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MATRIX ANALYSIS OF COUPLED DAMPED VIBRATIONS – TWO DEGREE-OF-FREEDOM ANALYTIC SOLUTION

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Abstract: The linear coupled damped free oscillator problem is solved by means of matrix analysis. The solution can be written by means of exponentials of time evolution matrices multiplied by time. These time evolution matrices are determined as the solutions of quadratic matrix equations. The intricacies of non-proportional damping become particularly clear in this treatment. For the two degree-of-freedom problem an explicit analytic solution is given in terms of a root of a cubic equation. The existence of an algebraically simpler special case (other than proportional damping) is pointed out.

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1. Introduction

It is well known that the conservative linear coupled oscillation problem is best understood in terms of normal modes. When there is viscous damping, however, normal modes no longer exist, unless one has the fortuitous case of proportional damping. The reason is the impossibility of simultaneous diagonalization of more than two quadratic forms, in the general case. Here a matrix analytic expression for the solution is derived. This expression gives new insight into the qualitative behavior of the solution. In the two degree-of-freedom case it yields a practically useful complete analytic solution. The two degree-of-

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freedom case yields a fourth degree characteristic equation. This is the highest degree equation that can be solved algebraically, in the general case, but the expressions for the roots are long and complicated. The matrix method used here reduces the problem to a cubic equation. The roots are then given by relatively much simpler formulas by Cardano. The existence of a simplifying special case is pointed out, and analyzed in some detail.

The systems that we will treat here have equations of motion of the form,

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + 2\boldsymbol{D}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{D}$, and $\boldsymbol{K}$, are $n \times n$ real symmetric, mass, damping, and stiffness matrices, respectively, while $\boldsymbol{x}$ and $\boldsymbol{0}$ are single column $n \times 1$ matrices representing a position vector and the zero vector, respectively. We briefly discuss how one can rotate and scale the mass matrix to the unit matrix. This gives the equations of motion

$$\ddot{\boldsymbol{y}} + 2\boldsymbol{\zeta}\dot{\boldsymbol{y}} + \boldsymbol{\omega}^2\boldsymbol{y} = \boldsymbol{0}, \tag{2}$$

with $\boldsymbol{y} = \boldsymbol{M}^{1/2}\boldsymbol{x}$. One can proceed to diagonalize $\boldsymbol{\omega}$, the matrix of angular frequencies, or $\boldsymbol{\zeta}$, the (damping) matrix of time constants, but not both. It turns out that coordinates, $\boldsymbol{q} = \widetilde{\boldsymbol{U}}_{\Omega_0}\boldsymbol{y}$, such that, $\boldsymbol{\Omega}_0^2 = \boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2$ is diagonal, are advantageous, and these will be used below.

This inability to make both matrices diagonal means that the differential equations are coupled in an essential way that can not be transformed away. This makes analytic solution and qualitative interpretation of results more difficult. Our main task will be to find the solution of the differential equations (2) using an ansatz of the form, $\boldsymbol{q} \sim \exp(\boldsymbol{\lambda}t)\,\boldsymbol{a}$, where $\boldsymbol{\lambda}$ is an $n \times n$ time evolution matrix and $\boldsymbol{a}$ a (constant) vector, and to find an expression that determines $\boldsymbol{\lambda}$.

There are many excellent texts on vibration problems. Den Hartog [7], Chen [5], and Meirovitch [16] are noteworthy examples. Some that concentrate on matrix methods are Gantmacher and Krein [8], Bishop et al [3], and, above all, Lancaster [14]. Recent important work by Angeles et al [2, 1] has clarified many aspects of the coupled damped vibration problem. The central results of the present work appear to be new, however.

The material is organized as follows. First the familiar case of coupled undamped conservative oscillators is treated with a matrix formalism. After that the damped case is considered and the basic matrix equation for this case is derived. The special case of proportional damping is then briefly discussed. After that the analytical solution in the two degree-of-freedom case is given in three steps. First the quadratic matrix equations are solved. Then eigenvalues and the diagonalizing similarity transformation of the resulting time evolution matrix is given. Finally the general solution, and its dependence on initial values, is presented.

The general solution is algebraically quite complicated so the existence of a special case – the "45-degree damping" – that is much simpler, is of interest. This case is analyzed in more detail and some relations are illustrated graphically.

Most technical as well as some pedagogical mathematical material is placed in substantial mathematical Appendices. The simultaneous diagonalization problem, the symmetry and antisymmetry of various matrix combinations, the explicit diagonalization of real symmetric and of arbitrary two by two matrices, as well as the solution of the cubic equation, are treated in extensive Appendices. These are intended to make the article self contained.

2. Free Coupled Oscillators

Consider the problem of free coupled oscillators,

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}. \tag{3}$$

We first diagonalize and scale the mass matrix (see Appendix A). We thus introduce the new variables $\boldsymbol{y}$, defined by,

$$\boldsymbol{y} \equiv \boldsymbol{M}^{1/2}\boldsymbol{x}, \iff \boldsymbol{x} = \boldsymbol{M}^{-1/2}\boldsymbol{y}. \tag{4}$$

Here $\boldsymbol{M}^{1/2}$ is the symmetric square root defined through equations (85) and (86). We then multiply equation (3) by $\boldsymbol{M}^{-1/2}$ from the left and insert $\boldsymbol{1} = \boldsymbol{M}^{-1/2}\boldsymbol{M}^{1/2}$ to the left of $\boldsymbol{x}$, and $\ddot{\boldsymbol{x}}$. Since $\boldsymbol{M}^{-1/2}\boldsymbol{0} = \boldsymbol{0}$, we get

$$\ddot{\boldsymbol{y}} + \boldsymbol{\omega}^2\boldsymbol{y} = \boldsymbol{0}, \tag{5}$$

where,

$$\boldsymbol{\omega}^2 \equiv \boldsymbol{M}^{-1/2}\boldsymbol{K}\boldsymbol{M}^{-1/2}, \tag{6}$$

is the *symmetric* matrix of squared angular frequencies. It corresponds to the single degree-of-freedom quantity $\omega^2 = k/m$. Note that one cannot just multiply equation (3) by $\boldsymbol{M}^{-1}$ from the left since $\boldsymbol{M}^{-1}\boldsymbol{K}$ is not a symmetric matrix.

Equation (5) can be solved by means of the ansatz,

$$\boldsymbol{y}(t) = \exp(\mathrm{i}\boldsymbol{\omega}t)\boldsymbol{a}_+ + \exp(-\mathrm{i}\boldsymbol{\omega}t)\boldsymbol{a}_-, \tag{7}$$

and one easily finds that given initial conditions, $\boldsymbol{y}_0 = \boldsymbol{y}(0)$ and $\dot{\boldsymbol{y}}_0 = \dot{\boldsymbol{y}}(0)$, are satisfied by the solution

$$\boldsymbol{y}(t) = \frac{1}{2}\left\{\exp(\mathrm{i}\boldsymbol{\omega}t)\left[\boldsymbol{y}_0 - \mathrm{i}\boldsymbol{\omega}^{-1}\dot{\boldsymbol{y}}_0\right] + \exp(-\mathrm{i}\boldsymbol{\omega}t)\left[\boldsymbol{y}_0 + \mathrm{i}\boldsymbol{\omega}^{-1}\dot{\boldsymbol{y}}_0\right]\right\}. \qquad (8)$$

One also finds,

$$\boldsymbol{y}(t) = \cos(\boldsymbol{\omega}t)\,\boldsymbol{y}_0 + \sin(\boldsymbol{\omega}t)\,\boldsymbol{\omega}^{-1}\dot{\boldsymbol{y}}_0, \qquad (9)$$

for the corresponding real solution.

Normal modes are coordinates,

$$\boldsymbol{q} = \widetilde{\boldsymbol{U}}_\omega \boldsymbol{y}, \qquad (10)$$

which arise when rotating to diagonal $\boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\mathrm{d} = \widetilde{\boldsymbol{U}}_\omega \boldsymbol{\omega} \boldsymbol{U}_\omega. \qquad (11)$$

Multiply equation (9) from the left by $\widetilde{\boldsymbol{U}}_\omega$ and insert $\mathbf{1} = \boldsymbol{U}_\omega \widetilde{\boldsymbol{U}}_\omega$ before $\boldsymbol{y}_0$ and $\dot{\boldsymbol{y}}_0$. We then get

$$\widetilde{\boldsymbol{U}}_\omega \boldsymbol{y}(t) = \widetilde{\boldsymbol{U}}_\omega \cos(\boldsymbol{\omega}t) \boldsymbol{U}_\omega \widetilde{\boldsymbol{U}}_\omega\, \boldsymbol{y}_0 + \widetilde{\boldsymbol{U}}_\omega \sin(\boldsymbol{\omega}t)\,\boldsymbol{\omega}^{-1} \boldsymbol{U}_\omega \widetilde{\boldsymbol{U}}_\omega \dot{\boldsymbol{y}}_0, \qquad (12)$$

which can be written,

$$\boldsymbol{q}(t) = \cos(\boldsymbol{\omega}_\mathrm{d} t)\,\boldsymbol{q}_0 + \sin(\boldsymbol{\omega}_\mathrm{d} t)\,\boldsymbol{\omega}_\mathrm{d}^{-1}\dot{\boldsymbol{q}}_0. \qquad (13)$$

Using equation (92) we now find that this corresponds to a set of decoupled solutions

$$q_j(t) = \cos(\omega_j t)\, q_j(0) + \sin(\omega_j t)\, \omega_j^{-1} \dot{q}_j(0), \quad j = 1, \ldots, n, \qquad (14)$$

that give the time dependencies of the normal modes.

3. Free Damped Coupled Oscillators

We now proceed to a damped problem, as defined by a mass, a damping and a stiffness matrix, and given by equation (1):

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + 2\boldsymbol{D}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \mathbf{0}. \qquad (15)$$

Roemer and Mook [19] discuss methods for finding these matrices. Assuming that they are known, we will now solve this equation, for given initial conditions, explicitly in terms of matrix functions.

