International Journal of Pure and Applied Mathematics
Volume 36 No. 1 2007, 13-26

APPLICATION OF ONCE INTEGRATED SEMIGROUPS FOR THE ABSTRACT BOUNDARY CONTROL

Tadeusz Kowalski[1], Wawrzyniec Sadkowski[2 §]
[1,2]Faculty of Mathematics and Information Sciences
Warsaw University of Technology
Pl. Politechniki 1, Warsaw, 00-661, POLAND

Abstract: An equivalent condition for approximate controlability of boundary abstract control problem, with A as a generator of a once integrated semigroup has been given. The approximate controlability of some hyperbolic problems with Neumann and Dirichlet boundary conditions have been proved.

AMS Subject Classification: 93C25
Key Words: once integrated semigroups, the abstract boundary control problem, approximate controlability of boundary control problems for some hyperbolic problems with Neumann and Dirichlet boundary conditions

1. Introduction

This paper consists of four parts. In the first one, some properties of once integrated semigroups have been presented. Once integrated semigroups are generalisation of C_0 semigroups [2], [4], [5]. In [6], the n-times integrated semigroups have been applied to the existence and uniqueness of classical, mild and integrated solutions of abstract Cauchy problems.

In the second part some theorems related to the abstract control boundary

$$\begin{cases} \dfrac{dz}{dt} = \Lambda z(t),\ t \geq 0\ z(0) = z_0, \\ \beta z(t) = u(t) \end{cases} \tag{1}$$

problems were proved in the case when the operator $A : D(A) \mapsto Z$ with $D(A) = D(\Lambda) \cap \ker(\beta)$ and $Az = \Lambda z$ for $z \in D(A)$ is the generator of a non-

Received: August 14, 2006
§Correspondence author

degenerate once integrated semigroups on Z and β is boundary operator. For C_0 semigroups such a problem has been investigated in [1].

In the third part, a once integrated semigroup has been used to prove the existence and the uniqueness of the integrated solutions for the following problems

$$\text{(2a)} \qquad \begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} \quad \text{for } x \in \langle 0,1\rangle \text{ and } t \in (0,T\rangle, \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \ \dfrac{\partial w(1,t)}{\partial x} = u(t) \quad \text{for } t \in \langle 0,T\rangle, \\ w(x,0) = w_1(x), \ w_t(x,t)|_{t=0} = w_2(x) \quad \text{for } x \in \langle 0,1\rangle, \end{cases}$$

$$\text{(2b)} \qquad \begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} \quad \text{for } x \in \langle 0,1\rangle \text{ and } t \in (0,T\rangle, \\ w(0,t) = 0, \ w(1,t) = u(t) \quad \text{for } t \in \langle 0,T\rangle \\ w(x,0) = w_1(x), \ w_t(x,t)|_{t=0} = w_2(x) \quad \text{for } x \in \langle 0,1\rangle. \end{cases}$$

In the last part, an approximate controlability of the systems (2a) and (2b) were proved.

2. Integrated Solutions to Cauchy Problem

Let X be a Banach space.

Definition 1. A strongly continuous family of operators $\{S(t)\}_{,t\geq 0} \subset \mathcal{L}(X)$ is called once integrated semigroups if

$$S(0) = 0 \tag{3}$$

and

$$S(t)S(s) = \int_t^{s+t} S(r)dr - \int_0^s S(r)dr \quad \text{for all} \quad s,t \geq 0. \tag{4}$$

Definition 2. $\{S(t)\}_{,t\geq 0}$ is non-degenerate if, whenever $S(t)x = 0$, for all $t \geq 0$, then x must equal to 0.

Definition 3. An once integrated semigroup $\{S(t)\}_{t\geq 0}$ is exponentially bounded if there exist $M > 0$ and $\omega \in R$ such that $||S(t)||_{\mathcal{L}(X)} \leq Me^{\omega t}$, for $t \geq 0$.

Definition 4. An operator valued function $R(\lambda)$, mapping $(\omega,\infty) \mapsto \mathcal{L}(X)$ for some $\omega \in R$, is said to be a Laplace transform if there exists a strongly continuous operator valued function $S(t)$ mapping $[0,\infty) \mapsto \mathcal{L}(X)$ satisfying $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, $t \geq 0$, for some $M > 0$, such that $R(\lambda) = \int_0^\infty e^{-\lambda t} S(t) dt$ for $\lambda > \omega$.

