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EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR WIENER PROCESSES WITH RANDOM INITIAL STATES

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Abstract: Wiener processes with initial states having a generalized Pareto distribution are considered in one and two dimensions. Exact formulae are given for the probability densities of these processes, and it is shown that when their variance parameters are small, a more generalized Pareto distribution is a very good approximation to the exact distribution.

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1. Introduction

Let $X(t)$ be a Wiener process with infinitesimal parameters μ and σ^2 (>0). Assume that $X(0)$ is a random variable having a generalized Pareto distribution (GPD) (see Pickands [3]); namely,

$$f_{X(0)}(x_0) = \frac{\alpha}{\beta}\left(1+\frac{x_0}{\beta}\right)^{-(\alpha+1)} \quad \text{for } x_0 \geq 0, \tag{1}$$

where α and β are positive parameters. In Section 2, we will give the exact formula for the probability density function (p.d.f.) of $X(t)$. Then, we will show that the (more) generalized Pareto distribution

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$$f_{X(t)}(x) = \frac{\alpha}{\beta}\left(1 - \frac{\mu t}{\beta} + \frac{x}{\beta}\right)^{-(\alpha+1)} \quad \text{for } x \geq \mu t, \tag{2}$$

which is the exact p.d.f. of $X(t)$ when σ decreases to 0, is a very good approximation to the exact function $f_{X(t)}$ when σ is relatively small.

Next, let $(X(t), Y(t))$ be a two-dimensional Wiener process defined by the stochastic differential equations (see, for example, Øksendal [2])

$$dX(t) = \mu_1\, dt + \sigma_1 dW_1(t), \tag{3}$$
$$dY(t) = \mu_2\, dt + \sigma_2 dW_2(t), \tag{4}$$

where we assume that the correlation coefficient between the standard Brownian motions $W_1(t)$ and $W_2(t)$ is equal to -1. In Section 3, we will compute in a particular instance the exact joint p.d.f. of $(X(t), Y(t))$ when $(X(0), Y(0))$ is a random vector having a GPD:

$$f_{X(0),Y(0)}(x_0, y_0) = \frac{\alpha(\alpha-1)}{\beta^2}\left(1 + \frac{(x_0+y_0)}{\beta}\right)^{-(\alpha+1)} \quad \text{for } x_0, y_0 \geq 0. \tag{5}$$

As in the one-dimensional case above, we will see that a GPD constitutes a very good approximation to the exact distribution of $(X(t), Y(t))$ for small variance parameters.

2. The One-Dimensional Case

As is well known (see Cox and Miller [1], for instance), conditional on $X(0) = x_0$, the Wiener process with infinitesimal parameters μ and σ^2 has a Gaussian distribution:

$$X(t)|\{X(0) = x_0\} \sim \mathrm{N}(x_0 + \mu t, \sigma^2 t).$$

Hence, conditional on $X(0)$, we can state the following lemma.

Lemma 2.1. *When $X(0)$ has a generalized Pareto distribution, defined in (1), the p.d.f. of $X(t)$ can be written as*

$$f_{X(t)}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi t}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2 t}(x - x_0 - \mu t)^2\right\} \times \frac{\alpha}{\beta}\left(1 + \frac{x_0}{\beta}\right)^{-(\alpha+1)} dx_0. \tag{6}$$

Corollary 2.1. *When the variance parameter σ decreases to 0, the p.d.f. of $X(t)$ tends to the density function defined in (2).*

Proof. The result follows at once from the fact that

$$\lim_{\sigma \downarrow 0} f_{X(t)|\{X(0)=x_0\}}(x) = \delta(x - x_0 - \mu t),$$

where $\delta(\cdot)$ is the Dirac delta function. □

Remarks. i) Notice that in the case when μ is equal to zero, the p.d.f.'s in (1) and (2) are identical.

