International Journal of Pure and Applied Mathematics

Volume 35 No. 4 2007, 485-490

CONTROLLABILITY SUBSPACES OF SINGULAR SYSTEMS

Mª Isabel García-Planas

Departament de Matemàtica Aplicada, I
Universitat Politècnica de Catalunya
C. Minería 1, Esc C, 1º-3ª
Barcelona, 08038, SPAIN
e-mail: maria.isabel.garcia@upc.edu

Abstract: In this paper we construct generalized invariant subspaces of controllability for triples of matrices representing singular systems. These subspaces permit us to divide the system in two independent subsystems a controllable system and a non-controllable one.

AMS Subject Classification: 15A21, 93B52
Key Words: singular systems, feedback, controllability, invariant subspaces

1. Introduction

Invariant subspaces for transformations from $\mathbb{C}^{m+n}$ into $\mathbb{C}^n$ was introduced by Gohberg, Lancaster, Rodman [3], as a generalization of similarity called block-similarity.

Our objective is to develop a generalization of the notion of invariant subspace that will apply to such transformations and to reduce to the notion when we have a map defined over a subspace.

Remember that a subspace $G \in \mathbb{C}^n$, is invariant under (A, B) as a map from $\mathbb{C}^{n+m}$ into $\mathbb{C}^n$ if and only if

$$AG \subset G + \operatorname{Im} B. \tag{1}$$

In this paper, we consider triples of matrices (E, A, B), $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, representing singular systems $E\dot{x} = Ax + Bu$, that we can see as a pair of maps (E, B), (A, B) defined modulo a subspace (see García-Planas [2]), classified under the following equivalence relation: two triples

Received: November 17, 2006

(E,A,B),(E',A',B) are equivalent if and only if the following equality holds:

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & & \\ & P & \\ F_E & F_A & B \end{pmatrix}, \tag{2}$$

where $Q, P \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, $F_E, F_A \in M_{m\times n}(\mathbb{C})$.

The main result is the decomposition of a regularizable singular system in a two independent systems, a controllable system $E_1\dot{x}_1 = A_1x_1 + B_1$ and a singular system $E_2\dot{x}_2 = A_2x_2$. Remember that a system is called regular if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha,\beta) \in \mathbb{C}^2$ and that guarantees the existence and uniqueness of classical solutions, and a system is called regularizable if there exist feedbacks F_E and F_A such that the system $(E + BF_E, A + BF_A, B)$ is regular. Regularizable condition is equivalent to rank $\begin{pmatrix} E & A & B \end{pmatrix} = n$.

2. Invariant (E,A,B)-Subspaces

In this section we try to generalize definition of invariant subspace under (A,B)-map, to the case of triples of matrices.

Let (E,A,B) be a standardizable triple in M, so there exists feedback F_E such that $E + BF_E$ is invertible, and it permit us to obtain the following standard system $((E + BF_E)^{-1}A, (E + BF_E)^{-1}B)$.

Applying definition given in the introduction, a subspace $G \in \mathbb{C}^n$ is invariant under $((E + BF_E)^{-1}A, (E + BF_E)^{-1}B)$ if and only if

$$(E + BF_E)^{-1}AG \subset G + \operatorname{Im}(E + BF_E)^{-1}B$$

and we can deduce the following proposition.

Proposition 1. *Let G be a vector subspace of $\mathbb{C}^n$ and (E,A,B) a standardizable triple of matrices. Then the followings are equivalent:*

a) *For any feedback $F_E \in M_{m\times n}(\mathbb{C})$ standardizing the triple,*

$$(E + BF_E)^{-1}AG \subset G + \operatorname{Im}(E + BF_E)^{-1}B\,.$$

b) *For any feedback $F_E \in M_{m\times n}(\mathbb{C})$,*

$$AG \subset (E + BF_E)G + \operatorname{Im} B\,.$$

c) $AG \subset (E + BF_E)G + \operatorname{Im} B$.

d) $AG \subset EG + \operatorname{Im} B$.