Definition 5. $G_1(X) \equiv \{A \in \mathcal{L}(X)$: the class of linear, not necessarily bounded operators in X, $\rho(A) \neq \theta$, and $R_1(\lambda, A) \equiv \frac{R(\lambda,A)}{\lambda}$ is a Laplace transform$\}$. An operator $A \in G_1(X)$ is called the generator of a strongly continuous once integrated semigroup.

Consider the inhomogeneous Cauchy problem

$$\frac{dw(t)}{dt} = Aw(t) + f(t), \quad 0 < t < T, \; w(0) = w_0. \tag{5}$$

Integrating twice the equation (5) we obtain

$$v(t) - tw_0 = A \int_0^t v(r) dr + \int_0^t (t-s) f(s) ds, \tag{6}$$

where $v(t) = \int_0^t w(s) ds$.

Definition 6. Let $f : [0,T) \to X$ be Bochner integrable. A continuous function $v : [0,T) \to X$ is called an integrated solution to (5) iff $\int_0^t v(s) ds \in D(A)$, and v satisfies the integral equation

$$v(t) - tw_0 = A \int_0^t v(r) dr + \int_0^t (t-s) f(s) ds \quad \text{for } 0 \leq t < T.$$

Theorem 1. *Let $f : [0,T) \to X$ be Bochner integrable and A be the generator of a non-degenerate once integrated semigroup $\{S(t)\}_{,t\geq 0}$. Then the following formula*

$$v(t) = S(t) w_0 + \int_0^t S(t-r) f(r) dr \tag{7}$$

gives the unique integrated solution to (5) *for any $w_0 \in X$,* [6].

3. Abstract Boundary Control Problems

Let Z, U be separable Hilbert spaces. Consider of the abstract boundary control problem of (1), where $\Lambda : D(\Lambda) \subset Z \mapsto Z$ is a operator, $u(t) \in U$ and the boundary operator $\beta : D(\beta) \subset Z \mapsto U$ satisfies condition $D(\Lambda) \subset D(\beta)$. Λ and β are linear and closed operators. In order to reformulate system (1) as a Cauchy problem, the additional conditions on the abstract boundary system will be imposed.

Definition 7. The control system of (1) is a boundary control system if the followings hold:

a) The operator $A : D(A) \mapsto Z$ with $D(A) = D(\Lambda) \cap \ker(\beta)$ and

$$Az = \Lambda z \quad \text{for } z \in D(A) \tag{8}$$

is the generator of a non-degenerate once integrated semigroups $S(t)$ on Z;

b) There exists a $B \in \mathcal{L}(U, Z)$ such that for all $u \in U$, $Bu \in D(\Lambda)$, the operator ΛB is an element of $\mathcal{L}(U, Z)$ and

$$\beta B u = u, \quad \text{for } u \in U. \tag{9}$$

Theorem 2. *If A is the generator of a non-degenerate once integrated semigroups $S(t)$, $t \geq 0$ on Z; B and ΛB are bounded linear operators, $w_0 \in Z$, $\dot{u}, u \in L_p([0,T]; U)$ for some $p \geq 1$, then the abstract Cauchy problem*

$$\begin{cases} \dfrac{dw(t)}{dt} = Aw(t) - B\dfrac{du(t)}{dt} + \Lambda B u(t)\,, \\ w(0) = w_0 \end{cases} \tag{10}$$

has a unique integrated solution $v(t)$ and it is given by the formula

$$v(t) = S(t)w_0 + \int_0^t S(t-s)[-B\frac{du(s)}{ds} + \Lambda B u(s)]ds. \tag{11}$$

This solution satisfies also the integral equation

$$v(t) - tw_0 = A\int_0^t v(p)dp + tBu(0) + \int_0^t (t-s)\Lambda B u(s)ds - \int_0^t Bu(s)ds. \tag{12}$$

Proof. Theorem 1 guarantes the existence of the unique integrated solution $v(t)$ given by (11). It also satisfies the integral equation

$$v(t) - tw_0 = A\int_0^t v(p)dp + \int_0^t (t-s)[-B\dot{u}(s)ds + \Lambda B u(s)]ds.$$

The previous equality implies (12). □

Definition 8. A continuous function $r : [0,T) \to Z$ is called an integrated solution of (1) iff $\int_0^t r(s)ds \in D(\Lambda)$, r satisfies the integral equation $r(t) = \Lambda \int_0^t r(p)dp + tz_0$ and boundary conditions are satisfied in the following way $\beta \int_0^t r(s)ds = \int_0^t (t-s)u(s)ds$.

