International Journal of Pure and Applied Mathematics
Volume 35 No. 4 2007, 479-484

CAPITULATION OF 2-IDEAL CLASSES OF $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$ IN THE GENUS FIELD OF k WHERE p IS PRIME SUCH THAT $p \equiv 1 \bmod 8$

Abdelmalek Azizi[1 §], Mohammed Taous[2]
[1,2]Department of Mathematics
Faculty of Science
Mohamed First University
Oujda, MOROCCO
[1]e-mail: abdelmalekazizi@yahoo.fr
[2]e-mail: taousm@hotmail.com

Abstract: Let p be a prime such that $p \equiv 1 \bmod 8$ and $i = \sqrt{-1}$. Let $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$ and $\mathbf{k}^*$ be the genus field of $\mathbf{k}$. Our goal is to study the problem of the capitulation of 2-ideal classes of $\mathbf{k}$ in $\mathbf{k}^*$.

AMS Subject Classification: 11R27, 11R29, 11R37
Key Words: units, class groups, class field, capitulation

1. Introduction

Let $\mathbf{k}$ be a number field of finite degree over $\mathbf{Q}$ and $Cl_{\mathbf{k}}$ (resp. $Cl_{\mathbf{k},2}$) be the class (resp. 2-class) group of $\mathbf{k}$. Let $\mathbf{K}$ be an unramified extension of $\mathbf{k}$ of finite degree and let $\mathcal{O}_{\mathbf{K}}$ be its ring of integers. We say that an ideal $\mathfrak{a}$ (or the class ideal of $\mathfrak{a}$) of $\mathbf{k}$ capitulates in $\mathbf{K}$ if it becomes principal in $\mathbf{K}$ (i.e., if $\mathfrak{a}\mathcal{O}_{\mathbf{K}}$ is principal in $\mathbf{K}$). The Hilbert class (resp. 2-class) field $\mathbf{k}^{(1)}$ (resp. $\mathbf{k}_2^{(1)}$) of $\mathbf{k}$ is the maximal Abelian unramified extension (resp. 2-extension) of $\mathbf{k}$. If $\mathbf{k}/\mathbf{F}$ is an arbitrary finite extension of number fields, the genus field $(\mathbf{k}/\mathbf{F})^*$ of $\mathbf{k}$ is defined to be the maximal field of type $\mathbf{Lk}$ such that $\mathbf{L}/\mathbf{F}$ is Abelian and $\mathbf{Lk}/\mathbf{k}$ is unramified. By F. Terada (cf. [11]) we know that if $\mathbf{k}/\mathbf{F}$ is cyclic extension, then the ambiguous ideal classes of $\mathbf{k}/\mathbf{F}$ capitulate already in $(\mathbf{k}/\mathbf{F})^*$.

Received: November 15, 2006
§Correspondence author

In this paper, we suppose that $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$, where p is a prime such that $p \equiv 1 \bmod 8$ and $i = \sqrt{-1}$, then $(\mathbf{k}/\mathbf{Q})^* := \mathbf{k}^* = \mathbf{Q}(\sqrt{2}, \sqrt{p}, i)$ and rank $Cl_{\mathbf{k},2} = 2$. In particular, $[(\mathbf{k}/\mathbf{Q}(i))^* : \mathbf{k}] = 4$ and $(\mathbf{k}/\mathbf{Q}(i))^*$ is quadratic unramified extension of $\mathbf{k}^*$. Our goal is to study the problem of the capitulation of 2-classes of $\mathbf{k}$ in $\mathbf{k}^*$ and to give the condition so that the ambiguous ideal classes of $\mathbf{k}/\mathbf{Q}(i)$ capitulate in $\mathbf{k}^*$.

2. The Number of 2-Ideal Classes that Capitulate from k to k*

Let p be a prime such that $p \equiv 1 \bmod 8$ and $\mathbf{k}^*$ be the genus field of $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$. We designate by $h(m)$ the 2-class number of $\mathbf{Q}(\sqrt{m})$ and by $h(K)$ the 2-class number of any number field K. If K be a biquadratic number field, we denote by Q_K the unit index of K. We state the following well known result: if K/k is a cyclic unramified extension of prime power degree, the number of ideal classes that capitulate from k to K is equal to $[K : k][E_k : \mathcal{N}_{K/k}(E_K)]$ where E_k (resp. E_K) is the unit group of k (resp. K) and $\mathcal{N}_{K/k}$ is the norm of K/k (cf. [5]). Since the genus field of $\mathbf{k}$ is $\mathbf{k}^* = \mathbf{Q}(\sqrt{2}, \sqrt{p}, i)$, we seek a fundamental system of units (F.S.U.) of $k_0 = \mathbf{Q}(\sqrt{2}, \sqrt{p})$ to find a F.S.U. of $\mathbf{k}^*$ (for more details on this method, see [1]).

