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ON GREEN RELATION $\mathcal{H}$ ON le-Γ-SEMIGROUPS

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Abstract: In this paper we give several properties that hold in every $\mathcal{H}$-class of an $le-\Gamma$-semigroup. We give a necessary and sufficient condition for an $\mathcal{H}$-class H of an $le-\Gamma$-semigroup M to be a sub-Γ-group of M and we provide several conditions for an $\mathcal{H}$-class to forms a sub-Γ-semigroup.

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1. Introduction and Preliminaries

In 1981, Sen introduced the concepts of Γ-semigroups [6]. Kwon and Lee gave some properties of special elements in $po-\Gamma$-semigroup and $le-\Gamma$-semigroup [3, 4]. We [1, 2] gave some other characterizations in le-Γ-semigroups in general and regular $le-\Gamma$-semigroups in particular and gave several properties that hold in every $\mathcal{H}$-classes of an le-Γ-semigroup satisfying Green's condition. In this paper we give some other properties that hold in every $\mathcal{H}$-classes of an le-Γ-semigroup. We give a necessary and sufficient condition when an $\mathcal{H}$-class H of an $le-\Gamma$-semigroup M is a sub-Γ-group of M. We also provide several conditions that ensure that an $\mathcal{H}$-class forms a sub-Γ-semigroup.

Sen [6] defined Γ-semigroup as follows.

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Definition 1.1. Let M and Γ be any two non-empty sets. Then M is called a $\Gamma-$*groupoid* if $a\alpha b \in M$ and $\alpha a\beta \in \Gamma$ for all $a, b \in M$ and for all $\alpha, \beta \in \Gamma$.

Definition 1.2. Let $M = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then M is called a Γ-semigroup if there exist mappings $M \times \Gamma \times M \to M$, written as $(a, \alpha, b) \longmapsto a\alpha b$ and $\Gamma \times M \times \Gamma \to \Gamma$ written as $(\alpha, a, \beta) \longmapsto \alpha a\beta$, satisfying the following identities

$$(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c) \text{ for all } a, b, c \in M \text{ and for all } \alpha, \beta \in \Gamma.$$

Sen and Saha [7] weakened the defining conditions of Γ-semigroup and defined Γ-semigroup by replacing the above conditions by follows:
(1) $a\alpha b \in M$
(2) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.
We call a $\Gamma-$*semigroup* M *both sided* if it further satisfies the identities:

$$\alpha a(\beta b\gamma) = (\alpha a\beta)b\gamma = \alpha(a\beta b)\gamma \text{ for all } a, b \in M \text{ and for all } \alpha, \beta, \gamma \in \Gamma.$$

In this paper we consider only both sided Γ-semigroup and call it simply Γ-semigroup.

An element a of a Γ-semigroup M is called an *idempotent* if $a\alpha a = a$ for all $\alpha \in \Gamma$.

Kwon and Lee [3] introduced the concept of a *po*-Γ-semigroup and of ideals in a *po*-Γ-semigroup and obtained some results.

Definition 1.3. A *po*-Γ-groupoid is an ordered set M at the same time Γ-groupoid such that

$$a \leq b \Rightarrow a\alpha c \leq b\alpha c, \quad c\alpha a \leq c\alpha b$$

for all $c \in M$ and for all $\alpha \in \Gamma$.

When M is a Γ-semigroup, then M is called a $po-\Gamma-semigroup$. A *poe*-Γ-semigroup is a $po-\Gamma$-semigroup M with a greatest element "e" (i.e., $e \geq a, \forall a \in M$).

In a $po-\Gamma$-groupoid M, a is called a *right* (resp. *left*) *ideal element* if $a\alpha b \leq a$ (resp. $b\alpha a \leq a$) for all $b \in M$ and for all $\alpha \in \Gamma$. And a is called an *ideal element* if it is both a right and left ideal element. In a $poe-\Gamma$-groupoid M, a is called *right* (resp. *left*) *ideal element* if $a\alpha e \leq a$ (resp. $e\alpha a \leq a$) for all $\alpha \in \Gamma$.

For $A \subseteq M$ we denote

$$(A] = \{t \in M | t \leq a, \text{ for some } a \in A\}$$

An element a of a *po*$-\Gamma$-semigroup M is called *regular* if there exists $b \in M$ such that $a \leq a\alpha b\beta a$ for some $\alpha, \beta \in \Gamma$. A *po*$-\Gamma$-semigroup M is called *regular* if every element of M is regular. The following are equivalent:

1. For every $A \subseteq M, A \subseteq (A\Gamma M\Gamma A]$,
2. For every element $a \in M, a \in (a\Gamma M\Gamma a]$.