Theorem 3. *If A is the generator of a non-degenerate once integrated semigroups $S(t)$, $t\geq 0$ on Z, B and ΛB are bounded linear operators, $w_0 = z_0 - Bu(0) \in Z$, $\dot{u}, u \in L_p([0,T];U)$ for some $p\geq 1$, then the integrated solutions $r(t)$, $w(t)$ of* (1) *and* (10)*, respectively, are related by*

$$v(t) = r(t) - \int_0^t Bu(s)ds. \tag{13}$$

Furthemore, the integrated solution of (1) *is unique.*

Proof. Let us denote $r(t)$, $w(t)$ integrated solutions of (1) and (10), respectively. The assumptions of this theorem quarantees the existence of the unique integrated solution $v(t)$ of (10) and it is given by (11). By (13) and (12) we have

$$\begin{aligned} r(t) &= v(t) + \int_0^t Bu(s)ds = tw_0 + A\int_0^t v(s)ds + tBu(0) \\ &\quad + \int_0^t (t-s)\Lambda Bu(s)ds - \int_0^t Bu(s)ds + \int_0^t Bu(s)ds \\ &= t[w_0 + Bu(0)] + A\int_0^t v(s)ds + \Lambda B\int_0^t\int_0^s u(p)dpds \\ &= t[w_0 + Bu(0)] + \Lambda\int_0^t v(s)ds + \Lambda\int_0^t\int_0^s Bu(p)dpds \\ &= tz_0 + \Lambda\int_0^t [v(s) + \int_0^s Bu(p)dp]ds \\ &= tz_0 + \Lambda\int_0^t r(p)dp. \end{aligned}$$

We have used the property of the integrated solution $\int_0^t v(s)ds \in D(A)$ and the equality (8). Thus if $v(t)$ is an integrated solution of (10), then $r(t)$ is an integrated solution of (1). In a similar way we can prove that if $r(t)$ is an inegrated solution of (1), then $v(t)$ is an integrated solution of (10). The property $\int_0^t v(s)ds \in D(A)$ and assumptions $D(A) \subset D(\Lambda) \subset D(\beta)$, $Bu(t) \in D(\beta)$, $\beta B = I$, imply

$$\begin{aligned} \beta(\int_0^t r(s)ds) &= \beta\Big(\int_0^t v(s)ds\Big) + \beta\Big(\int_0^t\int_0^s Bu(p)dpds\Big) \\ &= \beta\int_0^t (t-s)Bu(s)ds = \int_0^t (t-s)u(s)ds. \end{aligned}$$

So the boundary condition for the integrated solution $r(t)$ is satisfied. □

Remark 1. The integrated solution $r(t)$ of (1) is given by the formula

$$\begin{aligned} r(t) &= S(t)z_0 - S(t)Bu(0) + \int_0^t Bu(s)ds \\ &\quad - \int_0^t S(t-s)B(s)\dot{u}(s)ds + \int_0^t S(t-s)\Lambda Bu(s)ds. \end{aligned} \tag{14}$$

Definition 9. Let $R_b = \{z \in Z \mid$ there exists a $\tau > 0$ and a differentiable control function u such that $u(0) = 0$, u, $\dot{u} \in L_2([0,\tau];U)$ and

$$z = \int_0^\tau Bu(s)ds - \int_0^\tau S(\tau-s)B\dot{u}(s)ds + \int_0^\tau S(\tau-s)\Lambda Bu(s)ds\}.$$

The boundary control system is called approximately controlable on $[0,\tau]$ if R_b is dense in Z.

Lemma. *Let V, W, Z be Banach spaces and $F \in \mathcal{L}(V,Z)$ and $G \in \mathcal{L}(W,Z)$. The following conditions are equivalent*

a) $\ker(G^*) \subset \ker(F^*)$

b) $\overline{\mathrm{range}(F)} \subset \overline{\mathrm{range}(G)}$, see [2].