Proof. a)⇒b). Let v be a vector in G, then $Av \in AG$. Applying the condition a), we have $(E+BF_E)^{-1}Av \in G + \operatorname{Im}(E+BF_E)^{-1}B$, that is to say $(E+BF_E)^{-1}Av = u + (E+BF_E)^{-1}Bw$ and $Av = (E+BF_E)u + Bw \in (E+BF_E)G + \operatorname{Im} B$

b)⇒a). Let x be a vector in $(E+BF_E)^{-1}AG$, then there exists $u \in G$ such that $x = (E+BF_E)^{-1}AGu$, so $(E+BF_E)x = AGu$. Taking into account that $AGu \in (E+BF_E)G + \operatorname{Im} B$, there exists $v \in G$, $w \in \mathbb{C}^m$ such that $AGu = (E+BF_E)v + Bw$ then $(E+BF_E)x = (E+BF_E)v + Bw$ and $x = v + (E + BF_E^{-1}Bw \in G + \operatorname{Im}(E+BF_E)^{-1}B$

d)⇒b). Let x be a vector in AG, then $x = EGu + Bw$ so $x = EGu + BF_Eu - BF_Eu + Bw = (EG + BF_E)u + B(-F_Eu + w)$.

b)⇒d). Let x be in AG, $x = AGu \in (E + BF_EG + \operatorname{Im} B$, $AGu = (E + BF_E)v + Bw$, $v \in G$ $w \in \mathbb{C}^m$, $AGu = Ev + BF_Ev + Bw = Ev + B(F_Ev + w) \in EG + \operatorname{Im} B$.

d)⇒c). Let $x \in (A+BF_A)G$, then $x = Au + BF_Au = Ev + Bw + BF_Au$, $Ev + BF_Ev - BF_Ev + Bw + BF_Au$, $(E+BF_E)v + B(-F_Ev + w + F_Au)$.

c)⇒d). Let x be a vector in AG, $x = Au$ so $Au = Au + F_Au - F_Au = (A+BF_A)u$. Taking into account that $(A+BF_A)u \in (A+BF_E)G$, we have $(A+BF_A)u = (E+BF_E)v$. So, $Au = (E+BF_Ev - F_Au = Ev + B(F_Ev - F_Au) \in EG + \operatorname{Im} B$. □

This proposition permits us to generalize the definition of invariant subspace to the triples of matrices.

Definition 1. *A subspace $G \in \mathbb{C}^n$ is invariant under (E, A, B) if and only if*

$$AG \subset EG + \operatorname{Im} B\,. \tag{3}$$

We observe that, if $E = I_n$, this definition coincides with definition of (A, B)-invariant subspace.

Proposition 2. *Let (E, A, B) be a triple of matrices. A subspace $G \subset \mathbb{C}^n$ is invariant under (E, A, B) if and only if is invariant under $(E+BF_E, A+BF_A, B)$ for all feedbacks F_E, $F_A \in M_{m\times n}(\mathbb{C})$.*

Proof. Suppose that $AG \subset EG + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $Ax = Ey + Bw$, so for any F_E, $F_A \in M_{m\times n}(\mathbb{C})$, we have

$$\begin{gathered} Ax + BF_Ax - BF_Ax = Ey + BF_Ey - BF_Ey + Bw\,, \\ (A+BF_A)x = (E+BF_E)y + B(F_Ax - F_Ey + w). \end{gathered}$$

Consequently, for all $x \in G$, $(A + BF_A)G \subset (E + BF_E)G + \operatorname{Im} B$.

Reciprocally. Suppose that $(A + BF_A)G \subset (E + BF_E)G + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $(A + BF_A)x = (E + BF_E)y + Bw$, so $Ax = Ey - BF_Ax + BF_Ey + Bw$ and $Ax = Ey + B(-F_Ax + F_Ey + w)$. Then, for all $x \in G$ we have $AG \subset EG + \operatorname{Im} B$. □

Proposition 3. *Let* (E_1, A_1, B_1), $(E_2, A_2, B_2) = (QE_1P+QB_1F_E, QA_1P+QB_1F_A, QB_1R)$ *be two equivalent triples. Then* $G \subset \mathbb{C}^n$ *is an invariant subspace under* (E_1, A_1, B_1) *if and only if* $P^{-1}G$ *is invariant under* (E_2, A_2, B_2).