An element a of a *poe*$-\Gamma$-grupoid is called a *quasi-ideal element* if $e\alpha a \vee a\alpha e$ exists and $a\alpha e \wedge e\alpha a \leq a$ for all $\alpha \in \Gamma$. We denote by $q(a)$ the quasi-ideal element of M generated by a, i.e., the least quasi-ideal element of M containing a. The *zero* of a *poe*$-\Gamma$-grupoid M is an element of M denoted by 0 such that $e \neq 0 \leq a$ and $0\alpha a = a\alpha 0 = 0$ for every $a \in M, \alpha \in \Gamma$. Let M be a *poe* $-\Gamma$-grupoid with 0. A quasi-ideal element a of M is called *minimal* if $a \neq 0$ and there exists no quasi-ideal element t of M such that $0 < t < a$.

Definition 1.4. Let M be a semilattice under $\vee$ with a greatest element e and at the same time a *po* $-\Gamma$-semigroup such that

$$a\alpha(b \vee c) = a\alpha b \vee a\alpha c$$

and

$$(a \vee b)\beta c = a\beta c \vee b\beta c$$

for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$. Then M is called a $\vee e - \Gamma$-*semigroup*.

A $\vee e - \Gamma$-semigroup which is also a lattice is called an *le*-Γ-*semigroup*.

Example 1.5. Let $(X, \leq)$ and $(Y, \leq)$ be two finite chains. Let M be the set of all isotone mappings from X into Y and Γ be the set of all isotone mappings from Y into X. Let $, f, g \in M$ and $\alpha \in \Gamma$. We define $f\alpha g$ to denote the usual mapping composition of f, α and g. Then M is a Γ-semigroup. For $f, g \in M$, the mappings $f \vee g$ and $f \wedge g$ are defined by letting, for each $a \in X$

$$(f \vee g)(a) = \max\{f(a), g(a)\}, \quad (f \wedge g)(a) = \min\{f(a), g(a)\}$$

The same is defined in Γ (the maximum and minimum are considered with respect to the order $\leq$ in X and Y). The greatest element e is the mapping that sends every $a \in X$ to the greatest element of finite chains $(Y, \leq)$. Then M is an $le - \Gamma$-semigroup.

Example 1.6. Let M be a *po* $-\Gamma$-semigroup. Let M_1 be the set of all ideals of M. Then $((M_1, \Gamma), \cap, \cup)$ is an $le - \Gamma$-semigroup.

Example 1.7. Let M be a *po*$-\Gamma$-semigroup. Let $M_1 = P(M)$ be the set of all subsets of M and $\Gamma_1 = P(\Gamma)$ the set of all subsets of Γ. Then (M_1, Γ_1) is a $po-\Gamma$-semigroup where

$$A\Lambda B = \begin{cases} (A](\Lambda](B] = (A\Lambda B] & \text{if } A, B \in M_1\backslash\{\emptyset\}, \Lambda \in \Gamma_1\backslash\{\emptyset\}\,, \\ \emptyset & \text{if } A = \emptyset \text{ or } B = \emptyset\,. \end{cases}$$

Then $((M_1, \Gamma_1), \subseteq, \cap, \cup)$ is an $le-\Gamma$-semigroup.

In a $\vee e-\Gamma$-semigroup M, we define two mappings r_α and l_β for each $\alpha, \beta \in \Gamma$ as follows:

$$r_\alpha : M \to M, r_\alpha(a) = a\alpha e \vee a,$$
$$l_\beta : M \to M, l_\beta(a) = e\beta a \vee a$$

for all $a \in M$.

In an arbitrary $le-\Gamma$-semigroup M, the Green's relations are defined as follows:

$$\mathcal{L} = \{((a, \alpha), (b, \beta)) \in (M \times \Gamma) \times (M \times \Gamma) | e\alpha a \vee a = e\beta b \vee b\}\,,$$

or

$$\mathcal{L} = \{(a, b) \in M | l_\alpha(a) = l_\beta(b)\},$$

$$\mathcal{R} = \{((a, \alpha), (b, \beta)) \in (M \times \Gamma) \times (M \times \Gamma) | a\alpha e \vee a = a\beta e \vee b\},$$

or

$$\mathcal{R} = \{(a, b) \in M | r_\alpha(a) = r_\beta(b)\}, \quad \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

We say [1] that a $\mathcal{H}$-class of M satisfy Green's condition if the class contains the product of pairs of its elements. The authors [7] gave the following definition.