Corollary 1. *Let $V = Z; U, Z$ be Hilbert spaces, $W = L_2([0,T],U)$ $F = I$ and $G \in \mathcal{L}(W,\mathcal{Z})$. Then $\overline{\mathrm{range}(G)} = Z$ iff $\ker(G^*) = \{0\}$.*

Theorem 4. *Let A be a generator non-degenerate, exponentially bounded, once integrated semigroup on state space Z, $B, \Lambda B$ bounded operators and U control space, u, $\dot{u} \in L_2([0,\tau];U)$. The boundary control system is approximately controllable on $[0,\tau]$ iff for almost all $s \in [0,\tau]$ $\ker ß_\tau^*(s) = \{0\}$, where $ß_\tau^*$ is the adjoint operator to the control operator $ß_\tau[u(.)] = \int_0^\tau Bu(s)ds - \int_0^\tau S(\tau-s)B\dot{u}(s)ds + \int_0^\tau S(\tau-s)\Lambda Bu(s)ds$.*

Proof. It follows from Collarry 1. □

3.1. The Existence of Integrated Solution of Hyperbolic Equation with Dirichlet Boundary Condition

The problem

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} & \text{for } x \in \langle 0,1\rangle \text{ and } t \in \langle 0,T\rangle, \\ w(0,t) = 0,\ w(1,t) = 0 & \text{for } t \in \langle 0,T\rangle, \\ w(x,0) = w_1(x), \quad w_t(x,t)|_{t=0} = w_2(x) & \text{for } x \in \langle 0,1\rangle\,, \end{cases}$$

will be set equivalently in an abstract form on the Hilbert space

$$Z = L_2(0,1) \otimes L_2(0,1),$$

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, U = \begin{bmatrix} w \\ w_t \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \tag{15}$$

with the domain $D(\mathcal{A}) = D(A_0) \otimes L_2(0,1)$, where $A_0 h = -\frac{d^2h}{dx^2}$, $h \in D(A_0) = \{h \in L_2(0,1) \mid h, \frac{dh}{dx}$, are absolutely continuous, $\frac{d^2h}{dx^2} \in L_2(0,1)$ and $h(0) = 0$, $h(1) = 0\}$.

Theorem 5. *The operator $\mathcal{A}$ with the domain $D(\mathcal{A})$ generates an exponentially bounded non-degenerate once integrated semigroup on $L_2(0,1) \otimes L_2(0,1)$. It is given by the formula*

$$S(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 2\times \begin{bmatrix} \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\langle w_1, \sin n\pi x\rangle \sin(n\pi t) + \frac{1}{n^2\pi^2}\langle w_2, \sin n\pi x\rangle (1-\cos(n\pi t))\right]\sin n\pi x \\ \sum_{n=1}^{\infty}\left[\langle w_1, \sin n\pi x\rangle(\cos(n\pi t)-1) + \frac{1}{n\pi}\langle w_2, \cos n\pi x\rangle \sin(n\pi t)\right]\sin n\pi x \end{bmatrix}, \tag{16}$$

for $w_1, w_2 \in L_2(0,1)$.

The proof is given in [3].

3.2. The Existence of Integrated Solution of Equation with Neumann Boundary Condition

The problem

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} & \text{for } x \in \langle 0,1\rangle \text{ and } \ t \in (0,T\rangle, \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \quad \dfrac{\partial w(1,t)}{\partial x} = 0 & \text{for } t \in \langle 0,T\rangle, \\ w(x,0) = w_1(x), \quad w_t(x,t)|_{t=0} = w_2(x) & \text{for } x \in \langle 0,1\rangle, \end{cases}$$

will be set equivalently in an abstract form on the Hilbert space $Z = L_2(0,1) \otimes L_2(0,1)$.

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, U = \begin{bmatrix} w \\ w_t \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \tag{17}$$

with the domain $D(\mathcal{A}) = D(A_0) \otimes L_2(0,1)$, where $A_0 h = -\frac{d^2h}{dx^2}$, $h \in D(A_0) = \{h \in L_2(0,1) \mid h, \frac{dh}{dx}$, are absolutely continuous, $\frac{d^2h}{dx^2} \in L_2(0,1)$, and $\frac{dh(0)}{dx} = 0$, $\frac{dh(1)}{dx} = 0\}$.

Theorem 6. *The operator $\mathcal{A}$ with the domain $D(\mathcal{A})$ generates exponentially bounded non-degenerate once integrated semigroup on $L_2(0,1) \otimes L_2(0,1)$. It is given by the formula*

$$S(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \left[\begin{array}{l} \sum_{n=1}^{\infty}[\frac{2}{n\pi}\langle w_1, \cos n\pi x\rangle \cos n\pi x \sin(n\pi t) \\ \sum_{n=1}^{\infty}[2\langle w_1, \cos n\pi x\rangle \cos n\pi x(1-\cos(n\pi t)) \end{array}\right. \left.\begin{array}{l} +\frac{2}{n^2\pi^2}\langle w_2, \cos n\pi x\rangle \cos n\pi x(1-\cos(n\pi t)] + t\langle w_1, 1\rangle + \frac{t^2}{2}\langle w_2, 1\rangle \\ +\frac{2}{n\pi}\langle w_2, \cos n\pi x\rangle \cos n\pi x \sin(n\pi t)] + t\langle w_2, 1\rangle \end{array}\right], \tag{18}$$

for $w_1, w_2 \in L_2(0,1)$.