We first diagonalize and scale the mass matrix as above. We thus again introduce the new variables

$$\boldsymbol{y} \equiv \boldsymbol{M}^{1/2}\boldsymbol{x}. \tag{16}$$

Multiply equation (15) by $\boldsymbol{M}^{-1/2}$ from the left and insert $\mathbf{1} = \boldsymbol{M}^{-1/2}\boldsymbol{M}^{1/2}$ to the left of $\boldsymbol{x}, \dot{\boldsymbol{x}}$, and $\ddot{\boldsymbol{x}}$. Since $\boldsymbol{M}^{-1/2}\mathbf{0} = \mathbf{0}$, this gives,

$$\ddot{\boldsymbol{y}} + 2\,\boldsymbol{\zeta}\,\dot{\boldsymbol{y}} + \boldsymbol{\omega}^2\boldsymbol{y} = \mathbf{0}, \tag{17}$$

where $\boldsymbol{\omega}^2$ is given by equation (6) and,

$$\boldsymbol{\zeta} \equiv \boldsymbol{M}^{-1/2}\boldsymbol{D}\boldsymbol{M}^{-1/2}, \tag{18}$$

is the matrix of time constants, $\boldsymbol{\zeta}$, that characterizes the damping.

We now start our search for a matrix solution of equation (17). We make the ansatz

$$\boldsymbol{y}(t) = \exp(\boldsymbol{\lambda}t)\,\boldsymbol{a}, \tag{19}$$

and look for an equation that determines the square matrix $\boldsymbol{\lambda}$. We first need

$$\dot{\boldsymbol{y}} = \boldsymbol{\lambda}\,\boldsymbol{y}(t)\,, \tag{20}$$

$$\ddot{\boldsymbol{y}} = \boldsymbol{\lambda}^2\boldsymbol{y}(t)\,. \tag{21}$$

Inserting this in equation (17) gives

$$\left(\boldsymbol{\lambda}^2 + 2\boldsymbol{\zeta}\boldsymbol{\lambda} + \boldsymbol{\omega}^2\right)\boldsymbol{y}(t) = \mathbf{0}. \tag{22}$$

The ansatz will thus give a solution if $\boldsymbol{\lambda}$ satisfies the matrix equation

$$\boldsymbol{\lambda}^2 + 2\boldsymbol{\zeta}\boldsymbol{\lambda} + \boldsymbol{\omega}^2 = \mathbf{0}. \tag{23}$$

By, $\mathbf{0}$, we denote a zero vector or a zero matrix, depending on the context.

Our matrix equation (23) can be rewritten as,

$$(\boldsymbol{\lambda} + \boldsymbol{\zeta})^2 - [\boldsymbol{\lambda}, \boldsymbol{\zeta}] + \boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2 = \mathbf{0}, \tag{24}$$

where we have used the commutator (95). Apart from preventing simultaneous diagonalization, the presence of the non-zero commutator here also prevents factorization of the differential operator – a method that can give interesting results in the single degree-of-freedom case (Rosu and Reyes [20]).

If we now define the matrices $\boldsymbol{\Omega}_0$ and $\boldsymbol{\Omega}_1$ through,

$$\boldsymbol{\Omega}_0^2 \equiv \boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2, \tag{25}$$

and,

$$\boldsymbol{\lambda}_\pm \equiv \pm\mathrm{i}\boldsymbol{\Omega}_1 - \boldsymbol{\zeta}, \tag{26}$$

respectively, we get from equation (24),

$$\boldsymbol{\Omega}_1^2 \pm \mathrm{i}[\boldsymbol{\Omega}_1, \boldsymbol{\zeta}] = \boldsymbol{\Omega}_0^2, \tag{27}$$

as the matrix equation that determines $\boldsymbol{\Omega}_1$. In what follows we will mainly assume that $\boldsymbol{\Omega}_0^2$ of equation (25) is positive definite; this corresponds to the (n-dimensional) underdamped, or weak damping, case. Though the results, in principle, are general our notation will be adapted to the underdamped case.

If we can solve the quadratic matrix equation (27) and get $\boldsymbol{\Omega}_1$ for given $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$, we obtain the solution

$$\boldsymbol{y}(t) = \exp[(\mathrm{i}\boldsymbol{\Omega}_1 - \boldsymbol{\zeta})t]\,\boldsymbol{a}_+ + \exp[(-\mathrm{i}\boldsymbol{\Omega}_1 - \boldsymbol{\zeta})t]\,\boldsymbol{a}_-, \tag{28}$$

to be compared with equation (7). Standard methods for the analytic solution of a quadratic matrix equation are not available, to my knowledge, but for the two degree-of-freedom case it can be handled analytically, as we will see below. Once the relevant matrices have been calculated there are efficient numerical methods for matrix exponentials (Golub and Van Loan [10]).

4. Proportional Damping

The intricacies of the quadratic matrix equation (27) can be avoided if $\boldsymbol{\omega}^2$ and $\boldsymbol{\zeta}$ are diagonalized by the same orthonormal transformation. One can then separate the problem into uncoupled one-dimensional normal modes.

Simultaneous diagonalization is, according to the Cayley-Hamilton Theorem [22, 6], only possible when,

$$\boldsymbol{\zeta} = \alpha_{n-1}\boldsymbol{\omega}^{n-1} + \ldots + \alpha_1\boldsymbol{\omega} + \alpha_0\mathbf{1}, \tag{29}$$

i.e. when $\boldsymbol{\zeta}$ can be expressed a linear combination of the powers of $\boldsymbol{\omega}$. This is usually called proportional damping, and the eigenvectors of $\boldsymbol{\omega}^2$ and $\boldsymbol{\zeta}$ are then the same. Angeles and Ostrovskaya [1] have pointed out that there is a certain best approximation of a given system to such a proportionally damped system. They also pointed out that this approximation need not be all that good.

In the proportional damping case, when $\boldsymbol{\omega}$ commutes with $\boldsymbol{\zeta}$ (Caughey and O'Kelly [4]), it is easy to see that $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_0 = (\boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2)^{1/2}$ is a solution of equation (27). Simultaneous diagonalization is then possible and also transformation to normal mode coordinates $\boldsymbol{q}$. In terms of these the general solution can then be written

$$q_j(t) = \exp\left[\left(\mathrm{i}\sqrt{\omega_j^2 - \zeta_j^2} - \zeta_j\right) t\right] q_{j+} + \exp\left[\left(-\mathrm{i}\sqrt{\omega_j^2 - \zeta_j^2} - \zeta_j\right) t\right] q_{j-}, \quad (30)$$

to be compared with equation (14).

In this commuting, or equivalently, proportional damping, case one therefore also gets complete decoupling of suitable normal modes. An exhaustive analytical analysis of a two degree-of-freedom proportional damping case has recently been published by Oniszczuk [18] who states that "there is no simple direct relationship between these roots and arbitrary values of the physical parameters characterizing the vibrating system". Oniszczuk then proceeds to investigate this relationship in the proportional damping case. In this paper we will calculate a such direct relationship for the general case. Although straightforward, it will unfortunately not be simple

5. Diagonalizing $\mathbf{\Omega_0}$

For both the undamped and the proportional damping cases it is advantageous to work with a diagonal $\boldsymbol{\Omega}_0$. In what follows we assume that we have transformed to coordinates

$$\boldsymbol{q} = \widetilde{\boldsymbol{U}}_{\Omega_0} \boldsymbol{y} \quad (31)$$

that diagonalize $\boldsymbol{\Omega}_0^2 = \boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2$. The basic differential equation (17) is then,

$$\ddot{\boldsymbol{q}} + 2\boldsymbol{\zeta}\dot{\boldsymbol{q}} + \boldsymbol{\omega}^2 \boldsymbol{q} = \mathbf{0}, \quad (32)$$

but, in general, neither $\boldsymbol{\omega}$, nor $\boldsymbol{\zeta}$, are diagonal themselves – only the combination $\boldsymbol{\Omega}_0$. The alternative form,

$$\left[\mathbf{1}\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\zeta}\right]^2 \boldsymbol{q} + \boldsymbol{\Omega}_0^2 \boldsymbol{q} = \mathbf{0}, \quad (33)$$

of equation (32) brings out the significance of $\boldsymbol{\Omega}_0$ more clearly. Should $\boldsymbol{\zeta} = \mathbf{0}$ this means that the $\boldsymbol{q}$ are ordinary normal mode coordinates. Should $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ be simultaneously diagonalizable $\boldsymbol{q}$ will stand for the proportional damping normal coordinates of the previous section.

6. The Matrix Equation

Let us now analyze the matrix equations, (27):

$$\boldsymbol{\Omega}_1^2 \pm \mathrm{i}[\boldsymbol{\Omega}_1, \boldsymbol{\zeta}] = \boldsymbol{\Omega}_0^2, \tag{34}$$

arising form the ansatz, $\exp(\boldsymbol{\lambda}_\pm t)\,\boldsymbol{a} = \exp[(\pm\mathrm{i}\boldsymbol{\Omega}_1 - \boldsymbol{\zeta})t]\,\boldsymbol{a}$, more closely. We know that $\boldsymbol{\zeta}$ and the right hand side, $\boldsymbol{\Omega}_0$, are real symmetric (and also non-negative in our main, underdamped, case). The left hand side must therefore also be real symmetric. Since the commutator of two symmetric matrices is antisymmetric (see Appendix C) we immediately find that $\boldsymbol{\Omega}_1$ cannot be real symmetric for non-zero commutator.

We now assume that we can write $\pm\mathrm{i}\boldsymbol{\Omega}_1$ in the form,

$$\pm\mathrm{i}\boldsymbol{\Omega}_1 \equiv \pm\mathrm{i}\boldsymbol{\Omega} + \boldsymbol{\eta}, \tag{35}$$

where we assume that $\boldsymbol{\Omega}$ and $\boldsymbol{\eta}$ are real. The matrix equation then splits into the two equations,

$$\boldsymbol{\Omega}^2 - \boldsymbol{\eta}^2 + [\boldsymbol{\eta}, \boldsymbol{\zeta}] \;=\; \boldsymbol{\Omega}_0^2, \tag{36}$$

$$(\boldsymbol{\Omega}\boldsymbol{\eta} + \boldsymbol{\eta}\boldsymbol{\Omega}) - [\boldsymbol{\Omega}, \boldsymbol{\zeta}] \;=\; \boldsymbol{0}, \tag{37}$$

corresponding to real and imaginary parts, respectively.