Proof. Suppose that $A_1G \subset E_1G + \operatorname{Im} B$. Then

$$\begin{aligned} A_2P^{-1}G &= (QA_1P + QB_1F_{A_1})P^{-1}G = Q(A_1G + B_1F_{A_1}P^{-1}G) \\ Q(E_1G + \operatorname{Im} B_1) &= Q((Q^{-1}E_2P^{-1} - Q^{-1}B_2R^{-1}F_EP^{-1})G + \operatorname{Im} Q^{-1}B_2R^{-1}) \\ &Q(Q^{-1}(E_2P^{-1} - B_2R^{-1}F_EP^{-1})G + Q^{-1}\operatorname{Im} B_2R^{-1}) \\ &= QQ^{-1}((E_2P^{-1} - B_2R^{-1}F_EP^{-1})G + \operatorname{Im} B_2R^{-1}) \\ &\subset (E_2 - B_2R^{-1}F_E)P^{-1}G + \operatorname{Im} B_2\,. \end{aligned}$$

Now, it suffices to apply Proposition 2. □

Proposition 4. *Let* (I_n, A, B) *be a standard triple. Then*

$$G = [B, AB, \dots, A^{n-1}B]$$

is a (I_n, A, B)*-invariant subespace.*

Proof.

$$AG = A[B, AB, \dots, A^{n-1}B] = [AB, A^2B, \dots, A^nB]$$

Cayley-Hamilton Theorem insures that $A^n = \sum_{i=0}^{n-1} \lambda^i A^i$, then $A^nB = \sum_{i=0}^{n-1} \lambda^i A^i B$. Consequently

$$AG \subset G \subset +\operatorname{Im} B.$$ □

Theorem 1. *Let*

$$C_r = \begin{pmatrix} E & & & & B & & & \\ A & E & & & & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & E & & & & B \\ & & & A & & & & & B \end{pmatrix} \in M_{nr\times(n(r-1)+mr)}(\mathbb{C})$$

be the r*-controllability matrix. Suppose* r *being the least such that* $\operatorname{rank} C_r < (n(r-1) + mr)$, *and let* $\begin{pmatrix} v_1 & \dots & v_r & w_1 & \dots & w_{r+1} \end{pmatrix} \in \operatorname{Ker} C_r$ $(v_i$ *are* *vectors in* $\mathbb{C}^n$ *and* w_i *vectors in* $\mathbb{C}^m$*). Then* $G = [v_1, \dots v_r]$ *is a* (E, A, B)*-invariant subspace.*

Proof. We consider $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_{r-1} v_{r-1} + \lambda_r v_r$. Then:

$$\begin{aligned} Av &= \lambda_1 Av_1 + \lambda_2 Av_2 + \dots + \lambda_{r-1} Av_{r-1} + \lambda_r Av_r = \lambda_1(-Ev_2 - Bw_2) \\ &+ \lambda_2(-Ev_3 - Bw_3) + \dots + \lambda_{r-1}(-Ev_r - Bw_r - \lambda_r Bw_{r+1} = E(\lambda_1 v_2 - \lambda_2 v_3 \\ &- \dots - \lambda_{r-1} v_r) + B(-\lambda_1 w_2 - \lambda_2 w_3 - \dots - \lambda_{r-1} w_r - \lambda_r w_{r+1}) \in EG + \operatorname{Im} B. \end{aligned}$$

Corollary 2. *Let* (E, A, B) *be a triple with* $E = I_n$*. In this case the invariant subspace* G *obtained in the above theorem, coincides with the controllability* (A, B)*-invariant subspaces* $[B, AB, \dots A^{r-1}B]$*.*

Proof. Making block-row elemental transformations to the matrix C_r we obtain the equivalent matrix

$$\begin{pmatrix} I_n & & & & B & & & & \\ 0 & I_n & & & -AB & B & & & \\ & \ddots & \ddots & & & & \ddots & & \\ & & & I_n & (-1)^{r-2}A^{r-2}B & & -AB & B & \\ & & & 0 & (-1)^{r-1}A^{r-1}B & & & -AB & B \end{pmatrix}. \qquad \square$$

Theorem 2. *Suppose* (E, A, B) *be a triple with* $\operatorname{rank}\begin{pmatrix} E & A & B \end{pmatrix} = n$*. Then the triple* (E, A, B) *is equivalent under equivalence relation considered, to* $(E_1, A_1, B_1$ *with* $E_1 = \begin{pmatrix} I_{n_1} & & \\ & I_{n_2} & \\ & & N_{n_3} \end{pmatrix}$, $A_1 = \begin{pmatrix} N_{n_1} & & \\ & J_{n_2} & \\ & & I_{n_3} \end{pmatrix}$, $B_1 = \begin{pmatrix} B_{n_1} \\ 0 \end{pmatrix}$, *and* $(I_{n_1}, N_{n_1}, B_{n_1})$ *is controllable in its Kronecker canonical reduced form,* J_{n_2} *a Jordan matrix and* N_{n_i} *a nilpotent matrix in its reduced form.*

Proof. Let G be a maximal controllable invariant subspace. Taking a basis in G the triple can be reduced to:

$$\left(\begin{pmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} \\ 0 \end{pmatrix} \right)$$

with (E_{11}, A_{11}, B_{11}) a controllable system.