The proof is similar as in the Dirichlet case in [3].

Theorem 7. *There exists a unique integrated solution*

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = S(t)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \int_0^t S(t-r)F(r)dr$$

of the following problem

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}\begin{bmatrix} w \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} w \\ w_t \end{bmatrix}(0) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{for any} \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in L_2(0,1) \otimes L_2(0,1),$$

$f \in L^1([0,T], L_2(0,1))$ which satifies the integral equation

$$v(t) - tx = \mathcal{A}\int_0^t v(r)dr + \int_0^t (t-s)F(s)ds \quad \text{for} \ \ 0 \le t < T, \tag{20}$$

with

$$x = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

The function $w(t) = v_2(t) + w_1$ satisfies the following integral equation

$$w(t) - w_1 - tw_2 = A_0 \int_0^t (t-s)w(s)ds + \int_0^t (t-s)f(s)ds \tag{21}$$

and w is the weak solution of

$$\begin{cases} \dfrac{d^2}{dt^2}w(t) + A_0 w(t) = f(t) \quad on\ [0,T]\,, \\ \dfrac{d}{dt}w(t)|_{t=0} = w_2, \quad w(0) = w_1\,, \end{cases}$$

in the following sense

$$\begin{cases} \dfrac{d^2}{dt^2}\langle w(t), w^*\rangle = \langle w(t), -A_0^* w^*\rangle + \langle f(t), w^*\rangle \text{ a.e.} \qquad on\ [0,T], \\ \dfrac{d}{dt}\langle w(t), w^*\rangle|_{t=0} = \langle w_2, w^*\rangle,\ w(0) = w_1 \quad for\ any\ w^* \in\ D(A_0^*). \end{cases}$$

The proof is omitted.

Example 1. The system (2a) can be reformulated in the form (1) by defining $Z = L_2(0,1) \otimes L_2(0,1)$, $U = C$, $\Lambda = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}$ with domain

$$D(\Lambda) = \left\{ \begin{array}{l} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in Z,\ z_1, \dfrac{dz_1}{dx} \text{ are absolutely continuous,} \\ \dfrac{d^2 z_1}{dx^2} \in L_2(0,1), \dfrac{dz_1(0)}{dx} = 0, \quad z_2 \in L_2(0,1) \end{array} \right\}$$

and the boundary operator $\beta : D(\beta) \subset Z \mapsto C$ by $\beta\binom{z_1}{z_2} = \frac{dz_1(1)}{dx}$ with $D(\beta) = D(\Lambda)$.

$$\left\{ \begin{array}{l} \dfrac{d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}{dt} = \Lambda \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad t \geq 0,\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}(0) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ \\ \beta \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}(t) = u(t) \end{array} \right\} \tag{22}$$

The system (22) is a boundary control system, since $A = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}$ with domain $D(A) = D(\Lambda) \cap \ker(\beta) = D(A_0) \otimes L_2(0,1)$ is the generator of once integrated semigroups and Bu defined by $Bu = b(x)u$, where $b(x)$ is choosen $\begin{bmatrix} \frac{1}{2}x^2 \\ 0 \end{bmatrix}$, is contained in the domain of Λ with $\beta\Lambda u = u$, and $\Lambda b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Theorem 8. *The system* (22) *is approximately controllable.*

Proof. By (14) and (18)

$$r(t) = \int_0^t Bu(s)ds - S(t)Bu(0) + S(t)z_0 - \int_0^t S(t-s)B\dot{u}(s)ds$$

$$+\int_0^t S(t-s)\Lambda Bu(s)ds = S(t)z_0 + \frac{2(-1)^n}{n^2\pi^2}\times$$
$$\begin{bmatrix} \sum_{n=1}^{\infty}\cos n\pi x(\int_0^t u(s)ds - \int_0^t u(s)\cos n\pi(t-s)ds) + \int_0^t \frac{(t-s)^2}{2}u(s)ds \\ \sum_{n=1}^{\infty}\cos n\pi x\int_0^t u(s)\sin n\pi(t-s)ds + \int_0^t (t-s)u(s)ds \end{bmatrix},$$

where $z_0 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in L_2(0,1)\otimes L_2(0,1)$.

