.3) *is*

$$\sum_{i=1}^{\mu_{n,k}} V_i^2(k) - \mu_{n,k}[(\frac{k+1}{k!})^{\frac{1}{k}}(\frac{n!^2}{n+1})^{\frac{1}{n}}]^k V^{\frac{2k}{n}} \ge \alpha_n \delta_k(\Omega, \overline{\Omega}) \ , \tag{1.5}$$

where $\alpha_n = \frac{\mu_{n,k}}{\mu_{n,k}-(n+1-k)}$.

The famous Zhang-Yang's inequalities involving simplex [11] is:

$$\prod_{i=1}^{n+1} V_i^2 \ge [\frac{n^{3n}}{(n+1)^{n-1} n!^2}]^{\frac{n+1}{n}} V^{\frac{2(n^2-1)}{n}} \ , \tag{1.6}$$

if and only if Ω is regular, then it is equal.

According of inequality (1.6), we have the next result.

Theorem 1.2. *Suppose* $a_i (i = 1, 2, \ldots, \mu_{n,1})$ *is the edge lengths of simplex* Ω, $\overline{\Omega}$ *is a regular simplex with the edge lengths* $\overline{a} = \frac{1}{\mu_{n,1}} \sum_{i=1}^{\mu_{n,1}} a_i$, $\delta(\Omega, \overline{\Omega})$ *is the partial positive metric, then* $\forall\, \varepsilon \ge 0$, *there is:*

$$\prod_{i=1}^{n+1} V_i^2 - [\frac{n^{3n}}{(n+1)^{n-1} n!^2}]^{\frac{n+1}{n}} V^{\frac{2(n^2-1)}{n}} \le \varepsilon \Rightarrow \delta_{n-1}(\Omega, \overline{\Omega}) \le \frac{\varepsilon}{\beta_n} \ , \tag{1.7}$$

or a stability versions of inequality (1.6) *is*

$$\prod_{i=1}^{n+1} V_i^2 - [\frac{n^{3n}}{(n+1)^{n-1} n!^2}]^{\frac{n+1}{n}} V^{\frac{2(n^2-1)}{n}} \ge \beta_n \delta_{n-1}(\Omega, \overline{\Omega}) \ , \tag{1.8}$$

where $\beta_n = \frac{n^{3n} V^{2(n-1)}}{(n-1)(n+1)^{n-1} n!^2}$ *(the volume of simplex* Ω *is constant).*

Suppose R_n and r_n is the radius of circumscribed hypersphere of simplex and internal connection hypersphere respectively. The following famous *Euler*'s inequality (1.9) is developed by M.S. Klamkin [1] in 1979:

$$R_n \ge n r_n \ , \tag{1.9}$$

if and only if Ω is regular, it is equal.

According inequality (1.9), we have the following theorem.

Theorem 1.3. *Suppose* $a_i (i = 1, 2, \cdots, \mu_{n,1})$ *is the edge lengths of simplex* Ω, $\overline{\Omega}$ *is a regular simplex with the edge lengths* $\overline{a} = \frac{1}{\mu_{n,1}} \sum_{i=1}^{\mu_{n,1}} a_i$, $\delta(\Omega, \overline{\Omega})$ *is the partial positive metric, then* $\forall\, \varepsilon \geq 0$, *there is:*

$$R_n^2 - (nr_n)^2 \leq \varepsilon \Rightarrow \delta(\Omega, \overline{\Omega}) \leq \frac{\varepsilon}{\gamma_n} \ , \tag{1.10}$$

or a stability versions of inequality (1.9) *is:*

$$R_n^2 - (nr_n)^2 \geq \gamma_n \delta(\Omega, \overline{\Omega}) \ , \tag{1.11}$$

where $\gamma_n = \frac{1}{n^2-1}$.

2. Some Useful Lemmas

For proving the theorems that is given above, first, we establish a few useful lemmas.

