International Journal of Pure and Applied Mathematics

Volume 35 No. 4 2007, 503-510

HYPERSURFACES OF A REAL SPACE FORM

Sharief Deshmukh[1], Falleh R. Al-Solamy[2 §]

[1]Department of Mathematics
College of Science
King Saud University
P.O. Box 2455, Riyadh, 11451, KINGDOM OF SAUDI ARABIA
e-mail: shariefd@ksu.edu.sa

[2]Department of Mathematics
King Abdul Aziz University
P.O. Box 80015, Jeddah, 21589, KINGDOM OF SAUDI ARABIA
e-mail: falleh@hotmail.com

Abstract: Let M be an n-dimensional orientable compact hypersurface of a constant mean curvature α in the unit sphere $S^{n+1}(1)$. If the hypersurface M lies an open hemisphere of $S^{n+1}(1)$, then it is shown that $M = S^n(c)$, $c > 1$ is a small hypersphere. We also study hypersurfaces in a real space form $\overline{M}(c)$ which are semiparallel that is the hypersurfaces whose curvature tensor R and the shape operator A commute and obtain interesting properties of these hypersurfaces.

AMS Subject Classification: 53C20, 53C45
Key Words: semiparallel, hypersurfaces, shape operator, mean curvature

1. Introduction

The class of compact hypersurfaces in a real space form $\overline{M}(c)$ (a Riemannian manifold of constant sectional curvature c) is quite large and therefore it is an interesting question in geometry to obtain conditions which characterize special types of hypersurfaces such as totally geodesic ones or totally umbilic ones in this class. This question has been of considerable interest to many geometers and had been approached using various invariants of the hypersurfaces. Most

Received: December 6, 2006

§Correspondence author

natural invariants of a hypersurface are the mean curvature, Ricci curvature and scalar curvature. Nomizu and Smyth have studied non-negatively curved hypersurfaces with constant mean curvature in real space form and in particular they have shown that such compact hypersurfaces in a Euclidean space are spheres (cf. [10], [11]). Hypersurfaces with constant mean curvature and higher order mean curvatures in a real space form have also been studied by Chen, Montiel-Ros, Ripoll, Ros and obtained different characterizations for extrinsic spheres (cf. [2], [3], [4], [9], [13], [14]). Similarly Ros [15] has studied compact embedded hypersurfaces with constant scalar curvature in an Euclidean space and proved that they are essentially spheres. We denote by R the curvature tensor field of the hypersurface M of the real space form $\overline{M}(c)$. Let A be the shape operator of the hypersurface, α its mean curvature. As mentioned above there are so many results those characterize totally geodesic spheres in the unit sphere $S^{n+1}(1)$, and therefore it is interesting to obtain characterization for small hypespheres in the unit sphere $S^{n+1}(1)$. Since the small hyperspheres lie in the open hemisphere of $S^{n+1}(1)$, it is natural to suppose that the hypersurface under investigation lies in an open hemisphere. This raises a question, under what condition a compact hypersurface of $S^{n+1}(1)$ that lies in an open hemisphere a small hypersphere? In this paper we answer this question by proving the following

Theorem. *Let M be an orientable compact hypersurface of the unit sphere $S^{n+1}(1)$ with constant mean curvature α. If the hypersurface M lies in an open hemisphere of $S^{n+1}(1)$, then M is a small hypersphere $S^n(c)$.*

We also study the hypersurfaces in a real space form $\overline{M}(c)$ whose curvature tensor field R and the shape operator A commute so-called the semiparallel hypersurfaces. Hypersurfaces with parallel shape operator are called parallel hypersurfaces, naturally parallel hypersurfaces are semiparallel. We obtain conditions for such hypersurfaces to be totally geodesic in the unit sphere $S^{n+1}(1)$. In particular, we also show that semiparallel minimal hypersurfaces in the hyperbolic space forms $H^{n+1}(-1)$ are totally geodesic.