Lemma 1. *Let p be a prime such that $p \equiv 1$ mod 8 and $k_0 = \mathbf{Q}(\sqrt{2}, \sqrt{p})$. Then the unit index Q_{k_0} is equal to 2.*

Proof. From [12] it follows that

$$h(k_0) = \frac{Q_{k_0} h(2) h(p) h(2p)}{4} = \frac{Q_{k_0} h(2p)}{4}\ .$$

Since k_0 is quadratic unramified extension of $\mathbf{Q}(\sqrt{2p})$ and the 2-class group of $\mathbf{Q}(\sqrt{2p})$ is cyclic, then $h(k_0) = \frac{h(2p)}{2}$. Therefore, we have the unit index Q_{k_0} is equal to 2. □

Lemma 2. *Let p be a prime such that $p \equiv 1$ mod 8, $\mathbf{k}^*$ the be genus field of $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$. Let ε_1, ε_2 and ε_3 be the fundamental units of $\mathbf{Q}(\sqrt{p})$, $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{2p})$ respectively. Then:*

⋆ *If the norm of ε_3 is equal to -1, then $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$ is a F.S.U. of k_0.*

⋆ *If the norm of ε_3 is equal to 1, then $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$ is a F.S.U. of k_0.*

In both cases a F.S.U. of k_0 is a F.S.U. of $\mathbf{k}^*$.

Proof. Since the unit index Q_{k_0} of $k_0 = \mathbf{Q}(\sqrt{2}, \sqrt{p})$ is equal to 2 and the norms of the two fundamental units ε_1 and ε_2 are equal to -1, then by Kuroda (cf. [7]) we know that $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3},\, \varepsilon_2,\, \varepsilon_3\}$ or $\{\varepsilon_1,\, \varepsilon_2,\, \sqrt{\varepsilon_3}\}$ is a F.S.U. of k_0 whether the norm of ε_3 is equal to -1 or 1. If the norm of ε_3 is equal to -1, then by [1], $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3},\, \varepsilon_2,\, \varepsilon_3\}$ is F.S.U. of $\mathbf{k}^*$ if and only if there are no integers $\alpha,\, \beta,\, \gamma \in \{0,\, 1\}$, such that $(2+\sqrt{2})\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}^{\alpha}\varepsilon_2^{\beta}\varepsilon_3^{\gamma}$ is a square in k_0. We suppose that $(2+\sqrt{2})\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}^{\alpha}\varepsilon_2^{\beta}\varepsilon_3^{\gamma} = X^2$ with $X \in k_0$ and the preceding conditions. Let ϱ the $\mathbf{Q}$-automorphism defined by $\sqrt{2} \longmapsto -\sqrt{2}$ and $\sqrt{p} \longmapsto \sqrt{p}$, then $(X\varrho(X))^2 = 2\varepsilon_1^{\alpha}(-1)^{\beta}(-1)^{\gamma} = 2\varepsilon_1^{\alpha}(-1)^{\beta+\gamma} = \pm 2\varepsilon_1^{\alpha}$. It follows that 2 is a square in $\mathbf{Q}(\sqrt{p})$ or $2\varepsilon_1$ is a square in $\mathbf{Q}(\sqrt{p})$. And this is not the case. If the norm of ε_3 is equal to 1, we take again the same demonstration and we find contradictions. □

Theorem 1. *Let p be a prime such that $p \equiv 1$ mod 8 and $\mathbf{k}^*$ be the genus field of $\mathbf{k} = \mathbf{Q}(\sqrt{2p},\, i)$. Then the number of 2-ideal classes that capitulate from $\mathbf{k}$ to $\mathbf{k}^*$ is equal to 2 or 4 if the norm of ε_3 is equal to -1 or 1 respectively.*