All four matrices here are assumed real. These quadratic matrix equations thus determine the unknowns, $\boldsymbol{\Omega}$, and $\boldsymbol{\eta}$, in terms of the given, real symmetric, $\boldsymbol{\Omega}_0$, and $\boldsymbol{\zeta}$. It is consistent with these equations to assume that $\boldsymbol{\Omega}$ is symmetric and $\boldsymbol{\eta}$ is antisymmetric. This is proved in Appendix C. In the two by two case these symmetries turn out to be necessary consequences of the equations, as we will see below. In what follows we adopt these assumptions. Equation (36) is then an equation of symmetric matrices while equation (37) is antisymmetric.

In what follows we will write,

$$\boldsymbol{\lambda}_+ \equiv \boldsymbol{\lambda} \;=\; \mathrm{i}\boldsymbol{\Omega} + \boldsymbol{\eta} - \boldsymbol{\zeta}, \tag{38}$$

$$\boldsymbol{\lambda}_- \equiv \overline{\boldsymbol{\lambda}} \;=\; -\mathrm{i}\boldsymbol{\Omega} + \boldsymbol{\eta} - \boldsymbol{\zeta}, \tag{39}$$

denoting complex conjugation with an over-bar. The general solution (28) of equation (32), now takes the form,

$$\boldsymbol{q}(t) \;=\; \mathrm{e}^{\boldsymbol{\lambda} t}\boldsymbol{a}_+ + \mathrm{e}^{\overline{\boldsymbol{\lambda}} t}\boldsymbol{a}_- \tag{40}$$

$$\;=\; \exp[(\mathrm{i}\boldsymbol{\Omega} + \boldsymbol{\eta} - \boldsymbol{\zeta})t]\,\boldsymbol{a}_+ + \exp[(-\mathrm{i}\boldsymbol{\Omega} + \boldsymbol{\eta} - \boldsymbol{\zeta})t]\,\boldsymbol{a}_-. \tag{41}$$

Use of $\boldsymbol{q}(t)$ implies that we are working with diagonal $\boldsymbol{\Omega}_0$. Here one observes that, since $\boldsymbol{\eta}$ is antisymmetric, $\exp(\boldsymbol{\eta}t)$ is a rotation matrix [17] (see Appendix D). This presence of an extra rotation is seen to distinguish the general case from the proportional damping case, where $\boldsymbol{\eta} = \mathbf{0}$. We will not go further in the general analysis of the system of quadratic matrix equations. Instead we will concentrate on the two degree-of-freedom case.

7. Solution of the Two by Two Matrix Equations

We now try to find an analytic solution of equation (32) in the two-dimensional case. We first introduce suitable notation for the known input quantities of $\boldsymbol{\Omega}_0$ and $\boldsymbol{\zeta}$. We put

$$\boldsymbol{\Omega}_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{42}$$

since we are working with a diagonal effective angular frequency matrix $\boldsymbol{\Omega}_0$. Appendix B gives the formulas for real symmetric two by two matrix diagonalization. Then the damping matrix $\boldsymbol{\zeta}$ cannot be taken diagonal, so we put

$$\boldsymbol{\zeta} = \begin{pmatrix} \frac{1}{2}(\gamma+\alpha) & \beta \\ \beta & \frac{1}{2}(\gamma-\alpha) \end{pmatrix}. \tag{43}$$

The difference between the diagonal elements of $\boldsymbol{\zeta}$ is given by $\alpha = \zeta_{11} - \zeta_{22}$. The eigenvalues of $\boldsymbol{\zeta}$ are given by (see Appendix B)

$$\zeta_{1,2} = \frac{1}{2}\left(\gamma \pm \sqrt{\alpha^2 + 4\beta^2}\right). \tag{44}$$

Since $\boldsymbol{\zeta}$ only occurs in commutators in the equations (36)-(37), the parameter, $\gamma = \operatorname{trace}(\boldsymbol{\zeta})$, will not appear. The four input parameters to our matrix algebra problem are thus a, b, α, and β.

If we initially assume that $\boldsymbol{\Omega}$ and $\boldsymbol{\eta}$ are arbitrary real matrices, equations (36)-(37) and some algebra show that, in fact, $\boldsymbol{\Omega}$ must be symmetric and $\boldsymbol{\eta}$ antisymmetric. We therefore parameterize these matrices as,

$$\boldsymbol{\Omega} = \begin{pmatrix} A & U \\ U & B \end{pmatrix}, \tag{45}$$

and

$$\boldsymbol{\eta} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix}. \tag{46}$$

The four parameters A, B, U and Z are thus our unknowns.

If we put all this into equations (36)-(37) we get four equations: one from the antisymmetric (37), one from the off-diagonal symmetric, and two from the diagonal of the symmetric (36). These are, respectively,

$$Z(A+B) - \beta(A-B) + \alpha U = 0, \quad (47)$$

$$U(A+B) - \alpha Z = 0, \quad (48)$$

$$A^2 + U^2 + Z^2 + 2\beta Z = a^2, \quad (49)$$

$$B^2 + U^2 + Z^2 - 2\beta Z = b^2. \quad (50)$$

The first two of these are easily solved to give,

$$U = \frac{\alpha\beta(A-B)}{(A+B)^2 + \alpha^2}, \quad (51)$$

$$Z = \frac{\beta(A+B)(A-B)}{(A+B)^2 + \alpha^2}, \quad (52)$$

in terms of A and B. One can now proceed an insert these into next two equations and solve for A and B. This is easily done by means of a computer algebra system but yields very long expressions. Inspection of our results, however, indicate that it might be advantageous to introduce the new variables,

$$X = A + B, \quad \text{and} \quad Y = A - B, \quad (53)$$

by means of the substitutions

$$A = (X+Y)/2, \quad \text{and} \quad B = (X-Y)/2, \quad (54)$$

into the equations.

This, indeed, gives fairly compact results. One finds that $X = \sqrt{w}$ is given by a root of the cubic equation,

$$w^3 - [2(a^2+b^2) - (\alpha^2 + 4\beta^2)]\, w^2 + [(a^2-b^2)^2 - 2(\alpha^2+4\beta^2)(a^2+b^2)]\, w + [\alpha^2(a^2-b^2)^2] = 0, \quad (55)$$

in w. For the underdamped case this equation will have three real roots, w_i, $(i = 1, 2, 3)$, and $X^2 = w_3 = (A+B)^2$ is the largest of them. This is motivated in Appendix F where also w_3 is given in equation (155). Thereby the explicit formula for,

$$X(a, b, \alpha, \beta) = \sqrt{w_3(a, b, \alpha, \beta)}, \quad (56)$$

is available. Once X has been calculated, Y can be found from

$$Y(a, b, \alpha, \beta) = \frac{(X^2+\alpha^2)(a^2-b^2)}{X[X^2 + (\alpha^2+4\beta^2)]}. \quad (57)$$

The computer algebra system *Maple*, see [12] has been used to check these results.

Having found X and Y we can go back to equation (51) to find the off-diagonal element of $\boldsymbol{\Omega}$,

$$U(a,b,\alpha,\beta) = \frac{\alpha\beta Y}{X^2+\alpha^2}, \tag{58}$$

and to and equation (52) for the element of the rotating matrix $\boldsymbol{\eta}$,

$$Z(a,b,\alpha,\beta) = \frac{\beta XY}{X^2+\alpha^2}. \tag{59}$$

The final result of this section is thus explicit algebraic expressions for $\boldsymbol{\Omega}(a,b,\alpha,\beta)$ and $\boldsymbol{\eta}(a,b,\alpha,\beta)$. The complete time evolution matrix of equation (38), can now be written

$$\boldsymbol{\lambda}(a,b,\alpha,\beta,\gamma) = \begin{pmatrix} \frac{\mathrm{i}}{2}(X+Y)-\frac{1}{2}(\gamma+\alpha) & \mathrm{i}U+Z-\beta \\ \mathrm{i}U-Z-\beta & \frac{\mathrm{i}}{2}(X-Y)-\frac{1}{2}(\gamma-\alpha) \end{pmatrix}, \tag{60}$$

for the two degree-of-freedom case.

8. Eigenvalues and the General Solution

Above we motivated that a solution of equation (32) in the form,

$$\boldsymbol{q}(t) = \exp(\boldsymbol{\lambda}\, t)\, \boldsymbol{a}_+ + \exp(\overline{\boldsymbol{\lambda}}\, t)\, \boldsymbol{a}_-, \tag{61}$$

in general should exist and we have discussed the structure of the equations that determine the time evolution matrices $\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}$. In the two degree-of-freedom case we have determined these matrices by explicit calculation. The main algebraic task involved the solution of a cubic equation.

The calculation of a matrix exponential is most efficient when the matrix is diagonal. We thus proceed to find the eigenvalues of the time evolution matrix $\boldsymbol{\lambda}$ of equation (60). The eigenvalues are simply the roots of the quadratic equation

$$|\boldsymbol{\lambda} - \lambda \mathbf{1}| = \begin{vmatrix} \lambda_{11}-\lambda & \lambda_{12} \\ \lambda_{21} & \lambda_{22}-\lambda \end{vmatrix} = 0. \tag{62}$$

For the $\boldsymbol{\lambda}$ of equation (60) these are (see Appendix E),

$$\lambda_{1,2} = \frac{\mathrm{i}}{2}\left[X \pm \sqrt{Y^2+4(U^2+Z^2+\beta^2)-\alpha^2+2\mathrm{i}(\alpha Y+4\beta U)}\right] - \frac{\gamma}{2}. \tag{63}$$

The eigenvalues of $\overline{\boldsymbol{\lambda}}$ are the obviously the complex conjugates of these.