2. Preliminaries

Let M be an orientable hypersurface of the real space form $\overline{M}(c)$. We denote the induced metric on M by g. Let $\overline{\nabla}$ be the Riemannian connection on the real space form $\overline{M}(c)$ and ∇ be the Riemannian connection on M with respect to the induced metric g. Let N be the unit normal vector field and A be the shape operator of M. Then the Gauss and Weingarten formulas for the hypersurface are (cf. [1])

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX, \qquad X, Y \in \mathfrak{X}(M)\,, \tag{2.1}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We also have the following Gauss and Codazzi equations

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} + g(AY,Z)AX - g(AX,Z)AY\,, \tag{2.2}$$

$$(\nabla A)(X,Y) = (\nabla A)(Y,X), \qquad X, Y \in \mathfrak{X}(M)\,, \tag{2.3}$$

where R is the curvature tensor field of the hypersurface and $(\nabla A)(X,Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, .., e_n\}$ is a local orthonormal frame on M. If $A = \lambda I$ holds for a constant λ, then the hypersurface is said to be totally umbilical. The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = \operatorname{tr} A^2.$$

From equation (2.2) we get the following expression for the Ricci tensor field $\operatorname{Ric}(X,Y) = \sum_i g(R(e_i, X)Y, e_i)$

$$\operatorname{Ric}(X,Y) = (n-1)cg(X,Y) + n\alpha g(AX,Y) - g(AX,AY)\,. \tag{2.4}$$

The scalar curvature S of the hypersurface is given by

$$S = n(n-1)c + n^2\alpha^2 - \|A\|^2\,. \tag{2.5}$$

3. Proof of the Main Theorem

Let M be an orientable compact hypersurface of constant mean curvature α in the unit sphere $S^{n+1}(1)$ and $F : M \to S^{n+1}(1)$ be its immersion. If $i : S^{n+1}(1) \to R^{n+2}$ is the standard embedding, then $\psi = i \circ F : M \to R^{n+2}$ makes M submanifold of R^{n+2}. Let N be the unit normal vector field of M in $S^{n+1}(1)$ and $\overline{N}$ be that of $S^{n+1}(1)$ in R^{n+2}. Then the second fundamental form of M in R^{n+2} is expressed as

$$h(X,Y) = g(AX,Y)N - g(X,Y)\overline{N}, \quad X, Y \in \mathfrak{X}(M)\,, \tag{3.1}$$

where A is the shape operator of M in $S^{n+1}(1)$ and $\mathfrak{X}(M)$ is Lie algebra of smooth vector fields on M. Since M has constant mean curvature α as hypersurface of $S^{n+1}(1)$, from equation (3.1) it follows that the mean curvature vector field $H = \frac{1}{n}\sum h(e_i, e_i)$ of M as submanifold of R^{n+2} is given by

$$H = \alpha N - \overline{N}\,. \tag{3.2}$$

For a smooth function $\phi : M \to R$, let $\Delta\phi = \operatorname{div}(\nabla\phi)$ be the Laplacian of ϕ, where $\nabla\phi$ is the gradient of the function ϕ. Let $\xi \in \mathfrak{X}(R^{n+2})$ be a constant unit vector field. Then we can express $\xi\mid_M = u + \varsigma$, where $u \in \mathfrak{X}(M)$ and $\varsigma \in \Gamma(\upsilon)$, υ being normal bundle of M in R^{n+2}. Then using Gauss and Weingarten formulas for submanifold M of R^{n+2} we get

$$\nabla_X u = BX,\ \ \nabla^{\perp}_X \varsigma = -h(X, u),\ \ X \in \mathfrak{X}(M)\,, \tag{3.3}$$

where $B = A_\varsigma$ is the shape operator of M with respect to the normal ς as submanifold of R^{n+2} and $\nabla^{\perp}$ is the connection in the normal bundle υ. Define the function f and φ on M by $f = g(N, \varsigma)$ and $\varphi = g(\overline{N}, \varsigma)$, where we denote by the letter g the Euclidean metric on R^{n+2} as well as that induced on the unit sphere S^{n+1} and that induced on the hypersurface M. Then using equation (3.2), we get

$$\operatorname{tr} B = \sum_i g(Be_i, e_i) = \sum_i g(h(e_i, e_i), \varsigma) = n\alpha f - n\varphi,$$

and consequently using equation (3.3), the divergence of the vector u is computed as

$$\operatorname{div} u = n\alpha f - n\varphi. \tag{3.4}$$

Note that as $\nabla^{\perp}_X N = 0$ and $\nabla^{\perp}_X \overline{N} = 0$, $X \in \mathfrak{X}(M)$, for $X \in \mathfrak{X}(M)$, operating $\nabla^{\perp}_X$ on $\varsigma = fN + \varphi\overline{N}$ and using equation (3.3) we arrive at