ii) Another way of obtaining the p.d.f. in (2) is the following: the function

$$f(x;t) \equiv f_{X(t)}(x)$$

satisfies the Kolmogorov forward equation

$$\frac{1}{2}\sigma^2 f_{xx} - \mu f_x = f_t.$$

Therefore, if σ is very small, we may write that

$$-\mu f_x \simeq f_t.$$

It follows that

$$f(x,t) \simeq h\big(t - \tfrac{x}{\mu}\big)$$

(for $\mu \neq 0$), where $h(\cdot)$ is a general function. Using the fact that

$$f(x,0) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} \quad \text{for } x \geq 0,$$

we deduce that

$$f^*(x,t) := \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta} - \frac{\mu t}{\beta}\right)^{-(\alpha+1)} \quad \text{for } x \geq \mu t$$

could serve as an approximation for the exact function $f(x,t)$.

iii) Since

$$\int_0^\infty \frac{\alpha}{\beta}\left(1 - \frac{\mu t}{\beta} + \frac{x}{\beta}\right)^{-(\alpha+1)} dx = (1+t)^{-\alpha},$$

we may write that

$$f_{X(t)|\{X(t)>0\}}(x) \simeq \frac{\alpha}{\beta}(1+t)^{\alpha}\left(1 - \frac{\mu t}{\beta} + \frac{x}{\beta}\right)^{-(\alpha+1)} \quad \text{for } x \geq 0.$$

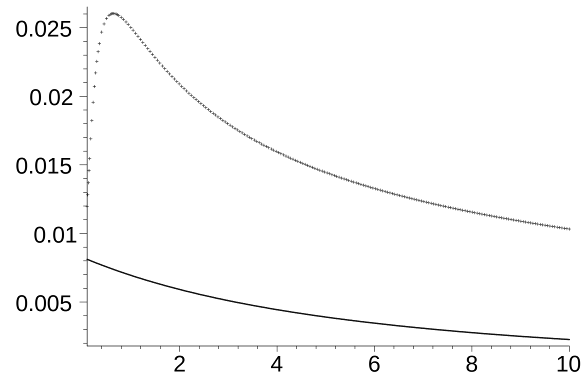

Figure 1: Comparison between the exact (see (6); dots) and approximate (see (2); line) solutions in the case when $\mu = -1$, $\alpha = 1$, $x = 10$ and $\sigma = 10$

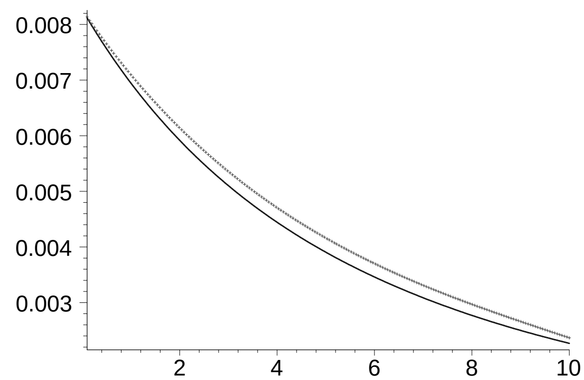

Figure 2: Comparison between the exact (see (6); dots) and approximate (see (2); line) solutions in the case when $\mu = -1$, $\alpha = 1$, $x = 10$ and $\sigma = 1$

That is, $X(t)|\{X(t) > 0\}$ has an approximate distribution (for σ very small) which is a (further) generalized Pareto distribution.

The results above are straightforward. However, what is interesting is to look at how well the GPD in (2) approximates the exact p.d.f. of $X(t)$ when σ is relatively small. To do so, we compute the exact (formula (6)) and approximate (formula (2)) functions in four particular instances (see Figures 1-4). We observe that when σ is equal to 0.1, which is not that small, the approximation (for the particular case considered) is truly excellent.