Proof. Let ε_1, ε_2 and ε_3 be the fundamental units of $\mathbf{Q}(\sqrt{p})$, $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{2p})$ respectively. Let $\mathcal{N}_{\mathbf{k}^*/\mathbf{k}}$ denote the norm of $\mathbf{k}^*/\mathbf{k}$. If the norm of ε_3 is equal to -1, we have that $E_{\mathbf{k}^*}$ is generated by $\{\zeta_8,\, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3},\, \varepsilon_2,\, \varepsilon_3\}$ (ζ_8 is the 8-th root of the unit), and we have also that $E_{\mathbf{k}}$ and $\mathcal{N}_{\mathbf{k}^*/\mathbf{k}}(E_{\mathbf{k}^*})$ are generated by $\{i,\, \varepsilon_3\}$. We conclude that two 2-ideal classes that capitulate from $\mathbf{k}$ to $\mathbf{k}^*$. In the same way, if the norm of ε_3 is equal to 1, we find that $E_{\mathbf{k}^*}$, $E_{\mathbf{k}}$ and $\mathcal{N}_{\mathbf{k}^*/\mathbf{k}}(E_{\mathbf{k}^*})$ are generated by $\{\zeta_8,\, \varepsilon_1,\, \varepsilon_2,\, \sqrt{\varepsilon_3}\}$, $\{i,\, \sqrt{i\varepsilon_3}\}$ and $\{i,\, \varepsilon_3\}$ respectively. Then $[E_{\mathbf{k}} : \mathcal{N}_{\mathbf{k}^*/\mathbf{k}}(E_{\mathbf{k}^*})] = 2$ and we have four 2-ideal classes that capitulate from $\mathbf{k}$ to $\mathbf{k}^*$. □

3. The 2-Ideal Classes of k which Capitulate in k*

Let L/K be a quadratic extension of number fields with the Galois group generated by σ. An ideal $\mathfrak{a}$ of L is called ambiguous (with respect to K) if it is fixed by σ: $\sigma(\mathfrak{a}) = \mathfrak{a}$. An ideal class $[\mathfrak{a}]$ of L is called ambiguous (with respect to K) if it is fixed by σ: $\sigma([\mathfrak{a}]) = [\mathfrak{a}]$. The group of ambiguous ideal classes is denoted by $\mathrm{Am}\,(L/K)$. An ideal class $[\mathfrak{a}]$ of L is called strongly ambiguous (with respect to K) if it contains an ambiguous ideal. The group of strongly ambiguous ideal classes is denoted by $\mathrm{Am}_s(L/K)$. Clearly, $\mathrm{Am}_s(L/K) \subseteq \mathrm{Am}\,(L/K)$. Moreover; if the number of classes of K is equal to 1, then by the ambiguous class number formula we have $|\mathrm{Am}\,(L/K)| = 2^r$ and rank $Cl_{L,2} = r$.

Lemma 3. *Let p be a prime such that $p \equiv 1 \bmod 8$ and $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$. Then $|\mathrm{Am}\,(\mathbf{k}/\mathbf{Q}(i))| = 4$. Moreover, $|\mathrm{Am}_s(\mathbf{k}/\mathbf{Q}(i))| = 2$ or $|\mathrm{Am}_s(\mathbf{k}/\mathbf{Q}(i))| = 4$ if the norm of the fundamental unit of $\mathbf{Q}(\sqrt{2p})$ is equal to -1 or 1 respectively.*

Proof. Since we have 3 primes of $\mathbf{Q}(i)$ which ramify in $\mathbf{k}$, then by the ambiguous class number formula we have rank $Cl_{\mathbf{k},2} = 2 - e$ such that 2^e is the number of the units of $\mathbf{k}$ consisting of norms from $\mathbf{k}$ to $\mathbf{Q}(i)$ of elements of $\mathbf{k}^\times$ (multiplicative group of $\mathbf{k}$). Since $p \equiv 1 \bmod 8$, then from Lemma 1 of [8] we have $e = 0$, which proves that $|\mathrm{Am}\,(\mathbf{k}/\mathbf{Q}(i))| = 4$. It is easy to see that every unit of $\mathbf{Q}(i)$ which is a relative norm of a number of $\mathbf{k}$ is a relative norm of a unit of $\mathbf{k}$ if and only if the norm of the fundamental unit of $\mathbf{Q}(\sqrt{2p})$ is equal to 1. The the ambiguous class number formula:$|\mathrm{Am}\,(\mathbf{k}/\mathbf{Q}(i))| = |\mathrm{Am}_s(\mathbf{k}/\mathbf{Q}(i))|[E_{\mathbf{k}} \cap \mathcal{N}_{\mathbf{k}/\mathbf{Q}(i)}(\mathbf{k}^\times) : \mathcal{N}_{\mathbf{k}/\mathbf{Q}(i)}(E_{\mathbf{k}})]$ and this remark finishes the proof. □