$$-h(X, u) = X(f)N + X(\varphi)\overline{N}\,.$$

Comparing this equation with (3.1) we conclude

$$\nabla f = -Au, \qquad \nabla\varphi = u, \tag{3.5}$$

where ∇f and $\nabla\varphi$ are gradients of the functions f and φ respectively. Then using the fact that the hypersurface M has constant mean curvature α, we compute the Laplacians Δf and $\Delta\varphi$ of f and φ using equations (3.4) and (3.5) as

$$\Delta f = -\operatorname{tr} AB, \qquad \Delta\varphi = n\alpha f - n\varphi, \tag{3.6}$$

where we use the fact that

$$\operatorname{div} Au = \sum_i e_i g(u, Ae_i) = \sum_i [g(\nabla_{e_i} u, Ae_i) + g(u, (\nabla A)(e_i, e_i))]$$
$$= \operatorname{tr} AB + nu(\alpha).$$

However, as $B = A_\varsigma = fA - \varphi I$ we get $\operatorname{tr} AB = f\|A\|^2 - n\varphi\alpha$, using this together with equations (3.6) and above equation, we get

$$\Delta(f + \alpha\varphi) = f(n\alpha^2 - \|A\|^2). \tag{3.7}$$

Since M lies on an open hemisphere of $S^{n+1}(1)$, there exists a constant unit vector field ξ on R^{n+2} such that $f = g(\varsigma, N) = g(\xi, N) > 0$. Also by Schwarz inequality we have $\|A\|^2 \geq n\alpha^2$ with equality holding if and only if $A = \alpha I$. Thus as M being compact, using Hopf Lemma in equation (3.7), we get $\Delta(f + \alpha\varphi) = 0$, which gives $\|A\|^2 = n\alpha^2$ that is $A = \alpha I$. Hence by equation (2.2) we get $M = S^n(1 + \alpha^2)$ that is M is small hypersphere.

We also have the following result for the estimate of the first nonzero eigenvalue λ_1 of the Laplacian operator Δ of the hypersurface.

Corollary 3.1. *Let M be an orientable compact hypersurface of the unit sphere $S^{n+1}(1)$ with constant mean curvature α. Then the first nonzero eigenvalue λ_1 of the Laplacian operator acting on smooth functions of the hypersurface M satisfies $\lambda_1 \leq n(1 + \alpha^2)$.*

Proof. As this hypersurface M of the unit sphere $S^{n+1}(1)$ is also a submanifold of the Euclidean space R^{n+2} with mean curvature vector field H given by equation (3.2). We have by equation (3.2) that $\|H\|^2 = 1 + \alpha^2$. Using the fact that α is a constant in Reilly's Theorem (cf. [12]) we get the result. □

4. Semiparallel Hypersurfaces

Let M be an n-dimensional orientable hypersurface of the $(n+1)$-dimensional real space form $\overline{M}(c)$. The hypersurface is said to parallel if the shape operator is parallel that is $(\nabla A)(X, Y) = 0$, $X, Y \in \mathfrak{X}(M)$. We call a hypersurface semiparallel if it satisfies

$$R(X, Y)AZ = A(R(X, Y)Z), \quad X, Y, Z \in \mathfrak{X}(M). \tag{4.1}$$

Clearly totally umbilical hypersurfaces and in general parallel hypersurfaces are semiparallel. Parallel hypersurfaces of a real space form $\overline{M}(c)$ are completely determined (cf [8]). In this section we study semiparallel hypersurfaces of a real space form $\overline{M}(c)$. First we prove the following theorem.

Theorem 4.1. *Let M be an n-dimensional orientable semiparallel hypersurface of the $(n+1)$-dimensional real space form $\overline{M}(c)$. Then the mean curvature α and the shape operator A of M satisfies*

$$n\alpha A^2 + (nc - \|A\|^2)A - n\alpha c I = 0.$$

Proof. Let $\{e_1, e_2, .., e_n\}$ be a local orthonormal frame on M. Then using equation (4.1) we get

$$\mathrm{Ric}\,(Y, AZ) = \sum_i g(R(e_i, Y)Z, Ae_i),$$

which together with equations (2.2) and (2.4) gives

$$\begin{aligned} &(n-1)cg(Y, AZ) + n\alpha g(AY, AZ) - g(AY, A^2 Z) \\ &\qquad\qquad c\{g(Y, Z)n\alpha - g(AY, Z)\} + g(AY, Z)\,\|A\|^2 - g(AY, A^2 Z). \end{aligned}$$