Lemma 1. *Suppose* $\tau = \{A_1, A_2, \cdots, A_{n+1}\}$ *is the set of top clusters of the simplex* Ω, $\tau_k = \{A_{i_1}, A_{i_2}, \cdots, A_{i_{k+1}}\} \subset \tau$ *is the set of top clusters of the* $k-$*dimensional subcomplex* $\Omega_{i_1 i_2 \cdots i_{k+1}} \in \Omega$, $V_j(k) (j = 1, 2, \cdots, \mu_{n,k})$ *is the volume of* $\Omega_{i_1 i_2 \cdots i_{k+1}} \in \Omega$, *if* $M_k = \prod_{j=1}^{\mu_{n,k}} V_j(k)$, $(1 \leq k < l \leq n$), *then*

$$(\frac{k!}{\sqrt{k+1}} M_k^{\frac{1}{\mu_{n,k}}})^{\frac{1}{k}} \geq (\frac{l!}{\sqrt{l+1}} M_l^{\frac{1}{\mu_{n,l}}})^{\frac{1}{l}} \ , \tag{2.1}$$

if and only if Ω *is regular, it is equal.*

Proof. According of inequality (1.6), there is

$$V \leq \sqrt{n+1} (\frac{(n-1)!^2}{n^{3n-2}})^{\frac{1}{2(n-1)}} \prod_{i=1}^{n+1} V_i^{\frac{n}{n^2-1}} \ , \tag{2.2}$$

and, if and only if Ω is regular, it is equal.

By (2.2), we have

$$(\prod_{i=1}^{n+1} V_i (n-1))^{\frac{n}{n^2-1}} \geq \frac{1}{\sqrt{n+1}} (\frac{n^{3n-2}}{(n-1)!^2})^{\frac{1}{2(n-1)}} V \ . \tag{2.3}$$

Utilizing (2.3) in $\Omega_{i_1 i_2 \cdots i_{k+1}}$, then

$$(\prod_{i=1}^{k+1} V_i(k-1))^{\frac{k}{k^2-1}} \geq \frac{1}{\sqrt{k+1}}(\frac{k^{3k-2}}{(k-1)!^2})^{\frac{1}{2(k-1)}} V_j(k) \ ,$$

where $j = 1, 2, \cdots, \mu_{n,k}$. Take plus of $\mu_{n,k}$ inequalities above, we could obtain

$$(\prod_{i=1}^{\mu_{n,k-1}} V_i(k-1))^{\frac{k(k+1)\mu_{n,k}}{(k^2-1)\mu_{n,k-1}}} \geq (k+1)^{-\frac{\mu_{n,k}}{2}}(\frac{k^{3k-2}}{(k-1)!^2})^{\frac{\mu_{n,k}}{2(k-1)}} \prod_{j=1}^{\mu_{n,k}} V_j(k) \ .$$

Note that

$$M_k = \prod_{i=1}^{\mu_{n,k}} V_i(k) \ , M_{k-1} = \prod_{i=1}^{\mu_{n,k-1}} V_i(k-1) \ ,$$

thus

$$M_{k-1}^{\frac{1}{(k-1)\mu_{n,k-1}}} \geq (k+1)^{-\frac{1}{2k}}(\frac{k^{3k-2}}{(k-1)!^2})^{\frac{1}{2k(k-1)}} M_k^{\frac{1}{k\mu_{n,k}}} \ .$$

Replaced k by $k-1$, then have

$$M_k^{\frac{1}{k\mu_{n,k}}} \geq (k+2)^{-\frac{1}{2(k+1)}}(\frac{(k+1)^{3k+1}}{k!^2})^{\frac{1}{2k(k+1)}} M_{k+1}^{\frac{1}{(k+1)\mu_{n,k+1}}} \ . \tag{2.4}$$

From (2.4), we obtain

$$M_k^{\frac{1}{k\mu_{n,k}}} \geq \prod_{i=1}^{l-k}[(k+i+1)^{-\frac{1}{2(k+i)}}(\frac{(k+i)^{3(k+i)-2}}{(k+i-1)!^2})^{\frac{1}{2(k+i-1)(k+i)}}] M_l^{\frac{1}{l\mu_{n,l}}} \ . \tag{2.5}$$

We could have (2.1) from (2.5) by some simple calculation and reset. □

Lemma 2. *Suppose V is the volume of simplex Ω, $V_i(k)$ $(1 \leq k \leq n-1)$ is volume of $k-$dimensional side of simplex Ω, then*

$$V \leq \frac{\sqrt{n+1}}{n!}(\frac{k!}{\sqrt{k+1}})^{\frac{n}{k}}(\prod_{i=1}^{\mu_{n,k}} V_i(k))^{\frac{n}{k\mu_{n,k}}} \ , \tag{2.6}$$

if and only if Ω is regular, it is equal.