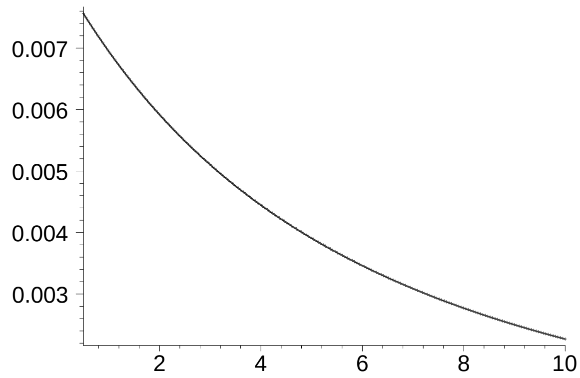

Figure 3: Comparison between the exact (see (6); dots) and approximate (see (2); line) solutions in the case when $\mu = -1$, $\alpha = 1$, $x = 10$ and $\sigma = 0.1$

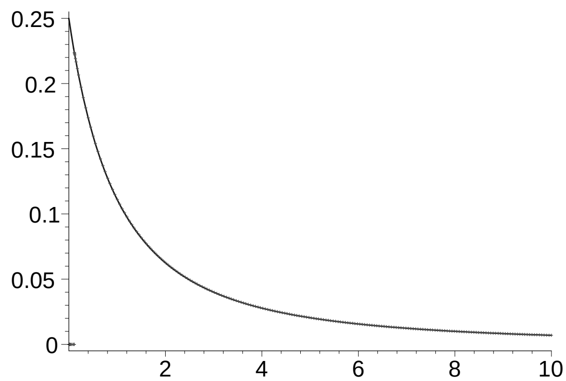

Figure 4: Comparison between the exact (see (6); dots) and approximate (see (2); line) solutions in the case when $\mu = -1$, $\alpha = 1$, $x = 1$ and $\sigma = 0.1$

3. The Two-Dimensional Case

We now consider the two-dimensional Wiener process defined by (3), (4). Because $\rho_{W_1(t),W_2(t)} = -1$, we may write that

$$W_2(t) = aW_1(t) + b,$$

where $a < 0$. It follows, $W_1(t)$ and $W_2(t)$ being *standard* Brownian motions, that

$$\mathrm{Var}[W_2(t)] = a^2 \mathrm{Var}[W_1(t)] \quad \Longrightarrow \quad t = a^2 t \quad \forall\, t > 0,$$

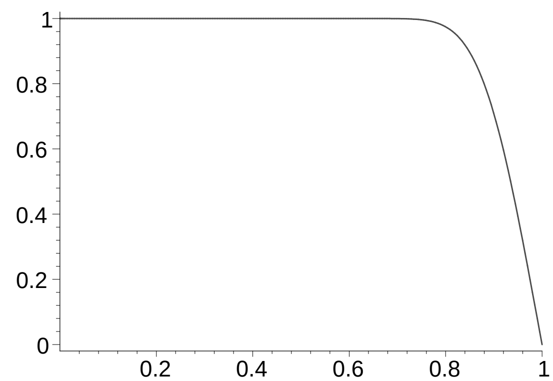

Figure 5: Calculation of the function h defined in (14) in the case when $c_1 = c_2 = -1$, $x = y = 1$ and $\sigma = 0.1$

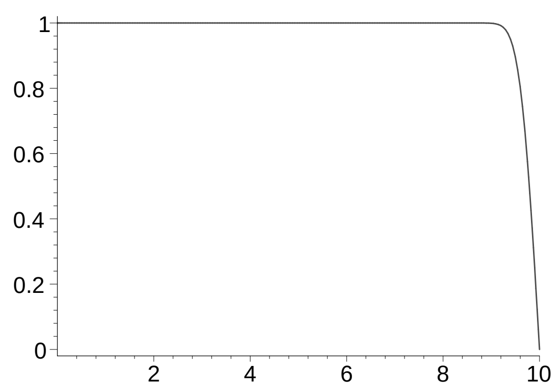

Figure 6: Calculation of the function h defined in (14) in the case when $c_1 = c_2 = -1$, $x = y = 10$ and $\sigma = 0.1$

which implies that

$$a = -1.$$

Moreover, since $W_i(0) = 0$ for $i = 1, 2$, we may also write that $b = 0$. The system (3), (4) becomes

$$\begin{aligned} dX(t) &= \mu_1 \, dt + \sigma_1 dW_1(t), \\ dY(t) &= \mu_2 \, dt - \sigma_2 dW_1(t). \end{aligned}$$