This proves the result. □

Corollary 4.1. *Let M be an n-dimensional orientable connected minimal semiparallel hypersurface of the $(n+1)$-dimensional real space form $\overline{M}(c)$. Then M has constant scalar curvature either $n(n-2)c$ or $n(n-1)c$.*

Proof. Since M is minimal by above theorem we have

$$(nc - \|A\|^2)A = 0\,.$$

If $c = 0$ the result is obvious. Therefore suppose $c \neq 0$ and consider the subsets $U = \left\{p \in M : \|A\|_p^2 = nc\right\}$ and $V = \{p \in M : A_p = 0\}$ which are closed subsets of M. Moreover we have $U \cap V = \phi$ and this together with the fact that M is connected we get that either $\|A\|^2 = nc$ or else $A = 0$. In both cases we get that M has constant scalar curvature either $S = n(n-2)c$ or $S = n(n-1)c$. □

For compact minimal hypersurface M in a unit Sphere $S^{n+1}(1)$, Simons result states that if the shape operator of the hypersurface satisfies $\|A\|^2 < n$, then M is totally geodesic (cf. [16]). We have the following corollary of Theorem 4.1 for semiparallel minimal hypersurface which need not be compact.

Corollary 4.2. *Let M be an orientable semiparallel minimal hypersurface of the unit sphere $S^{n+1}(1)$. If the shape operator A of the hypersurface M satisfies $\|A\|^2 < n$, then M is totally geodesic.*

Also we have the following direct consequences of the Theorem 4.1, for semiparallel minimal hypersurfaces of the Euclidean space and the hyperbolic space forms.

Corollary 4.3. *Let M be an n-dimensional orientable minimal semiparallel hypersurface of the Euclidean space R^{n+1}. Then M is totally geodesic.*

Corollary 4.4. *Let M be an n-dimensional orientable minimal semiparallel hypersurface of the Hyperbolic $H^{n+1}(-1)$. Then M is totally geodesic.*

References

[1] B.Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific (1983).

[2] B.Y. Chen, O.J. Garay, Constant mean curvature hypersurfaces with constant δ-invariant, *Int. J. Math. Math. Sci.*, **67** (2003), 4205-4216.

[3] B.Y. Chen, Constant ratio hypersurfaces, *Soochow J. Math.*, **27** (2001), 353-362.

[4] C. Chen, H. Sun, L. Tang, On translation hypersurfaces with constant mean curvature in $(n+1)$-dimensional spaces, *J. Bejing Inst. Technol.*, **12** (2003), 322-325.

[5] R. Deszcz, M. Hotlos, Z. dSenturk, On Ricci-pseudosymmetric hypersurfaces in space forms, *Demo. Math.*, **37** (2004), 203-214.

[6] R. Deszcz, M. Glogowska, Examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces, *Colloq. Math.*, **94** (2002), 87-101.

[7] S. Kobayashi, K. Nomizu, *Foundations in Differential Geometry*, Wiley-Interscience, N.Y. (1969).

[8] H.B. Lawson, Local rigidity theorems for minimal hypersurfaces, *Ann. Math.*, **89** (1969), 187-197.

[9] S. Montiel, A. Ros, Compact hypersurfaces: the Alexandrov Theorem for higher order mean curvatures, *Differential Geometry, Pitman Monogr. Surveys Pure Appl. Math. Loongman Sci. Tech. Harlow.*, **52** (1991), 279-296.

[10] K. Nomizu, B. Smyth, A formula of simon's type and hypersurfaces with constant mean curvature, *J. Diff. Geom.*, **3** (1969), 367-377.

[11] K. Nomizu, B. Smyth, On the Gauss mapping for hypersurfaces of constant mean curvature in the sphere, *Comm. Math. Helv.*, **44** (1969), 484-490.

[12] R. Reilly, On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space, *Comm. Math. Helv.*, **52** (1977), 525-533.

[13] J.B. Ripoll, A characterization of spheres in Euclidean space, *Arch. Math.*, **51** (1988), 474-475.

[14] A. Ros, Compact hypersurfaces with constant scalar curvature and a congruence theorem, *J. Diff. Geom.*, **27** (1988), 215-223.

[15] A. Ros, Compact hypersurfaces with constant higher order mean curvatures, *Rev. Mat. Iberoamericana*, **3** (1987), 447-453.

[16] J. Simon, Minimal varieties in Riemannian manifolds, *Ann. of Math.*, **88** (1968), 62-105.