One notes that these four eigenvalues, $\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2$, are the same as those that come directly from the secular equation of equation (32),

$$\left|\mathbf{1}\lambda^2 + 2\boldsymbol{\zeta}\lambda + \boldsymbol{\omega}^2\right| = 0, \tag{64}$$

which is of the fourth degree, in the two dimensional case. It is easy to see that this must be true since according to equation (23) we have that the matrix

$$f(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^2 + 2\boldsymbol{\zeta}\boldsymbol{\lambda} + \boldsymbol{\omega}^2 \tag{65}$$

is the zero matrix. If this matrix now operates on one of the eigenvectors, $\boldsymbol{s}_j$, of $\boldsymbol{\lambda}$ we find that

$$f(\boldsymbol{\lambda})\boldsymbol{s}_j = (\mathbf{1}\lambda_j^2 + 2\boldsymbol{\zeta}\lambda_j + \boldsymbol{\omega}^2)\boldsymbol{s}_j = \mathbf{0}. \tag{66}$$

If this is to be true for all $\boldsymbol{s}_j$ we must clearly have equation (64) for the eigenvalues. The pattern with pairs of of complex conjugate eigenvalues presupposes the weak damping case (see Angeles et al [2], Oniszczuk [18]).

As long as one can find two linearly independent (complex) eigenvectors $\boldsymbol{s}_{1,2}$, or column matrices, from

$$(\boldsymbol{\lambda} - \lambda_j\mathbf{1})\,\boldsymbol{s}_j = \mathbf{0}, \quad (j = 1, 2), \tag{67}$$

one can also find a non-singular similarity transformation matrix, $\boldsymbol{S} = (\boldsymbol{s}_1\ \boldsymbol{s}_2)$, such that

$$\boldsymbol{\lambda} = \boldsymbol{S}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{S}^{-1} = \boldsymbol{S}\boldsymbol{\lambda}_{\rm d}\boldsymbol{S}^{-1}. \tag{68}$$

See Appendix E for explicit expressions for these matrices. Should such linearly independent eigenvectors not exist the above diagonalization is impossible. One can then resort to the Jordan normal form (Lancaster [14], Cullen [6], Strang [23]) but we will not consider this case here.

We now have explicit expressions for the eigenvalues and, via Appendix E, the diagonalizing similarity transformation matrix $\boldsymbol{S}$ and and its complex conjugate $\overline{\boldsymbol{S}}$. By means of these we can now write the exact general solution (61) of the two dimensional coupled damped oscillation problem (32) in the form

$$\boldsymbol{q}(t) = \boldsymbol{S}\begin{pmatrix} {\rm e}^{\lambda_1 t} & 0 \\ 0 & {\rm e}^{\lambda_2 t} \end{pmatrix}\boldsymbol{s}_+ + \overline{\boldsymbol{S}}\begin{pmatrix} {\rm e}^{\bar{\lambda}_1 t} & 0 \\ 0 & {\rm e}^{\bar{\lambda}_2 t} \end{pmatrix}\boldsymbol{s}_-, \tag{69}$$

where

$$\boldsymbol{s}_+ \equiv \boldsymbol{S}^{-1}\boldsymbol{a}_+, \quad \boldsymbol{s}_- \equiv \overline{\boldsymbol{S}}^{\,-1}\boldsymbol{a}_-. \tag{70}$$

Starting from (61) we have used (68) together with equations (93) and (92) of Appendix A. One also sees that the $\boldsymbol{s}_{\pm}$ vectors can be determined from the initial conditions by the formula

$$\begin{pmatrix} \boldsymbol{s}_+ \\ \boldsymbol{s}_- \end{pmatrix} = \begin{pmatrix} \boldsymbol{S} & \overline{\boldsymbol{S}} \\ \boldsymbol{S}\boldsymbol{\lambda}_{\rm d} & \overline{\boldsymbol{S}}\,\overline{\boldsymbol{\lambda}}_{\rm d} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{q}(0) \\ \dot{\boldsymbol{q}}(0) \end{pmatrix}. \tag{71}$$

Everything up till now has required only two by two matrices, and many of the ideas are the same as for undamped normal modes. The main difference is in the much more complicated algebra required to get the eigenvalues and the eigenvectors, and the fact that these eigenvectors are neither real nor orthonormal.

Since explicit analytic, albeit lengthy, formulas in this paper give the eigenvalues λ_j and the elements of the matrix $\boldsymbol{S}$ we now have an analytical solution of the two degree-of-freedom coupled damped problem. The length of the formulas is of no consequence as long as a computer algebra system is used, but it does prevent a qualitative understanding of the dependence of the solution on the input parameters $(a, b, \alpha, \beta, \gamma)$. The existence of a simplifying special case (other than proportional damping) is therefore of some interest.

9. The Special Case of "45-Degree" Damping

One may use the treatment above to look for special cases that simplify the algebra. The greatest simplification is achieved by putting $\beta = 0$ in $\boldsymbol{\zeta}$. This, however, is the well known case of proportional damping, since it means that $\boldsymbol{\zeta}$ is diagonal at the same time as $\boldsymbol{\Omega}_0$ and therefore as $\boldsymbol{\omega}$, and it hardly requires further elaboration.

The other alternative is to put $\alpha = 0$, and this case is discussed here. From equation (58) we then immediately find that, U, the off-diagonal element of $\boldsymbol{\Omega}$, is zero. equation (149) of Appendix F gives that

$$X(a, b, \beta) = \sqrt{a^2 + b^2 - 2\beta^2 + 2\sqrt{(a^2+\beta^2)(b^2+\beta^2)}}, \tag{72}$$

where $X = A + B$. Then equation (57) gives that

$$Y(a, b, \beta) = \frac{(a^2 - b^2)\sqrt{a^2 + b^2 - 2\beta^2 + 2\sqrt{(a^2+\beta^2)(b^2+\beta^2)}}}{a^2 + b^2 + 2\beta^2 + 2\sqrt{(b^2+\beta^2)\,(a^2+\beta^2)}}, \tag{73}$$

where $Y = A - B$. Finally equation (59)

$$Z(a,b,\beta) = \frac{\beta(a^2 - b^2)}{a^2 + b^2 + 2\beta^2 + 2\sqrt{(a^2+\beta^2)(b^2+\beta^2)}} \tag{74}$$

gives the elements of the matrix $\boldsymbol{\eta}$. $A(a,b,\beta), B(a,b,\beta)$, and $Z(a,b,\beta)$, are plotted as functions of β for $a = 8$ and $b = 6$ in Figure 1.

The time evolution matrix (60) is now,

$$\boldsymbol{\lambda}(a,b,\beta,\gamma) = \begin{pmatrix} \mathrm{i}A - \frac{\gamma}{2} & Z - \beta \\ -Z - \beta & \mathrm{i}B - \frac{\gamma}{2} \end{pmatrix}, \tag{75}$$

with, $A = (X+Y)/2$, $B = (X-Y)/2$, and its eigenvalues are,

$$\lambda_{1,2} = \frac{\mathrm{i}}{2}\left[X \pm Y\sqrt{1 + \frac{4(Z^2 - \beta^2)}{Y^2}}\right] - \frac{\gamma}{2}. \tag{76}$$

These can be expressed as,

$$\lambda_{1,2}(a,b,\beta,\gamma) = \frac{\mathrm{i}}{2}\left[X \pm \sqrt{\frac{(a^2-b^2)^2}{X^2+4\beta^2} - 4\beta^2}\right] - \frac{\gamma}{2}. \tag{77}$$

For $\beta = 0$ this becomes,

$$\lambda_{1,2}(a,b,\gamma) = \frac{\mathrm{i}}{2}\left[(a+b) \pm (a-b)\right] - \frac{\gamma}{2}. \tag{78}$$

When β increases one finds that the eigenvalues become degenerate when,

$$\beta = \beta_d \equiv \frac{(a+b)(a-b)}{2\sqrt{2(a^2+b^2)}}, \tag{79}$$

and then the eigenvalues both are,

$$\lambda_{1,2}(a,b,\gamma) = \frac{\mathrm{i}}{2}\sqrt{\frac{3a^4 + 10a^2b^2 + 3b^4}{2(a^2+b^2)}} - \frac{\gamma}{2}. \tag{80}$$

Up to this β-value the damping of both modes is described by $\exp(-\gamma t/2)$, or equivalently, by the time constant $\gamma/2$. For $\beta > \beta_d$ the square-root in equation (77) becomes imaginary and thus contributes to the damping instead of the angular frequency. Then the two angular frequencies are both $\mathrm{Im}\lambda_{1,2} = X/2$ but the damping splits into two different time constants

$$\mathrm{Re}\lambda_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{4\beta^2 - \frac{(a^2-b^2)^2}{X^2+4\beta^2}}. \tag{81}$$

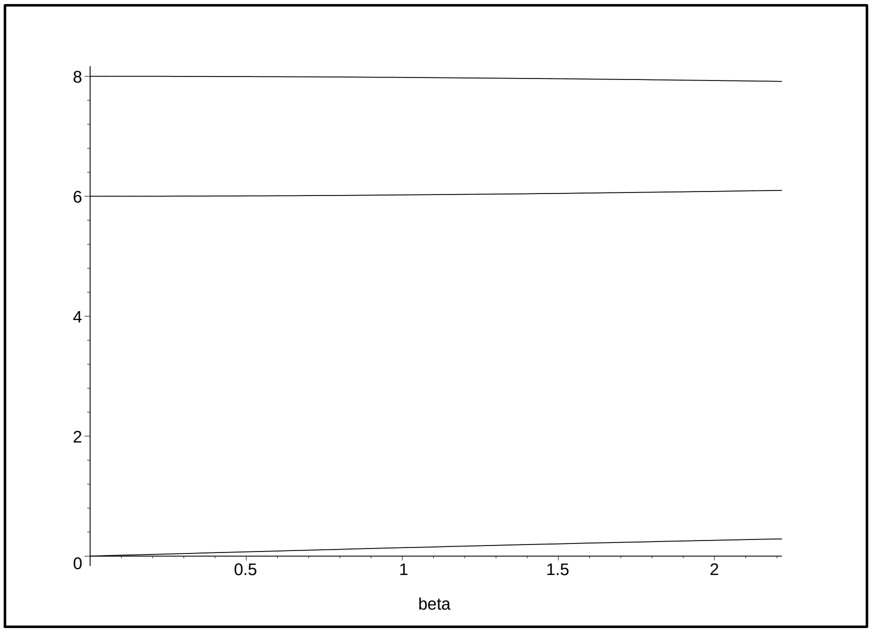

Figure 1: Plot of A, B, and Z as functions of β for $a = 8$, $b = 6$

For $\beta > \beta_m$ given by,

$$\beta_m \equiv \frac{1}{2}\sqrt{\frac{(a^2 - b^2)^2 + 2\gamma^2(a^2 + b^2) + \gamma^4}{2(a^2 + b^2) + \gamma^2}}, \tag{82}$$

the larger of the two real parts in equation (81) becomes positive. This means that we no longer have a damped problem, so $\beta > \beta_m$ must be un-physical, and β_m is the maximum possible β-value. At that value one of the modes is un-damped, i.e. $\mathrm{Re}\lambda_1 = 0$.