Definition 1.8. A Γ-semigroup M is called a Γ-*group* if M_α is a group for some (hence for all) $\alpha \in \Gamma$.

2. Main Results

Let M be a $le-\Gamma$-semigroup. We denote by q_H[1] the representative quasi-ideal element of an $\mathcal{H}$-class H of a, i.e. $q_H = r_\alpha(a) \wedge l_\alpha(a)$, $a \in M, \alpha \in \Gamma$.

Proposition 2.1. *Let M be a $le-\Gamma$-semigroup. An $\mathcal{H}$-class H of M is a sub-Γ-group of M if and only if H consists of a single idempotent.*

Proof. ($\Rightarrow$) Let H be a sub-Γ-group of M, that is H_α is a group [7], $\alpha \in \Gamma$. Let $q = q_H$. Then $q = q\alpha q, \alpha \in \Gamma$. Indeed: We denote by i the identity element of H_α. Then $i \leq q$. By Theorem 3(1) from [1], $q\alpha q \in H$, whence $q\alpha q \leq q = q\alpha i \leq q\alpha q$. Thus, $q = q\alpha q$ and $q = i$.

Let a be an arbitrary element of H. We denote by a^{-1} the inverse element of a in H_α. Then $a^{-1} \leq q$. Then we obtain $q = a\alpha a^{-1} \leq a\alpha q = a$. On the other hand, $a \leq q$, thus $a = q$.

($\Leftarrow$) This part is obvious. □

Theorem 2.2. *Let H be an $\mathcal{H}$-class of M. Then the following are equivalent:*

1. H is a sub-Γ-semigroup of M;

2. $a\alpha a \in H, \forall a \in H, \forall \alpha \in \Gamma$;

3. H satisfies Green's condition and $a\alpha q = q\alpha q = q\alpha a, \forall a \in H, \forall \alpha \in \Gamma$, where $q = q_H$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $a \in H, \alpha \in \Gamma$. Since by (2), $(a, a\alpha a) \in \mathcal{H}$, by Lemma 6 from [1] we take $a\alpha e = (a\alpha a)\alpha e$ and $e\alpha a = e\alpha(a\alpha a)$. Similarly, since $(a, q) \in \mathcal{H}$, we get $a\alpha e = q\alpha e$ and $e\alpha a = e\alpha q$. By Theorem 3(4) from [1], $q\alpha q = q\alpha e\alpha q$. Thus, we have

$$a\alpha(q\alpha q) = a\alpha(q\alpha e\alpha q) = a\alpha(a\alpha e\alpha q) = (a\alpha a\alpha e)\alpha q = (a\alpha e)\alpha q = q\alpha e\alpha q = q\alpha q\,,$$

$$(q\alpha q)\alpha a = (q\alpha e\alpha q)\alpha a = (q\alpha e\alpha a)\alpha a = q\alpha(e\alpha a\alpha a) = q\alpha(e\alpha a) = q\alpha e\alpha q = q\alpha q\,.$$

Since $a, q\alpha q \in H$, we have $a \leq q$ and $q\alpha q \leq q$. Therefore,

$$\begin{aligned} q\alpha q &= a\alpha(q\alpha q) \leq a\alpha q \leq q\alpha q \quad \text{whence} \quad a\alpha q = q\alpha q\,,\\ q\alpha q &= (q\alpha q)\alpha a \leq q\alpha a \leq q\alpha q \quad \text{whence} \quad q\alpha a = q\alpha q\,. \end{aligned}$$

(3) $\Rightarrow$ (1) Let $a, b \in H, \alpha \in \Gamma$. Then $r_\alpha(b) = b\alpha e \vee b = q\alpha e \vee q = r_\alpha(q)$. Then we obtain $r_\alpha(a\gamma b) = a\gamma b\alpha e \vee a\gamma b = a\gamma q\alpha e \vee a\gamma q = r_\alpha(a\gamma q)$, $\gamma \in \Gamma$. By (3) we have $a\gamma q = q\gamma q$ whence $r_\alpha(a\gamma b) = r_\alpha(q\gamma q)$.