So taking $Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & I_{n-n_1} \end{pmatrix}$, $P = \begin{pmatrix} P_{11} & 0 \\ 0 & I_{n-n_1} \end{pmatrix}$, $F_E = \begin{pmatrix} F_{E_{11}} & 0 \end{pmatrix}$, $F_A = \begin{pmatrix} F_{A_{11}} & 0 \end{pmatrix}$ and R such that $Q_{11} \begin{pmatrix} E_{11} & A_{11} & B_{11} \end{pmatrix} \begin{pmatrix} P_{11} & & \\ & P_{11} & \\ F_{E_{11}} & F_{A_{11}} & R \end{pmatrix}$ is in its Kronecker reduced form $(I_{n_1}, N_{n_1}, B_{n_1})$, the triple is reduced to

$$\left(\begin{pmatrix} I_{n_1} & E_{12} \\ 0 & E_{22} \end{pmatrix}, \begin{pmatrix} N_{n_1} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} B_{n_1} \\ 0 \end{pmatrix} \right).$$

Now, we consider $Q = \begin{pmatrix} I_{n_1} & \\ & Q_{22} \end{pmatrix}$ and $P = \begin{pmatrix} I_{n_1} & \\ & P_{22} \end{pmatrix}$ such that $Q \begin{pmatrix} E_{22} & A_{22} \end{pmatrix} \begin{pmatrix} P_{22} & \\ & P_{22} \end{pmatrix}$ is in its canonical form as a pencil we have that the triple is reduced to

$$\left(\begin{pmatrix} I_{n_1} & X & Y \\ & I_{n_2} & \\ & & N_{n_3} \end{pmatrix}, \begin{pmatrix} N_{n_1} & Z & T \\ & J_{n_2} & \\ & & I_{n_3} \end{pmatrix}, \begin{pmatrix} B_{n_1} \\ 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} I_{n_1} & \overline{X} \\ & \overline{E}_2 \end{pmatrix}, \begin{pmatrix} N_{n_1} & \overline{Y} \\ & \overline{A}_2 \end{pmatrix}, \begin{pmatrix} B_{n_1} \\ 0 \end{pmatrix} \right).$$

Finally, observing that the system

$$\begin{pmatrix} I_{n_2} & Q_2 \\ 0 & I_{n-n_1} \end{pmatrix} \begin{pmatrix} I_n & \overline{X} & N_{n_1} & \overline{Y} & B_{11} \\ 0 & \overline{E}_2 & 0 & \overline{A}_2 & 0 \end{pmatrix} \begin{pmatrix} I_{n_1} & P_2 & & & \\ 0 & I_{n-n_1} & & & \\ & & I_{n_1} & P_2 & \\ & & 0 & I_{n-n_1} & \\ F_1 & F_2 & F_3 & F_4 & R \end{pmatrix} = \begin{pmatrix} I_n & 0 & N_{n_1} & 0 & B_{11} \\ 0 & \overline{E}_2 & 0 & \overline{A}_2 & 0 \end{pmatrix}$$

has a solution we can conclude that this triple is equivalent to

$$\left(\begin{pmatrix} I_{n_1} & & \\ & I_{n_2} & \\ & & N_{n_3} \end{pmatrix}, \begin{pmatrix} N_{n_1} & & \\ & J_{n_2} & \\ & & I_{n_3} \end{pmatrix}, \begin{pmatrix} B_{n_1} \\ 0 \end{pmatrix} \right). \qquad \square$$

References

[1] Mª I. García-Planas, Controllability indices for multi-input singular systems, *ICM* (2006).

[2] Mª I. García-Planas, Similarity of pairs of linear maps defined modulo a subspace, *Linear and Multilinear Algebra*, **54**, No. 5 (2006).

[3] I. Gohberg, P. Lancaster, L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley (1986).

[4] P. Lancaster, M. Tismenetsky, *The Theory of Matrices*, Academic Press, San Diego (1985).