Figure 1 shows how the elements, A, B, Z, of $\mathrm{i}\boldsymbol{\Omega} + \boldsymbol{\eta}$ changes with β for the specific values $a = 8$, $b = 6$. They clearly behave very smoothly. Figures 2 and 3 show the behavior of the real and imaginary parts of the eigenvalues of $\boldsymbol{\lambda}$, as functions of β. Here $a = 8$, $b = 6$, $\gamma = 4$. This means that $\beta_d = 0.9899494934$ and that $\beta_m = 2.215266893$. The change in the system from having two different angular frequencies and a single damping time constant up to β_d and then, for larger β, a single angular frequency and two different damping time constants, is clearly seen. In these figures $0 < \beta < \beta_m$ and the fact that the larger damping becomes zero at β_m is seen in Figure 2.

Clearly the value $\beta = \beta_d$ represents a classical bifurcation of the type occur-

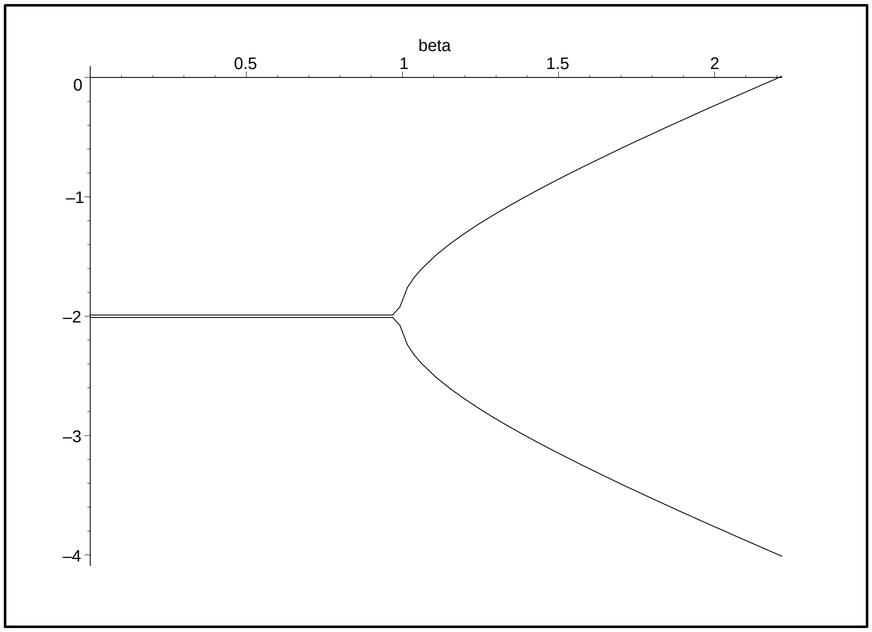

Figure 2: Plot of $\mathrm{Re}\lambda_1$, and $\mathrm{Re}\lambda_2$ as functions of β for $a = 8$, $b = 6, \gamma = 4$

ring in non-linear systems. One of the conclusions to be drawn from this paper is that these linear mechanical systems have extremely non-linear behavior as analytic functions of the parameters of the input mass, damping, and stiffness matrices.

A concrete example of a damped two degree-of-freedom system in which one of the modes is undamped was given by Wilms and Pinkney [25]. Their system, which is an example of what here is called "45-degree damping", has been further discussed in [21, 11]. In Appendix G a class of concrete systems is given and the dependence of the physical parameters on $a, b, \alpha, \beta, \gamma$ is calculated.

10. Conclusions

Since the characteristic equation for a coupled damped two degree-of-freedom vibration problem is of the fourth degree it should be obvious that one can find an analytic solution to such a problem – the quartic equation being the highest degree equation that can be solved algebraically in the general case.

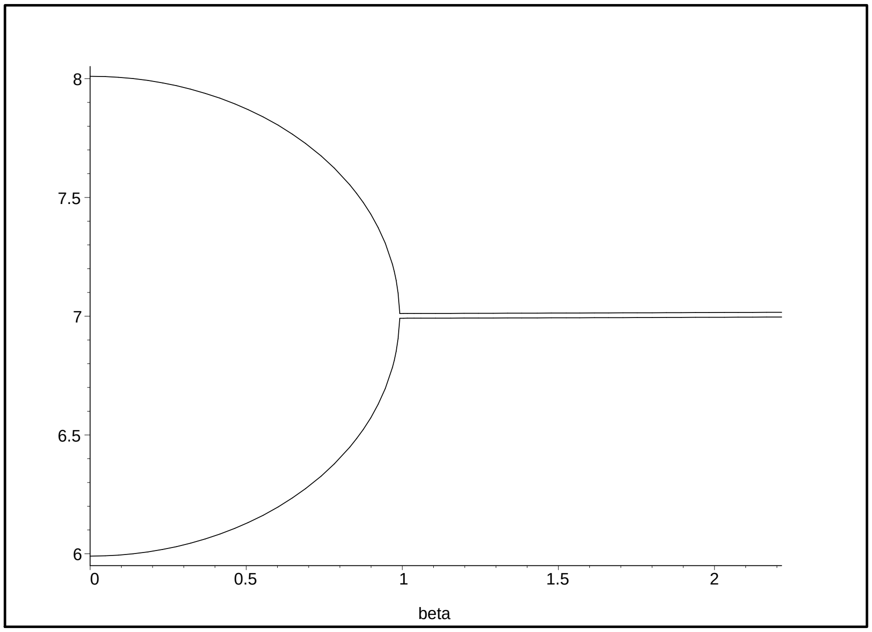

Figure 3: Plot of Imλ_1, and Imλ_2 as functions of β for $a = 8$, $b = 6, \gamma = 4$

What has been achieved in the present study is the solution of this problem by means of merely a cubic equation and this is important since it reduces the algebraic complexity considerably. The cubic equation arose from an ansatz with the exponential of a time evolution matrix, instead of the usual ansatz for the solution as a sum of ordinary exponential functions.

The algebraic simplification that this entails makes it easier to understand how solutions of the two degree-of-freedom damped vibration problem depend on the physical parameters. The discovery of the "45-degree damping" special case is one outcome of the improved insight that the new solution method gives. Our designation of this special case is due to the fact that the damping matrix must be rotated by 45 degrees to become diagonal (relative to the coordinates that diagonalize $\mathbf{\Omega}_0$).

The method for the solution of coupled damped vibration problems presented here deserves further study. The performance of our time-evolution matrix method for systems with more than two degrees-of-freedom may be worth investigating. Above all, however, the method gives insight into the intricacies and subtleties of non-proportional damping and clearly shows how the solution necessarily rotates between temporary “normal modes”.

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A. Appendix: Diagonalization and Matrix Functions

We assume that the matrices of our problem, $\boldsymbol{M}, \boldsymbol{D}$, and $\boldsymbol{K}$, are real symmetric and non-negative. This means that they can all be diagonalized by means of orthonormal matrices. If $\boldsymbol{A}$ is such a matrix and $\boldsymbol{U}_a$ is the appropriate orthonormal matrix we have

$$\boldsymbol{A}_{\rm d} = \widetilde{\boldsymbol{U}}_a \boldsymbol{A} \boldsymbol{U}_a \,, \tag{83}$$

or, explicitly,

$$\begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} u_{11} & u_{21} & \cdots \\ u_{12} & u_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots \\ u_{21} & u_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

for the diagonal matrix $\boldsymbol{A}_{\rm d}$. The numbers a_j are the eigenvalues (or characteristic values) of the matrix. Note that since $\boldsymbol{U}_a$ is orthonormal its transpose is equal to the inverse: $\widetilde{\boldsymbol{U}}_a = \boldsymbol{U}_a^{-1}$. The matrix $\boldsymbol{A}$ can thus be written

$$\boldsymbol{A} = \boldsymbol{U}_a \boldsymbol{A}_{\rm d} \widetilde{\boldsymbol{U}}_a \,. \tag{84}$$

Since we are dealing with non-negative matrices the eigenvalues $a_j \geq 0$ and we can define powers of the diagonal matrix through

$$\boldsymbol{A}_{\rm d}^{\beta} \equiv \begin{pmatrix} a_1^{\beta} & 0 & \cdots \\ 0 & a_2^{\beta} & \cdots \\ \vdots & \vdots & \end{pmatrix} \tag{85}$$

for positive real β in the non-negative case, and for any real β in the positive case ($a_j > 0$ for all $j = 1, \ldots, n$). Using this, powers of the matrix $\boldsymbol{A}$ itself can be defined through

$$\boldsymbol{A}^{\beta} = \boldsymbol{U}_a \boldsymbol{A}_{\rm d}^{\beta} \widetilde{\boldsymbol{U}}_a . \tag{86}$$

Note that these are all real symmetric matrices just as $\boldsymbol{A}$ itself.