Proof. Note that

$$M_k = \prod_{i=1}^{\mu_{n,k}} V_i(k) \ , M_n = V \ ,$$

Let $l = n$ in (2.1), so the proof is complete. □

Lemma 3. *Suppose* a_i $(i=1,2,\cdots,\mu_{n,1})$ *and* R_n *is the edge lengths and the radius of circumscribed hypersphere of simplex respectively, then*

$$R_n^2-\frac{1}{(n+1)^2}\sum_{i=1}^{\mu_{n+1}}a_i^2\geq 0\ , \tag{2.7}$$

if and only if the center of gravity of Ω *is at one point with the center point of the circumscribed hypersphere, it is equal.*

The proof of Lemma 3 is obvious, here we omit its proof.

Lemma 4. *Suppose* V, r_n *is the volume of simplex* Ω *and the radius of the internal connection hypersphere of simplex respectively. then*

$$V^{\frac{2}{n}}\geq n(n+1)^{\frac{(n+1)}{n}}(n!)^{-\frac{2}{n}}r_n^2\ . \tag{2.8}$$

Proof. In (2.2), utilizing the arithmetic-geometric average inequality for $\prod_{i=1}^{n+1}V_i^{\frac{n}{n^2-1}}$, then we have

$$V\leq\sqrt{n+1}\left(\frac{(n-1)!^2}{n^{3n-2}}\right)^{\frac{1}{2(n-1)}}\left(\frac{1}{n+1}\sum_{i=1}^{n+1}V_i\right)^{\frac{n}{n-1}}\ .$$

Note that

$$nV=r_n\sum_{i=1}^{n+1}V_i\ .$$

So, the lemma is proved. □

Lemma 5. *For n-dimensional simplex* Ω $(n\geq 2)$ *and* $\forall\,\alpha\in(0,1]$, *we have*

$$\left(\sum_{i=1}^{n+1}V_i^{\alpha}\right)^2-2\sum_{i=1}^{n+1}V_i^{2\alpha}\geq(n^2-1)\mu_n^{\alpha}V^{\frac{2(n-1)\alpha}{n}}\ , \tag{2.9}$$

where $\mu_n=\frac{n^3}{n+1}[\frac{\sqrt{n+1}}{n!}]^{\frac{2}{n}}$, *and if and only if* Ω *is regular, it is equal.*

The proof of Lemma 3 follows from [10], [11].

Lemma 6. *Suppose the volume of n-dimensioal simplex* Ω *and its k-dimensioal subsimplex* $(1\leq k\leq n-1)$ *is* V *and* $V_i(k)$ *respectively,* $\Omega\in E^n$, $n\geq 2$ *(especially,* $V_i(n-1)=V_i$, $V_i(1)=a_i$*),* $\alpha\in(0,1]$ *and* $\delta\leq n+1-k$, *then*

$$\left(\sum_{i=1}^{\mu_{n,k}}V_i^{\alpha}(k)\right)^2-\delta\sum_{i=1}^{\mu_{n,k}}V_i^{2\alpha}(k)$$

$$\geq \binom{n+1}{k+1}(\binom{n+1}{k+1} - \delta)[\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha} V^{\frac{2k\alpha}{n}}$$
$$= \varphi(n,k,\alpha,\delta) V^{\frac{2k\alpha}{n}} \ , \quad (2.10)$$

where $\varphi(n,k,\alpha,\delta) = \binom{n+1}{k+1}(\binom{n+1}{k+1} - \delta)[\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha}$, $\mu_{n,k} = \binom{n+1}{k+1}$, *if and only if* Ω *is regular, it is equal.*