Since H satisfies Green's condition, by Theorem 3(1) from [1] we have that $q\gamma q \in H$ whence $r_\alpha(q\gamma q) = r_\alpha(q)$. Hence we have $r_\alpha(a\gamma b) = r_\alpha(q)$ and $(a\gamma b, q) \in \mathcal{R}$. Similarly, $(a\gamma b, q) \in \mathcal{L}$. Thus, $(a\gamma b, q) \in \mathcal{H}$ and $a\gamma b \in H$. □

Corollary 2.3. *Let H be an $\mathcal{H}$-class of M. If q_H is a zero of H, then H is a sub-Γ-semigroup of M.*

Proof. Since a zero element is always an idempotent, $q_H = q_H \alpha q_H$, $\forall \alpha \in \Gamma$. Thus, H satisfies the condition (3) of Theorem 2.2. □

Lemma 2.4. *Let H be an $\mathcal{H}$-class of M satisfying Green's condition. If $q = q_H$ is an ideal element, then q is a zero of H.*

Proof. Since q is an ideal element, $q\alpha e \leq q$ and $e\alpha q \leq q$, $\forall \alpha \in \Gamma$. On the other hand, by Theorem 3(3) from [1], we have $q \leq q\alpha e$ and $q \leq e\alpha q$. Thus $q\alpha e = q = e\alpha q$. By Theorem 3(1) from [1], $q\alpha q \in H$ whence by Lemma 6 from [1], we have $e\alpha(q\alpha q) = e\alpha q$. Therefore

$$q\alpha e\alpha q = (q\alpha e)\alpha q = (e\alpha q)\alpha q = e\alpha(q\alpha q) = e\alpha q = q$$

Let a be an arbitrary element of H. By Lemma 6 from [1], we have $a\alpha e = q\alpha e$ and $e\alpha a = e\alpha q$. Therefore

$$a\alpha q = a\alpha e\alpha q = q\alpha e\alpha q = q \text{ and } q\alpha a = q\alpha e\alpha a = q\alpha e\alpha q = q.$$

The lemma is proved. □

By Corollary 2.3 and Lemma 2.4 we obtain the next result.

Corollary 2.5. *If an $\mathcal{H}$-class H of M satisfies Green's condition and q_H is an ideal element, then H is a sub-Γ-semigroup of M.*

Remark 2.6. We call a $le-\Gamma$-semigroup M to be a *duo* $le-\Gamma$-semigroup if the sets of its left ideal elements and of its right ideal elements coincide. It is clear that from Corollary 2.5 it follows that for duo $le-\Gamma$-semigroup, $\mathcal{H}$-classes satisfying Green's condition are sub-Γ-semigroups.

Corollary 2.7. *The $\mathcal{H}$-class H of the greatest element e of M is a sub-Γ-semigroup of M if and only if e is an idempotent.*

Proof. If H is a sub-Γ-semigroup of M, then it satisfies Green's condition. Since $q_H = e$ and e is an ideal element, applying Lemma 2.4, we take $e = e\alpha e$, $\forall \alpha \in \Gamma$.

Conversely, if $e = e\alpha e$ then H satisfies Green's condition. Since $q_H = e$ and e is an ideal element, applying Corollary 2.5 we have H is a sub-Γ-semigroup of M. □

Proposition 2.8. *Let H be an $\mathcal{H}$-class of M such that its representative quasi-ideal element $q = q_H$ is minimal in the set of all quasi-ideal elements of M. Then $H = (q] = \{a \in M | a \leq q\}$ and H is a sub-Γ-semigroup of M.*

Proof. If $a \in H$, then $a \leq q$. Let us suppose that $a \leq q$. Then $r_\alpha(a) \wedge l_\alpha(a) \leq r_\alpha(q) \wedge l_\alpha(q) = q$, $\forall \alpha \in \Gamma$. By Lemma 1 from [1] and Lemma 3 from [1] $r_\alpha(a) \wedge l_\alpha(a)$ is a quasi-ideal element. Since q is a minimal quasi-ideal element, $r_\alpha(a) \wedge l_\alpha(a)$ must coincide with q. However, $r_\alpha(a) \wedge l_\alpha(a)$ is the representative quasi-ideal element of the $\mathcal{H}$-class of a, and we conclude that $(a, q) \in \mathcal{H}$, that is $a \in H$. We have proved that $H = \{a \in M | a \leq q\}$.

By Theorem 2.2(2), from $a \leq q$ and $a \leq e$ we take

$$a\alpha a \leq q\alpha e \wedge e\alpha q \leq r_\alpha(q) \wedge l_\alpha(q) = q.$$

So, $a \leq q$ implies $a\alpha a \leq q$, that is H is a sub-Γ-semigroup of M. □

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