A real symmetric matrix corresponds to a quadratic form

$$g_A(\boldsymbol{x}) = \widetilde{\boldsymbol{x}} \boldsymbol{A} \boldsymbol{x} = \sum_{jk} a_{jk} x_i x_k, \tag{87}$$

and, when the matrix is positive definite the level surfaces, $g_A(\boldsymbol{x}) = c$, of such a quadratic form are n-dimensional ellipsoids. Rotating to make the matrix diagonal is equivalent to rotating the ellipsoids so that their axes coincide with the coordinate axes. With $\boldsymbol{x} = \boldsymbol{U}_a \boldsymbol{y}$, using (84) and $\widetilde{\boldsymbol{U}}_a \boldsymbol{U}_a = \mathbf{1}$, this gives

$$g_A(\boldsymbol{x}) = \widetilde{\boldsymbol{x}} \boldsymbol{A} \boldsymbol{x} = \widetilde{\boldsymbol{y}} \widetilde{\boldsymbol{U}}_a \boldsymbol{U}_a \boldsymbol{A}_{\mathrm{d}} \widetilde{\boldsymbol{U}}_a \boldsymbol{U}_a \boldsymbol{y} = \widetilde{\boldsymbol{y}} \boldsymbol{A}_{\mathrm{d}} \boldsymbol{y} = g_a(\boldsymbol{y}). \tag{88}$$

We now have,

$$g_a(\boldsymbol{y}) = \sum_j a_j y_j^2, \tag{89}$$

so, $g_a(\boldsymbol{y}) = c$, corresponds to an ellipsoid with axes aligned with the y-coordinate axes.

Clearly, in the general case a rotation will only make a single ellipsoid aligned with the coordinate axes. By also allowing scaling transformations, however, a second quadratic form can be made diagonal. Assume that the mass matrix $\boldsymbol{M}$ is positive definite. After a rotation has aligned its corresponding ellipsoid with the x-coordinate axes one can scale the lengths, $\sqrt{m}\, y_j = \sqrt{m_j}\, x_j$, along the different axes so that $g_m(\boldsymbol{x}) = \sum_j m_j x_j^2 = m \sum_j y_j^2$. Here m is e.g. the average mass $nm = \sum_j^n m_j$. The mass matrix then becomes $\boldsymbol{M} = m\mathbf{1}$ and its ellipsoid is an n-dimensional sphere, $S_n(\boldsymbol{y}) = \sum_j^n y_j^2$. A sphere is unaffected by rotations so now a second quadratic form can be diagonalized by rotation without affecting the mass matrix (Goldstein [9]). Mathematically this is equivalent to defining the scalar product with the mass matrix, but engineers are likely to prefer the above geometric reasoning.

Many texts on matrix theory point out that one can define functions of matrices, $f(\boldsymbol{A})$, by means of power series (Cullen [6], Stephenson [22], Miller [17]). It is then necessary that the eigenvalues of the matrix all lie inside the radius of convergence of the power series of $f(z)$. In particular the exponential of matrix can always be defined through

$$\exp \boldsymbol{A} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} \tag{90}$$

and this can be extended so that,

$$\exp(\mathrm{i}\boldsymbol{A}) = \cos \boldsymbol{A} + \mathrm{i} \sin \boldsymbol{A}, \tag{91}$$

always holds, by definition. For a diagonal matrix one has

$$f(\boldsymbol{A}_{\mathrm{d}}) = \begin{pmatrix} f(a_1) & 0 & \cdots \\ 0 & f(a_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{92}$$

and the eigenvalues of $f(\boldsymbol{A})$ are $f(a_i)$. A matrix $\boldsymbol{B}$ is said to be similar to $\boldsymbol{A}$

if, $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$, for some nonsingular matrix $\boldsymbol{S}$. A central result is that,

$$\exp(\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}) = \boldsymbol{S}^{-1}\exp(\boldsymbol{A})\boldsymbol{S}, \tag{93}$$

the exponential of the similarity transformed matrix is equal to the similarity transformed exponential of the original matrix. Similar matrices have the same eigenvalues

Many operations that can be done on numbers are also valid for matrices. The most important exception is the fact that matrix multiplication does not commute. Instead

$$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A} + [\boldsymbol{A}, \boldsymbol{B}], \tag{94}$$

where

$$[\boldsymbol{A}, \boldsymbol{B}] \equiv \boldsymbol{A}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{A} \tag{95}$$

is the commutator of the matrices. It is also important to realize that, in general,

$$\exp(\boldsymbol{A} + \boldsymbol{B}) \neq \exp(\boldsymbol{A})\exp(\boldsymbol{B}). \tag{96}$$

These matrices are equal only when the commutator $[\boldsymbol{A}, \boldsymbol{B}]$ is zero.

By means of these facts one can solve the first order differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{97}$$

and get

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}t)\boldsymbol{x}(0) \tag{98}$$

for the solution [6]. We generalize this result to vibration problems in this article.

B. Appendix: Real Symmetric Two by Two Matrices

The formulas for the diagonalization and the eigenvalues of a real symmetric two by two matrix are here briefly given for reference. The matrix

$$\boldsymbol{A} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{99}$$

is diagonalized by

$$\boldsymbol{U} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{100}$$

and

$$\boldsymbol{A}_{\rm d} = \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} = \widetilde{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U}. \tag{101}$$

This means that we must have

$$\boldsymbol{U}\boldsymbol{A}_{\rm d}\widetilde{\boldsymbol{U}} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{102}$$

$$= \begin{pmatrix} a_0\cos^2\alpha + b_0\sin^2\alpha & (a_0-b_0)\sin\alpha\cos\alpha \\ (a_0-b_0)\sin\alpha\cos\alpha & a_0\sin^2\alpha + b_0\cos^2\alpha \end{pmatrix} = \boldsymbol{A} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \tag{103}$$

and thus a_0, b_0, and the angle α are determined by,

$$a_0\cos^2\alpha + b_0\sin^2\alpha = a, \tag{104}$$

$$a_0\sin^2\alpha + b_0\cos^2\alpha = b, \tag{105}$$

$$(a_0-b_0)\sin\alpha\cos\alpha = c. \tag{106}$$

From this one finds that,

$$a_0 = \frac{1}{2}\left[(a+b) + \sqrt{(a-b)^2+4c^2}\right], \tag{107}$$

$$b_0 = \frac{1}{2}\left[(a+b) - \sqrt{(a-b)^2+4c^2}\right], \tag{108}$$

$$\cos\alpha = \sqrt{\frac{1}{2}\left(1 + \frac{|a-b|}{\sqrt{(a-b)^2+4c^2}}\right)}, \tag{109}$$

$$\sin\alpha = \frac{(a-b)c}{|(a-b)c|}\sqrt{\frac{1}{2}\left(1 - \frac{|a-b|}{\sqrt{(a-b)^2+4c^2}}\right)}. \tag{110}$$

Here a_0 and b_0 are the eigenvalues of $\boldsymbol{A}$.

C. Appendix: Matrix Symmetry and Antisymmetry

Consider the matrix equations (36) and (37):

$$\boldsymbol{\Omega}^2 - \boldsymbol{\eta}^2 + [\boldsymbol{\eta}, \boldsymbol{\zeta}] = \boldsymbol{\Omega}_0^2, \tag{111}$$

$$(\boldsymbol{\Omega}\boldsymbol{\eta} + \boldsymbol{\eta}\boldsymbol{\Omega}) - [\boldsymbol{\Omega}, \boldsymbol{\zeta}] = \boldsymbol{0}, \tag{112}$$

where it is known that $\boldsymbol{\Omega}_0^2$ and $\boldsymbol{\zeta}$ are real symmetric and that $\boldsymbol{\Omega}$ and $\boldsymbol{\eta}$ are real. We now show that these equations are consistent with the assumption that $\boldsymbol{\Omega}$ is symmetric and that $\boldsymbol{\eta}$ is antisymmetric.

Let $\boldsymbol{\sigma}$ denote a symmetric matrix,

$$\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \tag{113}$$

and $\boldsymbol{\varepsilon}$ an antisymmetric matrix,

$$\widetilde{\boldsymbol{\varepsilon}} = -\boldsymbol{\varepsilon}. \tag{114}$$

In our calculations below we will make frequent use of the matrix identity

$$\widetilde{\boldsymbol{A}\boldsymbol{B}} = \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{A}}, \tag{115}$$

as well as the commutativity of matrix addition: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$.

First we show that the product of a symmetric matrix $\boldsymbol{\sigma}$ with itself is symmetric. Put $\boldsymbol{A} = \boldsymbol{\sigma}^2$. Then

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{\sigma}^2} = \widetilde{\boldsymbol{\sigma}\boldsymbol{\sigma}} = \widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\boldsymbol{\sigma} = \boldsymbol{A}, \tag{116}$$

so $\widetilde{\boldsymbol{A}} = \boldsymbol{A}$ and the square is symmetric. This means that the first term in equation (111) is symmetric if $\boldsymbol{\Omega}$ is symmetric.

We now show that the square of an antisymmetric matrix $\boldsymbol{\varepsilon}$ is symmetric. Put $\boldsymbol{A} = \boldsymbol{\varepsilon}^2$. Then

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{\varepsilon}^2} = \widetilde{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}} = \widetilde{\boldsymbol{\varepsilon}}\widetilde{\boldsymbol{\varepsilon}} = (-\boldsymbol{\varepsilon})(-\boldsymbol{\varepsilon}) = \boldsymbol{A}, \tag{117}$$

so $\widetilde{\boldsymbol{A}} = \boldsymbol{A}$ and the square is symmetric. This means that the second term in equation (111) is symmetric if $\boldsymbol{\eta}$ is antisymmetric.