Proof. Let

$$f(\alpha,\delta) \equiv \sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) + 2 \sum_{1\leq i<j\leq \mu_{n,k}} V_i^{\alpha}(k)V_j^{\alpha}(k) - \delta \sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k)$$

$$= 2 \sum_{1\leq i<j\leq \mu_{n,k}} V_i^{\alpha}(k)V_j^{\alpha}(k) - (n-k)\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) + [(n+1-k)-\delta]\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k)$$

$$= \sum_{(i_1,i_2,\cdots,i_{k+2})\in\Gamma_1} [2 \sum_{1\leq p<q\leq k+2} V_{i_p}^{\alpha}(k)V_{i_q}^{\alpha}(k) - \sum_{p=1}^{k+2} V_{i_p}^{2\alpha}(k)]$$

$$+ 2 \sum_{(i_p,i_q)\in\Gamma_2} V_{i_p}^{\alpha}(k)V_{i_q}^{\alpha}(k) + (n+1-k-\delta)\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) \ , \quad (2.11)$$

where $\Gamma_1 = \{(i_1,i_2,\cdots,i_{k+2})|V_{i_1}(k),V_{i_2}(k),\cdots,V_{i_{k+2}}(k)\}$, then Γ_1 is the volume of every side of one $k+1$-dimensioal subsimplex in Ω, $1\leq i_1 < i_2 < \cdots < i_{k+2} \leq n+1$.

$\Gamma_2 = \{(i_p,i_q)|V_{i_p}(k),V_{i_q}(k)\}$ is the volume of two side of $k+1$-dimensioal subsimplex that is not the same one. Designate $|A|$ is the elements number of set A, so

$$|\Gamma_1| = \mu_{n,k+1} = \binom{n+1}{k+2} \ ,$$

$$|\Gamma_2| = \binom{\mu_{n,k}}{2} - \binom{n+1}{k+2}\binom{k+2}{2} = \frac{1}{2}\binom{n+1}{k+1}[\binom{n+1}{k+1} - (n-k)(k+1) - 1] \ .$$

When $(i_1,i_2,\cdots,i_{k+2})\in\Gamma_1$,let $\Omega_{i_1i_2\cdots i_{k+2}}$ denote the $k+1$-dimensioal subsimplex and its every sides volume is $V_{i_1}(k),V_{i_2}(k),\cdots,V_{i_{k+2}}(k)$, and volume of $k+1$-dimensioal subsimplex is $V_{i_1i_2\cdots i_{k+2}} = V_i(k+1)$. Designate the first term and second term and third term of (2.11) is Q_1,Q_2,Q_3 respectively. We deal with Q_1 by the arithmetic-geometric average inequality, then have

$$Q_1 \geq \sum_{(i_1,i_2,\cdots,i_{k+2})\in\Gamma_1} k(k+2)\mu_{k+1}^{\alpha}(V_{i_1i_2\cdots i_{k+2}})^{\frac{2k\alpha}{k+1}}$$

$$\geq k(k+2)\mu_{k+1}^{\alpha}\mu_{n,k+1}[\prod_{i=1}^{\mu_{n,k+1}} V_i(k+1)]^{\frac{2k\alpha}{(k+1)\mu_{n,k+1}}}.$$

Continuously using the following inequality in above inequality, we obtain:

$$[\prod_{i=1}^{\mu_{n,k+1}} V_i(k+1)]^{\frac{1}{\mu_{n,k+1}}} \geq \frac{\sqrt{k+2}}{(k+1)!}(\frac{n!}{\sqrt{n+1}})^{\frac{k+1}{n}} V^{\frac{k+1}{n}} \tag{2.12}$$

(the change form of Lemma 2), then

$$\begin{aligned}
&Q_1 \geq \\
&k(k+2)[\frac{(k+1)^3}{k+2}(\frac{\sqrt{k+2}}{(k+1)!})^{\frac{2}{k+1}}]^{\alpha}\cdot\mu_{n,k+1}\cdot[\frac{\sqrt{k+2}}{(k+1)!}(\frac{n!}{\sqrt{n+1}})^{\frac{k+1}{n}}]^{\frac{2k\alpha}{k+1}} V^{\frac{2k\alpha}{n}} \\
&= k(n-k)\mu_{n,k}(\frac{n!}{\sqrt{n+1}})^{\frac{2k\alpha}{n}}(\frac{\sqrt{k+1}}{k!})^{2\alpha} V^{\frac{2k\alpha}{n}} \\
&= k(n-k)\mu_{n,k}[\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha} V^{\frac{2k\alpha}{n}}.
\end{aligned}$$