Next, because $W_2(t) = -W_1(t)$, we have:

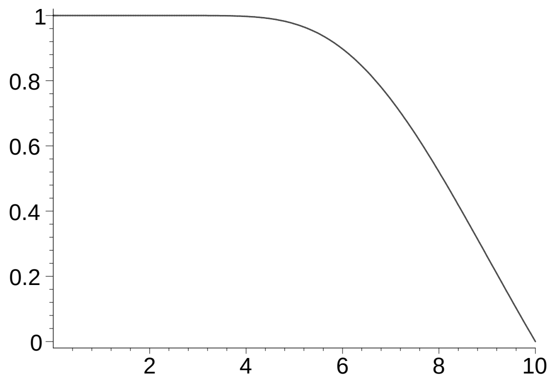

Figure 7: Calculation of the function h defined in (14) in the case when $c_1 = c_2 = -1$, $x = y = 10$ and $\sigma = 1$

$$
\begin{aligned}
&p(x, y, t; x_0, y_0) := \\
&\lim_{dx\downarrow 0}\lim_{dy\downarrow 0} \frac{P[X(t) \in (x, x+dx], Y(t) \in (y, y+dy] | X(0) = x_0, Y(0) = y_0]}{dxdy} \\
&= \lim_{dx\downarrow 0}\lim_{dy\downarrow 0} \frac{1}{dxdy} \{P[X(t) \in (x, x+dx] | X(0) = x_0, Y(0) = y_0] \\
&\quad \times P[Y(t) \in (y, y+dy] | X(t) = \in (x, x+dx], X(0) = x_0, Y(0) = y_0]\} \\
&= \frac{1}{\sqrt{2\pi t}\sigma_1} \exp\left\{-\frac{1}{2\sigma_1^2 t}(x - x_0 - \mu_1 t)^2\right\} \delta\left(y + \frac{\sigma_2}{\sigma_1}(x - x_0) - y_0 - \mu t\right), \qquad (7)
\end{aligned}
$$

where

$$\mu := \frac{\sigma_2}{\sigma_1}\mu_1 + \mu_2.$$

Indeed, we may write that

$$
\begin{aligned}
X(t) &= X(0) + \mu_1 t + \sigma_1 W_1(t), \qquad (8) \\
Y(t) &= Y(0) + \mu_2 t - \sigma_2 W_1(t), \qquad (9)
\end{aligned}
$$

so that

$$\frac{\sigma_2}{\sigma_1}X(t) + Y(t) = \frac{\sigma_2}{\sigma_1}X(0) + Y(0) + \left(\frac{\sigma_2}{\sigma_1}\mu_1 + \mu_2\right)t.$$

We can now state the following proposition.

Proposition 3.1. *The joint p.d.f. of the two-dimensional stochastic process $(X(t), Y(t))$, when $(X(0), Y(0))$ has the two-dimensional GPD defined in (5),* *is given by*

$$f(x,y,t) = \int_{\frac{\mu_2 t - y}{\sigma_2}}^{\frac{x-\mu_1 t}{\sigma_1}} \frac{\alpha(\alpha-1)}{\beta^2}\left[1+\frac{(x+y-(\mu_1+\mu_2)t+(\sigma_2-\sigma_1)w)}{\beta}\right]^{-(\alpha+1)} \times \frac{1}{\sqrt{2\pi t}} e^{-w^2/2t}\, dw. \tag{10}$$