Proposition 1. *Let p be a prime such that $p \equiv 1 \bmod 8$ and $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$. Let $\mathcal{H}$ be the prime ideal of $\mathbf{k}$ above $1 + i$. Then the class of $\mathcal{H}$ in $\mathbf{k}$ is of order 2. Moreover, $\mathcal{H}$ capitulates in the genus field of $\mathbf{k}$.*

Proof. Suppose by contradiction that $\mathcal{H} = (\alpha)$ for certain α in $\mathbf{k}$ which is equivalent to $\mathcal{H}^2 = (\alpha^2) = (1 + i)$. Consequently, there exists a unit ε of $\mathbf{k}$ such that $(1 + i)\varepsilon = \alpha^2$ and $\varepsilon = i^m \sqrt{i^r \varepsilon_3^s}^n$, where ε_3 is the fundamental unit of $\mathbf{Q}(\sqrt{2p})$ and $r = s = 1$ (resp. $r = 0$ and $s = 2$) if the norm of ε_3 is equal to 1 (resp. -1). If we take the norm of α in $\mathbf{k}/\mathbf{Q}(\sqrt{-2p})$, we obtain that 2 is square in $\mathbf{Q}(\sqrt{-2p})$, which leads to a contradiction. Let $\varepsilon_2 = 1 + \sqrt{2}$ the fundamental unit of $\mathbf{Q}(\sqrt{2})$ and $\mathbf{k}^* = \mathbf{Q}(\sqrt{2}, \sqrt{p}, i)$ be the genus field of $\mathbf{k}$. It is easy to prove that $(1 + i)\varepsilon_2 = (\frac{\varepsilon_2}{\sqrt{2}} + \frac{i}{\sqrt{2}})^2$, thus $(1 + i)\varepsilon_2$ is square in $\mathbf{k}^*$ and $((1 + i)\varepsilon_2) = \mathcal{H}^2 = ((\frac{\varepsilon_2}{\sqrt{2}} + \frac{i}{\sqrt{2}})^2)$, then $\mathcal{H} = (\frac{\varepsilon_2}{\sqrt{2}} + \frac{i}{\sqrt{2}})$ in $\mathbf{k}^*$. This means that $\mathcal{H}$ capitulates in $\mathbf{k}^*$. □

Proposition 2. *Let p be a prime such that $p = a^2 + 16b^2 \equiv 1 \bmod 8$ and $\mathbf{k} = \mathbf{Q}(\sqrt{2p}, i)$. Let $\mathcal{P}$ a prime ideal of $\mathbf{k}$ above $a + 4bi$. Then the class of $\mathcal{P}$ in $\mathbf{k}$ is of order 2. Moreover, $\mathcal{P}$ capitulates in the genus field of $\mathbf{k}$.*

Proof. Suppose by contradiction that $\mathcal{H} = (\alpha)$ for certain α in $\mathbf{k}$, which is equivalent to $\mathcal{P}^2 = (\alpha^2) = (a + 4bi)$. Consequently, there exists a unit ε of $\mathbf{k}$ such that $(\pi)\varepsilon = \alpha^2$ and $\varepsilon = i^m \sqrt{i^r \varepsilon_3^s}^n$, where ε_3 is the fundamental unit of $\mathbf{Q}(\sqrt{2p})$ and $r = s = 1$ (resp. $r = 0$ and $s = 2$) if the norm of ε_3 is equal to 1 (resp. -1). If we take the norm of α in $\mathbf{k}/\mathbf{Q}(\sqrt{-2p})$, we obtain that p is square in $\mathbf{Q}(\sqrt{-2p})$, which leads to a contradiction. Let $\varepsilon_1 = \frac{1}{2^j}(x + y\sqrt{p})$ the fundamental unit of $\mathbf{Q}(\sqrt{2p})$, where $j = 0$ or 1. Since the norm of ε_1 is equal to -1, then $(x - 2^j i)(x + 2^j i) = py^2$. By the uniqueness of the decomposition in $\mathbf{Q}(i)$, we obtain

$$x - 2^j i = \pi y_1^2, \quad x + 2^j i = \pi' y_2^2,$$

with $\pi = a + 4bi$ or $i(a + 4bi)$, where π' is the complex conjugate of π and $y = y_1 y_2$. These conditions imply that there exists an integer $k = 0$ or 1 such that $(a + 4bi)\varepsilon_1 = (\frac{1}{2}(y_1\sqrt{2}^{1-j} i^k(a + 4bi) + y_2\sqrt{2}^{1-j}(-i)^k\sqrt{p}))^2$, hence $\mathcal{P}^2$ is generated by a square in $\mathbf{Q}(\sqrt{2},\, \sqrt{p},\, i)$. This proves that $\mathcal{P}$ capitulates in the genus field of $\mathbf{k}$. □