We now show:

Theorem 1. *The commutator of an antisymmetric matrix with a symmetric matrix is symmetric.*

Proof. Put $\boldsymbol{A} = [\boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$. Then

$$\begin{aligned} \widetilde{\boldsymbol{A}} = \widetilde{[\boldsymbol{\varepsilon}, \boldsymbol{\sigma}]} &= \widetilde{\boldsymbol{\varepsilon}\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}\boldsymbol{\varepsilon}} = \widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\varepsilon}} - \widetilde{\boldsymbol{\varepsilon}}\widetilde{\boldsymbol{\sigma}} \\ &= \boldsymbol{\sigma}(-\boldsymbol{\varepsilon}) - (-\boldsymbol{\varepsilon})\boldsymbol{\sigma} = \boldsymbol{\varepsilon}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}, \boldsymbol{\sigma}] = \boldsymbol{A}, \end{aligned} \tag{118}$$

so $\widetilde{\boldsymbol{A}} = \boldsymbol{A}$ and the commutator is symmetric. □

Since $\boldsymbol{\zeta}$ is symmetric this means that the third term in equation (111) is symmetric if $\boldsymbol{\eta}$ is antisymmetric. Thus all terms of equation (111) are symmetric under our assumptions.

We now move to equation (112) and show that both terms are antisymmetric. We first show:

Theorem 2. *The symmetrized product* $(\boldsymbol{A}, \boldsymbol{B}) \equiv \boldsymbol{AB} + \boldsymbol{BA}$ *of a symmetric and an antisymmetric matrix is antisymmetric.*

Proof. Put $\boldsymbol{A} = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$. We then have:

$$\widetilde{\boldsymbol{A}} = \widetilde{(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})} = \widetilde{\boldsymbol{\sigma\varepsilon}} + \widetilde{\boldsymbol{\varepsilon\sigma}} = \widetilde{\boldsymbol{\varepsilon}}\widetilde{\boldsymbol{\sigma}} + \widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\varepsilon}} \tag{119}$$

$$= -\boldsymbol{\varepsilon\sigma} - \boldsymbol{\sigma\varepsilon} = -(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = -(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = -\boldsymbol{A}. \tag{120}$$

The first term of equation (112) is thus antisymmetric if $\boldsymbol{\Omega}$ is symmetric and $\boldsymbol{\eta}$ is antisymmetric. □

We finally show:

Theorem 3. *The commutator of two symmetric matrices is an antisymmetric matrix.*

Proof. Put $\boldsymbol{A} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2]$. Then

$$\widetilde{\boldsymbol{A}} = [\widetilde{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2}] = \widetilde{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2} - \widetilde{\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1} = \widetilde{\boldsymbol{\sigma}_2}\widetilde{\boldsymbol{\sigma}_1} - \widetilde{\boldsymbol{\sigma}_1}\widetilde{\boldsymbol{\sigma}_2} \tag{121}$$

$$= \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = [\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1] = -[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] = -\boldsymbol{A}.$$

The commutator is thus antisymmetric. □

Since $\boldsymbol{\zeta}$ is symmetric the second term in equation (112) is antisymmetric assuming that $\boldsymbol{\Omega}$ is symmetric. All terms of equation (112) are thus antisymmetric under our assumptions. This concludes our proof.

D. Appendix: Rotation Matrices

The elements of a rotation matrix $\boldsymbol{U}$ are scalar products $(\boldsymbol{e}'_i \cdot \boldsymbol{e}_j)$ of the basis vectors of the original and of the rotated orthonormal bases. From this fact one easily concludes that a rotation matrix has the property,

$$\boldsymbol{U}^{-1} = \widetilde{\boldsymbol{U}}, \tag{122}$$

i.e. the inverse is equal to the transpose.

The exponential of an antisymmetric matrix,

$$\boldsymbol{A} = \exp(\boldsymbol{\varepsilon}), \tag{123}$$

obeys,

$$\boldsymbol{A}^{-1} = \exp(-\boldsymbol{\varepsilon}) = \exp(\widetilde{\boldsymbol{\varepsilon}}) = \widetilde{\boldsymbol{A}}. \tag{124}$$

It must therefore be a rotation matrix (also called an orthogonal matrix).

E. Appendix: Diagonalizing Two by Two Matrices

Consider the two by two matrix

$$\boldsymbol{A} = \begin{pmatrix} a & c \\ d & b \end{pmatrix}. \tag{125}$$

Here we give formulas for its eigenvalues, a_1, a_2, and the similarity transformation $\boldsymbol{S}$ that transforms $\boldsymbol{A}$ to its diagonal form,

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \boldsymbol{A}_{\mathrm{d}}, \tag{126}$$

and some conditions for this to be possible. See Iooss and Joseph [13] for a fuller discussion.

The eigenvalues are found from the secular equation

$$|\boldsymbol{A} - x\mathbf{1}| = \begin{vmatrix} a-x & c \\ d & b-x \end{vmatrix} = 0, \tag{127}$$

and are found to be,

$$a_{1,2} = \frac{1}{2}\left[(a+b) \pm \sqrt{(a-b)^2 + 4cd}\right]. \tag{128}$$

If we can find eigenvectors (column matrices), $\boldsymbol{v}_1, \boldsymbol{v}_2$, corresponding to these eigenvalues they can be used to construct the similarity transformation matrices, as we now show.

The eigenvectors are solutions of

$$(\boldsymbol{A} - a_j\mathbf{1})\,\boldsymbol{v}_j = \mathbf{0} \qquad (j = 1, 2), \tag{129}$$

and since the determinant $|\boldsymbol{A} - a_j\mathbf{1}|$ is zero, one of the two component equations suffices to determine the components of $\boldsymbol{v}_j$. For example, we may simply take,

$$\boldsymbol{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} c \\ -(a-a_1) \end{pmatrix}, \tag{130}$$

and

$$\boldsymbol{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} -(b-a_2) \\ d \end{pmatrix}. \tag{131}$$

Such solutions are clearly defined only modulo arbitrary non-zero factors. A convenient choice is to take this factor as the inverse square root of the determinant of the matrix

$$\boldsymbol{V} = (\boldsymbol{v}_1\ \boldsymbol{v}_2) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}. \tag{132}$$

This determinant is given by

$$\det \boldsymbol{V} \equiv V = \frac{1}{2}(a-b)\left[(a-b) - \sqrt{(a-b)^2 + 4cd}\right] + 2cd. \tag{133}$$

and the thus normalized eigenvectors are then,

$$\boldsymbol{s}_1 = \frac{1}{\sqrt{V}}\boldsymbol{v}_1 = \begin{pmatrix} s_{11} \\ s_{21} \end{pmatrix} = \begin{pmatrix} c/\sqrt{V} \\ -(a-a_1)/\sqrt{V} \end{pmatrix}, \tag{134}$$

and,

$$\boldsymbol{s}_2 = \frac{1}{\sqrt{V}}\boldsymbol{v}_2 = \begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix} = \begin{pmatrix} -(b-a_2)/\sqrt{V} \\ d/\sqrt{V} \end{pmatrix}. \tag{135}$$

If we now define,

$$\boldsymbol{S} = (\boldsymbol{s}_1\ \boldsymbol{s}_2) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \tag{136}$$

we see that

$$\det \boldsymbol{S} = 1, \tag{137}$$

and finally that,

$$\boldsymbol{AS} = (\boldsymbol{As}_1\ \boldsymbol{As}_2) = (a_1\boldsymbol{s}_1\ a_2\boldsymbol{s}_2), \tag{138}$$

is the result of multiplying $\boldsymbol{S}$ by $\boldsymbol{A}$ from the left.

We now invert $\boldsymbol{S}$. Since the determinant is unity the inverse is simply given given by

$$\boldsymbol{S}^{-1} = \begin{pmatrix} \widetilde{\boldsymbol{s}}^1 \\ \widetilde{\boldsymbol{s}}^2 \end{pmatrix} = \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix}. \tag{139}$$

Here, $\widetilde{\boldsymbol{s}}^1, \widetilde{\boldsymbol{s}}^2$, are row matrices that correspond to left eigenvectors of $\boldsymbol{A}$.

If we multiply equation (138) by $\boldsymbol{S}^{-1}$ from the left we find

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{S}^{-1}(a_1\boldsymbol{s}_1\; a_2\boldsymbol{s}_2) = \begin{pmatrix} a_1\widetilde{\boldsymbol{s}}^1\boldsymbol{s}_1 & a_2\widetilde{\boldsymbol{s}}^1\boldsymbol{s}_2 \\ a_1\widetilde{\boldsymbol{s}}^2\boldsymbol{s}_1 & a_2\widetilde{\boldsymbol{s}}^2\boldsymbol{s}_2 \end{pmatrix}, \tag{140}$$

and since $\boldsymbol{S}^{-1}\boldsymbol{S} = \boldsymbol{1}$ we see that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \boldsymbol{A}_{\rm d}. \tag{141}$$

The matrix $\boldsymbol{S}$ thus, indeed, diagonalizes $\boldsymbol{A}$.

The construction of the matrix $\boldsymbol{S}$ and its inverse will fail if the determinant is zero. The determinant must be proportional to V given by equation (133) and $V = 0$ if either, $cd = 0$, or, $cd = -(a-b)^2/4$. If both c and d are zero the matrix is already diagonal so a diagonalization by the the above procedure, will fail if c or d is zero, but not both. The other failing case, $cd = -(a-b)^2/4$, corresponds to the two eigenvalues being equal.

F. Appendix: The Cubic Equation and its Roots

We will here first make plausible that the cubic equation (55), that determines X through $w = X^2$, namely,

$$w^3 - Rw^2 + Sw + T = 0, \tag{142}$$

with

$$R \equiv 2(a^2+b^2) - (\alpha^2 + 4\beta^2), \tag{143}$$

$$S \equiv (a^2-b^2)^2 - 2(\alpha^2+4\beta^2)(a^2+b^2), \tag{144}$$

$$T \equiv \alpha^2(a^2-b^2)^2, \tag{145}$$

has three real roots in the underdamped case, and that $X = \sqrt{w}$ is given by the largest of these roots. We then give formulas for these roots.