Proof. We condition on the initial state $(X(0), Y(0))$. Making use of the function $p(x,y,t;x_0,y_0)$ given in (7), we may write that

$$\begin{aligned} f(x,y,t) &= \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi t}\sigma_1} \exp\left\{-\frac{1}{2\sigma_1^2 t}(x-x_0-\mu_1 t)^2\right\} \\ &\times \delta\left(y+\frac{\sigma_2}{\sigma_1}(x-x_0)-y_0-\mu t\right)\frac{\alpha(\alpha-1)}{\beta^2}\left[1+\frac{(x_0+y_0)}{\beta}\right]^{-(\alpha+1)} dy_0 dx_0 \\ &= \int_0^{x+\frac{\sigma_1}{\sigma_2}(y-\mu t)} \frac{1}{\sqrt{2\pi t}\sigma_1} \exp\left\{-\frac{1}{2\sigma_1^2 t}(x-x_0-\mu_1 t)^2\right\} \\ &\times \frac{\alpha(\alpha-1)}{\beta^2}\left[1+\frac{\left(x_0+y+\frac{\sigma_2}{\sigma_1}(x-x_0)-\mu t\right)}{\beta}\right]^{-(\alpha+1)} dx_0\,. \end{aligned}$$

Making the change of variable

$$w = \frac{x-x_0-\mu_1 t}{\sigma_1},$$

we obtain the formula (10). □

In the particular case when $\sigma_1 = \sigma_2$, we obtain the following important corollary.

Corollary 3.1. *When $\sigma_1 = \sigma_2$, the joint p.d.f. $f(x,y,t)$ simplifies to*

$$f(x,y,t) = \frac{\alpha(\alpha-1)}{\beta^2}\left[1+\frac{(x+y-(\mu_1+\mu_2)t)}{\beta}\right]^{-(\alpha+1)} \times \left[\Phi\left(\frac{x-\mu_1 t}{\sigma_1\sqrt{t}}\right)-\Phi\left(\frac{\mu_2 t-y}{\sigma_1\sqrt{t}}\right)\right], \tag{11}$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal random variable.

Remarks. i) The function $f(x, y, t)$ satisfies the Kolmogorov forward equation

$$\frac{1}{2}\sigma_1^2 f_{xx} + \frac{1}{2}\sigma_2^2 f_{yy} + \sigma_{12} f_{xy} - \mu_1 f_x - \mu_2 f_y = f_t, \tag{12}$$

where $\sigma_{12} = -\sigma_1\sigma_2$ because $\rho_{W_1(t),W_2(t)} = -1$. Next, let

$$g(x, y, t) = \frac{\alpha(\alpha-1)}{\beta^2}\left[1 + \frac{(x + y - (\mu_1 + \mu_2)t)}{\beta}\right]^{-(\alpha+1)}. \tag{13}$$

Since g is a function of $x + y$, we have:

$$\frac{1}{2}\sigma_1^2 g_{xx} + \frac{1}{2}\sigma_2^2 g_{yy} - \sigma_1\sigma_2 g_{xy} = \left(\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - \sigma_1\sigma_2\right) g_{xx}.$$

Hence, we find that, if $\sigma_1 = \sigma_2$, the function $g(x, y, t)$ is a solution of the Kolmogorov forward equation (12). Furthermore, notice that

$$g(x, y, 0) = \frac{\alpha(\alpha-1)}{\beta^2}\left[1 + \frac{(x + y)}{\beta}\right]^{-(\alpha+1)},$$

which is the appropriate initial condition for the joint p.d.f. of $(X(t), Y(t))$. However, looking at the formula (11), we see that $f(x, y, t)$ is *not* a function of $x + y$.

ii) Formula (10) can also be obtained by conditioning on $W_1(t)$ (which is assumed independent of $X(0)$ and $Y(0)$). Making use of (8), (9), we may write that

$$X(t)|\{W_1(t) = w\} = x \iff X(0) = x^* := x - \mu_1 t - \sigma_1 w$$

and

$$Y(t)|\{W_1(t) = w\} = y \iff Y(0) = y^* := y - \mu_2 t + \sigma_2 w.$$

Next, $W_1(t)$ being a standard Brownian motion, we have:

$$\begin{aligned} &f(x, y, t) \\ &= \lim_{dx\downarrow 0}\lim_{dy\downarrow 0}\frac{1}{dxdy}\int_{-\infty}^{\infty} P[X(t) \in (x, x + dx], Y(t) \in (y, y + dy]|W_1(t) = w] \\ &\qquad\qquad \times f_{W_1(t)}(w)\, dw \\ &= \lim_{dx\downarrow 0}\lim_{dy\downarrow 0}\frac{1}{dxdy}\int_{-\infty}^{\infty} P[X(0) \in (x^*, x^* + dx], Y(0) \in (y^*, y^* + dy]] \\ &\qquad\qquad \times \frac{1}{\sqrt{2\pi t}} e^{-w^2/2t}\, dw. \end{aligned}$$