Theorem 2. *Let p be a prime such that $p \equiv 1$ mod 8 and $\mathbf{k} = \mathbf{Q}(\sqrt{2p},\, i)$. Then the strongly ambiguous ideal classes capitulate in the genus field of $\mathbf{k}$.*

Proof. If the norm of ε_3 is equal to -1, then by Lemma 3 the group of strongly ambiguous ideal classes is generated by the classes of $\mathcal{H}$, and we have that $\mathcal{H}$ capitulates in the genus field of $\mathbf{k}$. If the norm of ε_3 is equal to 1, Lemma 3 implies that the group of strongly ambiguous ideal classes is generated by the classes of $\mathcal{H}$ and $\mathcal{P}$. Using the same proof of Proposition 1, we show that the class of $\mathcal{PH}$ in $\mathbf{k}$ is of order 2. By Proposition 1 and Proposition 2 our theorem is established. □

Theorem 3. *Let p be a prime such that $p \equiv 1$ mod 8 and $\mathbf{k} = \mathbf{Q}(\sqrt{2p},\, i)$. Let $\mathbf{k}_2^{(1)}$ be the Hilbert 2-class field of $\mathbf{k}$ and $\mathbf{k}_2^{(2)}$ that of $\mathbf{k}_2^{(1)}$. Then $\mathbf{k}_2^{(1)} \neq \mathbf{k}_2^{(2)}$ if and only if $p = x^2 + 32y^2$.*

Proof. We use the same notions of the previous section. From [12] it follows that

$$h(\mathbf{k}^*) = \frac{q(\mathbf{k}^*/\mathbf{Q})}{2^5} h(2)h(p)h(-1)h(-2)h(-p)h(2p)h(-2p).$$

Since $h(2) = h(p) = h(-1) = h(-2) = 1$, and $h(\mathbf{k}) = \frac{h(2p)h(-2p)}{2O_{\mathbf{k}}}$, then $h(\mathbf{k}^*) = \frac{q(\mathbf{k}^*/\mathbf{Q})h(-p)h(\mathbf{k})}{2^4 O_{\mathbf{k}}}$, where $q(\mathbf{k}^*/\mathbf{Q}) = [E_{\mathbf{k}^*} :< i,\, \epsilon_1,\, \epsilon_2,\, \epsilon_3 >]$, with ε_1, ε_2 and ε_3 the fundamental units of $\mathbf{Q}(\sqrt{p})$, $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{2p})$ respectively. In both cases of the norm of ϵ_3, it is easy to see that $q(\mathbf{k}^*/\mathbf{Q}) = 4$. Since $\mathbf{k}^*$ be an unramified extension of $\mathbf{k}$ and rank $Cl_{\mathbf{k},2} = 2$, then according to [4] we have $\mathbf{k}_2^{(1)} = \mathbf{k}_2^{(2)} \Leftrightarrow h(\mathbf{k}^*) = h(\mathbf{k})/2 \Leftrightarrow h(-p) = 2Q_{\mathbf{k}}$ which is equivalent to ($h(-p) = 4$ and $Q_{\mathbf{k}} = 2$) or ($h(-p) = 2$ and $Q_{\mathbf{k}} = 1$). However, if $h(-p) = 4$, thus $h(2p) = 2$ (cf. [6]), then the norm of ϵ_3 is equal to 1 (cf. [10]). Hence, $O_{\mathbf{k}} = 2$ (cf. [2]). In addition, P. Barruccand and H. Cohn showed in [3] that $h(-p) \equiv 0$ mod 4 if and only if $p \equiv 1$ mod 8. More precisely $h(-p) = 4$ if and only if $p \neq x^2 + 32y^2$. In other words $\mathbf{k}_2^{(1)} \neq \mathbf{k}_2^{(2)}$ if and only if $p = x^2 + 32y^2$. □

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