Similarly, by the same technique, we have

$$Q_2 \geq \mu_{n,k}[\mu_{n,k} - (n-k)(k+1) - 1](\prod_{i=1}^{\mu_{n,k}} V_i(k))^{\frac{2\alpha}{\mu_{n,k}}}$$

$$Q_3 \geq (n+1-k-\delta)\mu_{n,k}(\prod_{i=1}^{\mu_{n,k}} V_i(k))^{\frac{2\alpha}{\mu_{n,k}}}.$$

Then, plus Q_2 and Q_3, and we use the inequality (2.12) (let $k+1$ replace by k in inequality (2.12)), it follows:

$$\begin{aligned}
Q_1 + Q_2 &\geq \mu_{n,k}[\mu_{n,k} - (n-k)k - \delta](\prod_{i=1}^{\mu_{n,k}} V_i(k))^{\frac{2\alpha}{\mu_{n,k}}} \\
&\geq \mu_{n,k}[\mu_{n,k} - (n-k)k - \delta][\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha} V^{\frac{2k\alpha}{n}}.
\end{aligned}$$

Then

$$\begin{aligned}
f(\alpha,\delta) \equiv Q_1 + Q_2 + Q_3 &\geq k(n-k)\mu_{n,k}[\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha} V^{\frac{2k\alpha}{n}} \\
&+ \mu_{n,k}[\mu_{n,k} - (n-k)k - \delta][\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha} V^{\frac{2k\alpha}{n}}
\end{aligned}$$

$$= \mu_{n,k}(\mu_{n,k} - \delta)[\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^{2\alpha} V^{\frac{2k\alpha}{n}} .$$

The proof is complete. □

Lemma 7. *Suppose the volume of n-dimensional simplex is V and the volume of each one side of simplex is V_i $(i = 1, 2, \cdots, n+1)$, then $\forall\ \alpha \in (0, 1]$, there is:*

$$\prod_{i=1}^{n+1} V_i^{2\alpha} \geq \frac{1}{(n+1)^{(n-1)\alpha+1}}(\frac{n^{3n}}{n!^2})^{\alpha} V^{2(n-1)\alpha} \sum_{i=1}^{n+1} V_i^{2\alpha} , \tag{2.13}$$

and, if and only if Ω is regular, it is equal.

For the proof see [11].

3. The Proofs of Theorems

In this section, based on the lemmas given in the second section, we prove the stability theorems of Weitzenböck's and Zhang-Yang's inequalities for simplex.

First, we prove the Theorem 1.1. Because the inequality (1.5) implying the inequality (1.4), we only need prove the inequality (1.5). For $n \geq 2$, we have two cases:

Case 1. If $n = 2$, the inequality (1.5) is the famous Finsher-Hadwiger's inequality.

Case 2. If $n > 2$, for $1 \leq k \leq n-1$, there is $\mu_{n,k} \geq 4$. Based on (2.6) and the arithmetic-geometric average inequality, then

$$\frac{1}{\mu_{n,k}} \sum_{i=1}^{\mu_{n,k}} V_i(k) \geq \frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}} V^{\frac{k}{n}} . \tag{3.1}$$

Note that the arithmetic average inequality:

$$(\sum_{i=1}^{m} x_i)^2 = m \sum_{i=1}^{m} x_i^2 - \sum_{1 \leq i < j \leq m} (x_i - x_j)^2 , \tag{3.2}$$

and

$$\sum_{1 \leq i < j \leq m} (x_i - x_j)^2 = m \sum_{i=1}^{m} (x_i - \overline{x})^2 , \tag{3.3}$$

where $\overline{x} = \frac{1}{m} \sum\limits_{i=1}^{m} x_i$. then

$$(\sum_{i=1}^{m} x_i)^2 = m[\sum_{i=1}^{m} x_i^2 - \sum_{i=1}^{m} (x_i - \overline{x})^2] . \tag{3.4}$$

Let $(\mu_{n,k}-(n+1-k))$ times the both side of (1.5), then (1.5) equals of the following inequality

$$
\begin{aligned}
(\sum_{i=1}^{\mu_{n,k}} V_i(k))^2 - (n+1-k)\sum_{i=1}^{\mu_{n,k}} V_i^2(k) \\
&\geq \mu_{n,k}(\mu_{n,k}-(n+1-k))[(\frac{k+1}{k!^2})^{\frac{1}{k}}(\frac{n!^2}{n+1})^{\frac{1}{n}}]^k V^{\frac{2k}{n}} \\
&= \mu_{n,k}(\mu_{n,k}-(n+1-k))[\frac{\sqrt{k+1}}{k!}(\frac{n!}{\sqrt{n+1}})^{\frac{k}{n}}]^2 V^{\frac{2k}{n}} . \quad (3.5)
\end{aligned}
$$

It is the case of Lemma 6 (2.10) when $\alpha = 1,\ \delta = n+1-k$. So the Theorem 1.1 is proved.