Remembering that $X(0)$ and $Y(0)$ must be positive, we indeed retrieve the formula (10).

iii) Notice that the function g defined in (13), which is a GPD, is the *exact* solution if σ_1 $(= \sigma_2)$ decreases to zero or if $W_1(t) = 0$. That is, when $\sigma_1 = \sigma_2$,

$$\lim_{\sigma_1 \downarrow 0} f(x, y, t) = g(x, y, t)$$

and (see (8) and (9))

$$\lim_{dx \downarrow 0} \lim_{dy \downarrow 0} \frac{P[X(t) \in (x, x + dx], Y(t) \in (y, y + dy] | W_1(t) = 0]}{dxdy} = g(x, y, t).$$

Thus, every time the standard Brownian motion crosses the origin, the random vector $(X(t), Y(t))$ has a GPD. Also, if $\mu_1 + \mu_2 = 0$, this GPD is the same as the initial distribution.

iv) As in the preceding section, it is interesting to see how well the function $g(x, y, t)$ approximates the exact function $f(x, y, t)$ when σ_1 $(= \sigma_2)$ is small. We calculate the function

$$h(x, y, t) := \Phi\left(\frac{x - \mu_1 t}{\sigma_1 \sqrt{t}}\right) - \Phi\left(\frac{\mu_2 t - y}{\sigma_1 \sqrt{t}}\right) \tag{14}$$

in three particular cases (see Figures 5-7). We observe that, for $\sigma_1 = 0.1$, the function $h(x, y, t)$ is almost equal to 1 for most of the values of t that are valid when x and y are fixed. Therefore, the function g provides an excellent approximation to the exact joint p.d.f. of $(X(t), Y(t))$ in many instances.

4. Conclusion

In this note, we have considered one- and two-dimensional Wiener processes with random initial states, rather than deterministic ones, as is usually assumed. We have seen that when the initial state has a GPD, the p.d.f. of $X(t)$ or the joint p.d.f. of $(X(t), Y(t))$ is also that of a further generalized Pareto distribution. This result has been obtained for Wiener processes. We could look into the case when $(X(t), Y(t))$ is a two-dimensional Ornstein-Uhlenbeck process, for instance.

We have chosen a GPD as initial distribution because it is an important distribution, used, in particular, in extreme value statistics. An example of application of the results presented here is the following: suppose that a certain process, for example the variations of the flow of a river, behaves approximately like a Wiener process with small variance parameter σ^2. When the river flow is very high, generally in the springtime (in Canada), if we can show that its distribution is approximately GPD, then it will evolve, at least during a certain period of time, approximately like a further GPD (depending on t).

Finally, we could consider other possibilities for the initial distribution of $X(t)$ or $(X(t), Y(t))$. To obtain a nice formula like the one for the function $f(x, y, t)$ given in (11), this initial distribution $f_{X(0),Y(0)}(x_0, y_0)$ should be a function of $x_0 + y_0$. For example, $(X(0), Y(0))$ could have a two-dimensional exponential distribution given by

$$f_{X(0),Y(0)}(x_0, y_0) = \lambda^2 e^{-\lambda(x_0+y_0)} \quad \text{for } x_0 \geq 0, y_0 \geq 0,$$

where λ is a positive parameter. Then $X(0)$ and $Y(0)$ would be independent random variables, which is not the case when $(X(0), Y(0))$ has the GPD defined in (5).

Acknowledgments

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