First recall that α and β are essentially elements of the damping matrix $\boldsymbol{\zeta}$ of equation (43). For the underdamped case these must be small enough for $\boldsymbol{\Omega}_0^2 = \boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2$, of equation (42), to remain positive definite. If we put $\alpha = \beta = 0$ in (142) –(145) we get

$$w^3 - 2(a^2+b^2)w^2 + (a^2-b^2)^2 w = 0. \tag{146}$$

This equation has the roots

$$w_{1,2,3} = \begin{cases} 0\,, \\ (a-b)^2\,, \\ (a+b)^2\,, \end{cases} \tag{147}$$

and it is clear that the largest one corresponds to $X \equiv A + B = \sqrt{w_3} = a + b$.

One also notes that as long as $\alpha = 0$ the equation remains essentially quadratic, in that $w = 0$ is a trivial root. For $\alpha = 0$ but non-zero β equations (142)-(145) give

$$w^3 - [2(a^2+b^2) - 4\beta^2]w^2 + [(a^2-b^2)^2 - 8\beta^2(a^2+b^2)]w = 0 \tag{148}$$

and the non-zero roots are

$$w_{2,3} = a^2 + b^2 - 2\beta^2 \mp 2\sqrt{a^2b^2 + \beta^2(a^2+b^2) + \beta^4}. \tag{149}$$

One sees that $w_2 < (a-b)^2$ and that $w_3 > (a+b)^2$.

We now present the required solution of the cubic equation. For a recent pedagogical account, see McKelvey [15]. In our solution of equations (142)-(145) we assume that $R > 0$ and that S is considerably smaller than R, but possibly negative, and that T is quite small. If we make the substitution,

$$w = \frac{R}{3} + x, \tag{150}$$

we get,

$$x^3 - 3Px - 2Q = 0, \tag{151}$$

where,

$$P \equiv \frac{R^2}{9} - \frac{S}{3} \;\text{ and, }\; Q \equiv \frac{R^3}{27} - \frac{RS}{6} - \frac{T}{2}. \tag{152}$$

If we now put (assuming $P^3 \geq Q^2$),

$$\Xi \equiv \left(Q + \mathrm{i}\sqrt{P^3 - Q^2}\right)^{1/3}, \tag{153}$$

we get from Cardano's formula [24] for the roots of equation (142),

$$w_{1,2} \;=\; \frac{R}{3} - \mathrm{Re}\,\Xi \mp \sqrt{3}\,\mathrm{Im}\,\Xi, \tag{154}$$

$$w_3 \;=\; \frac{R}{3} + 2\,\mathrm{Re}\,\Xi, \tag{155}$$

in increasing order. Finally then $X(a, b, \alpha, \beta)$ is given by $X = \sqrt{w_3}$.

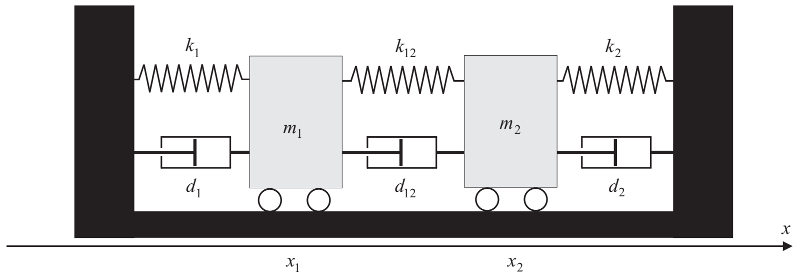

Figure 4: A specific two degree-of-freedom system. x_1 and x_2 represent the deviations of the masses from their equilibrium positions along the x-axis.

G. Appendix: A Class of Example Systems

Consider the system of Figure 4. For this system we have equations of motion

$$m_1\ddot{x}_1 = -k_1x_1 - d_1\dot{x}_1 - k_{12}(x_1 - x_2) - d_{12}(\dot{x}_1 - \dot{x}_2)\,, \quad (156)$$

$$m_2\ddot{x}_2 = -k_2x_2 - d_2\dot{x}_2 - k_{12}(x_2 - x_1) - d_{12}(\dot{x}_2 - \dot{x}_1)\,, \quad (157)$$

and the basic matrices of equation (1) are then,

$$\boldsymbol{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad (158)$$

$$2\boldsymbol{D} = \begin{pmatrix} d_1 + d_{12} & -d_{12} \\ -d_{12} & d_2 + d_{12} \end{pmatrix}, \quad (159)$$

and

$$\boldsymbol{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}. \quad (160)$$

We see that for this type of system the mass matrix is always diagonal. A non-diagonal mass matrix is only found in systems with constraints, such as the e.g. double pendulum.

The matrix $\boldsymbol{M}^{-1/2}$ simply has, $\sqrt{1/m_1}$, $\sqrt{1/m_2}$, as diagonal elements so, if we introduce

$$\bar{m} \equiv \sqrt{m_1 m_2}, \quad (161)$$

for the geometric mean mass, we get from equation (18) that

$$2\boldsymbol{\zeta} = \begin{pmatrix} (d_1 + d_{12})/m_1 & -d_{12}/\bar{m} \\ -d_{12}/\bar{m} & (d_2 + d_{12})/m_2 \end{pmatrix}, \tag{162}$$

and from equation (6) that

$$\boldsymbol{\omega}^2 = \begin{pmatrix} (k_1 + k_{12})/m_1 & -k_{12}/\bar{m} \\ -k_{12}/\bar{m} & (k_2 + k_{12})/m_2 \end{pmatrix}. \tag{163}$$

The relevant coordinates are now $\tilde{\boldsymbol{y}} = (y_1, y_2) = (\sqrt{m_1}\, x_1, \sqrt{m_2}\, x_2)$, according to equation (4).

To proceed we must now find $\boldsymbol{\Omega}_0^2 = \boldsymbol{\omega}^2 - \boldsymbol{\zeta}^2$ of equation (25). A simple calculation gives

$$\boldsymbol{\Omega}_0^2 = \begin{pmatrix} \frac{k_1+k_{12}}{m_1} - \left[\frac{d_1+d_{12}}{2m_1}\right]^2 - \frac{d_{12}^2}{4\bar{m}^2} & \frac{d_{12}}{4\bar{m}}\left[\frac{d_1}{m_1} + \frac{d_2}{m_2} + \frac{d_{12}}{\mu}\right] - \frac{k_{12}}{\bar{m}} \\ \frac{d_{12}}{4\bar{m}}\left[\frac{d_1}{m_1} + \frac{d_2}{m_2} + \frac{d_{12}}{\mu}\right] - \frac{k_{12}}{\bar{m}} & \frac{k_2+k_{12}}{m_2} - \left[\frac{d_2+d_{12}}{2m_2}\right]^2 - \frac{d_{12}^2}{4\bar{m}^2} \end{pmatrix}, \tag{164}$$

where we have introduced,

$$\mu \equiv m_1 m_2/(m_1 + m_2), \tag{165}$$

for the reduced mass.

Finally we should transform to coordinates $\boldsymbol{q}$ that diagonalize $\boldsymbol{\Omega}_0^2$ according to equation (31). To avoid further algebra here we may instead assume that the system is such that $\boldsymbol{\Omega}_0^2$ already is diagonal. This is the case when the off-diagonal elements of $\boldsymbol{\omega}^2$ and $\boldsymbol{\zeta}^2$ are equal; a kind of off-diagonal critical damping. Algebraically this means that

$$k_{12} = \frac{d_{12}}{4}\left(\frac{d_1}{m_1} + \frac{d_2}{m_2} + \frac{d_{12}}{\mu}\right), \tag{166}$$

or equivalently,

$$d_{12} = \frac{\mu}{2}\left[\frac{d_1}{m_1} + \frac{d_2}{m_2}\right]\left(\sqrt{1 + 16\left[\frac{d_1}{m_1} + \frac{d_2}{m_2}\right]^{-2}\frac{k_{12}}{\mu}} - 1\right). \tag{167}$$

Note that neither $\boldsymbol{\omega}^2$ nor $\boldsymbol{\zeta}$ need to be diagonal in order for $\boldsymbol{\Omega}_0^2$ to have this property. With the choice (166) inserted in (164) we get

$$\boldsymbol{\Omega}_0^2 = \begin{pmatrix} \frac{k_1}{m_1} - \frac{d_1^2}{4m_1^2} - \frac{d_{12}}{4m_1}\left[\frac{d_1}{m_1} - \frac{d_2}{m_2}\right] & 0 \\ 0 & \frac{k_2}{m_2} - \frac{d_2^2}{4m_2^2} - \frac{d_{12}}{4m_2}\left[\frac{d_1}{m_1} - \frac{d_2}{m_2}\right] \end{pmatrix}, \tag{168}$$

for the matrix of squared angular frequencies.

In Section 7 we introduced the notation,

$$\boldsymbol{\zeta} = \begin{pmatrix} (\gamma+\alpha)/2 & \beta \\ \beta & (\gamma-\alpha)/2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\Omega}_0^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}. \tag{169}$$

Comparing with (162) we see that

$$-\frac{d_{12}}{2\bar{m}} = \beta, \tag{170}$$

$$\frac{d_1}{m_1} = \gamma + \alpha + 2\frac{\bar{m}}{m_1}\beta, \tag{171}$$

$$\frac{d_2}{m_2} = \gamma - \alpha + 2\frac{\bar{m}}{m_2}\beta. \tag{172}$$

If we compare the expression for $\boldsymbol{\Omega}_0^2$ of (169) with (168) we find

$$\frac{k_1}{m_1} = a^2 + \frac{\gamma\alpha}{2} + \frac{\alpha^2}{4} + \frac{\gamma^2}{4} + \frac{\bar{m}}{m_1}\gamma\beta + \beta^2, \tag{173}$$

$$\frac{k_2}{m_2} = b^2 - \frac{\gamma\alpha}{2} + \frac{\alpha^2}{4} + \frac{\gamma^2}{4} + \frac{\bar{m}}{m_2}\gamma\beta + \beta^2. \tag{174}$$

This thus determines the physical parameters as functions of the five variables $a, b, \alpha, \beta, \gamma$. To get the "45-degree damping" case we simply put $\alpha = 0$ here to find how the physical parameters must vary with β if a, b, and γ are to remain constant.