Especially, taking $k = 1$, $\alpha = 1$ in (1.5), then we obtain the following result.

Corollary. *Suppose a_i $(i = 1, 2, \cdots, \mu_{n,1})$ and V is the edge lengths and volume of Ω respectively, then*

$$\sum_{i=1}^{\mu_{n,1}} a_i^2 - n[(n+1)^{n-1}n!^2]^{\frac{1}{n}} V^{\frac{2}{n}} \geq \alpha_n' \delta(\Omega, \overline{\Omega}) \ , \quad (3.6)$$

where $\alpha_n' = \frac{n+1}{n-1}$,$\overline{\Omega}$ is a regular simplex with edge lengths $\overline{a} = \frac{1}{\mu_{n,1}} \sum_{i=1}^{\mu_{n,1}} a_i$.

Remark. Let $n = 2$ in (3.6), then we obtain the Finsher-Hadwiger's inequality on triangle:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S_{\triangle} + (a-b)^2 + (b-c)^2 + (c-a)^2 \ .$$

Second, we prove the Theorem 1.2. Because the inequality (1.8) implying the inequality (1.7), we only need prove the inequality (1.8). Utilizing (2.13) (the case of $\alpha = 1$), it follows that:

$$\prod_{i=1}^{n+1} V_i^2 \geq \frac{n^{3n}}{(n+1)^n n!^2} V^{2(n-1)} \sum_{i=1}^{n+1} V_i^2 \ . \quad (3.7)$$

Let $k = n-1$ in (1.4), then $V_i(n-1) = V_1$, so we have:

$$\sum_{i=1}^{n+1} V_i^2 \geq (n+1)[(\frac{n}{(n-1)!^2})^{\frac{1}{n-1}}(\frac{n!^2}{n+1})^{\frac{1}{n}}]^{n-1} V^{\frac{2(n-1)}{n}}$$

$$+\frac{n+1}{n-1}\delta_{n-1}(\Omega,\overline{\Omega}), \quad (3.8)$$

where $\overline{\Omega}$ is a simplex with $n-1$ dimensional side's area $\overline{V}=\frac{1}{n+1}\sum_{i=1}^{n+1}V_i$.

Let (3.8) substituting into (3.7), we have:

$$\prod_{i=1}^{n+1}V_i^2 \geq (\frac{n^{3n}}{(n+1)^{n-1}n!^2})^{\frac{n+1}{n}}V^{\frac{2(n^2-1)}{n}}+\beta_n\delta_{n-1}(\Omega,\overline{\Omega}) \ .$$

At last, we prove the Theorem 1.3. Because of inequality (1.11) implying the inequality (1.10), we only need prove the inequality (1.11). Let the inequality (2.7) substituting into the inequality (3.6), we obtain:

$$R_n^2-\frac{n}{(n+1)^2}[(n+1)^{n-1}n!^2]^{\frac{1}{n}}V^{\frac{2}{n}} \geq \frac{\alpha_n'}{(n+1)^2}\delta(\Omega,\overline{\Omega}) \ , \quad (3.9)$$

and (2.8) substituting into (3.9), then

$$R_n^2-n^2r_n^2 \geq \frac{\alpha_n'}{(n+1)^2}\delta(\Omega,\overline{\Omega})=\frac{1}{(n^2-1)}\delta(\Omega,\overline{\Omega}) \ ,$$

or

$$R_n-nr_n \geq \frac{1}{(n^2-1)(R_n+nr_n)}\delta(\Omega,\overline{\Omega}) \geq \frac{1}{2R_n(n^2-1)}\delta(\Omega,\overline{\Omega}) \ ,$$

Let $D(\Omega)=2R_n$, $\gamma_n=\frac{1}{D(\Omega)(n^2-1)}$, then we have

$$R_n-nr_n \geq \gamma_n\delta(\Omega,\overline{\Omega}) \ .$$

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