The control operator is given by the formula

$$ß_t u = \begin{bmatrix} \int_0^t \frac{(t-s)^2}{2}u(s)ds + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}\cos n\pi x\Big(\int_0^t u(s)(1-\cos n\pi(t-s))ds\Big) \\ \int_0^t (t-s)u(s)ds + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\cos n\pi x\int_0^t u(s)\sin n\pi(t-s)ds \end{bmatrix},$$

and the adjoint to it is

$$ß_t^*\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(s) = \frac{(t-s)^2}{2}(v_1,1) + (t-s)(v_2,1) + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x) -$$
$$\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)\cos n\pi(t-s) + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}(v_2,\cos n\pi x)\sin n\pi(t-s).$$

Uniform convergence of the above series with respect to s for $v_1, v_2 \in L_2(0,1)$ guarantee that $ß_t^*\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(s)$ is continuous function of s. It can be also proven that $ß_t^* : L_2(0,1)\otimes L_2(0,1) \mapsto L_2(0,t)$ for $t > 0$.

The Fourier's expansions of functions $g(\tau) = \tau$, $h(\tau) = \tau^2$ for $0 < \tau < 2$ are given by the formulae

$$\tau = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi\tau, \quad \tau^2 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi\tau}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi\tau}{n}.$$

Putting these expansions to the formula

$$\Phi(\tau) = \frac{\tau^2}{2}(v_1,1) + \tau(v_2,1) + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)$$
$$-\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)\cos n\pi\tau + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}(v_2,\cos n\pi x)\sin n\pi\tau$$

we have

$$\begin{aligned}\Phi(\tau) &= \frac{2}{3}(v_1,1)+(v_2,1)+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)\\ &+\sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2}(v_1,1)-\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)\right)\cos n\pi\tau\\ &+\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n\pi}(v_2,\cos n\pi x)-\frac{2}{n\pi}(v_1,1)-\frac{2}{n\pi}(v_2,1)\right)\sin n\pi\tau.\end{aligned}$$

Let $\Phi(\tau)=0$ for $\tau\in(0,2)$ then

$$\begin{cases}\frac{2}{3}(v_1,1)+(v_2,1)+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)=0\,,\\ \frac{2}{n^2\pi^2}(v_1,1\ )-\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)=0\,,\\ \frac{2(-1)^n}{n\pi}(v_2,\cos n\pi x)-\frac{2}{n\pi}(v_1,1)-\frac{2}{n\pi}(v_2,1)=0 \quad \text{for } n\in N.\end{cases}$$

This system of equations is equivalent to the following one

$$\begin{cases}(v_1,1)+(v_2,1)=0,\\ (-1)^n(v_1,1)=(v_1,\cos n\pi x),\\ (v_2,\cos n\pi x)=0 \ \text{ for } \ n\in N.\end{cases}$$

For any $v_1\in L_2(0,1)$ $\lim_{n\to\infty}\int_0^1 v_1(x)\cos n\pi x dx=0$, so we have $(v_1,1)=0,(v_1,\cos n\pi x)=0$ and $(v_2,1)=0,(v_2,\cos n\pi x)=0$ for $n\in N$. The functions $\{1,\cos n\pi x\}$ are the basis in $L_2(0,1)$, it implies that $v_1=0$, $v_2=0$.

Let $\tau>2$ and $2k<\tau<2k+2$, $k\in$N, then

$$\tau=2k+1-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi\tau$$

and

$$\tau^2=\frac{4}{3}(3k^2+3k+1)+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi\tau}{n^2}-\frac{4+8k}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi\tau}{n},$$

so

$$\Phi(\tau)=\frac{4}{3}(v_1,1)(3k^2+3k+1)+(2k+1)(v_2,1)+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)$$

$$+\sum_{n=1}^{\infty}\cos n\pi\tau[\frac{2(v_1,1)}{n^2\pi^2}-\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)]$$
$$+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n\pi}(v_2,\cos n\pi x)-\frac{4k+2}{n\pi}(v_1,1)-\frac{2}{n\pi}(v_2,1)\right]\sin n\pi\tau.$$

In this case we have equalities

$$\begin{cases}\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)+\frac{2}{3}(v_1,1)(3k^2+3k+1)+(2k+1)(v_2,1)=0,\\ \frac{2(v_1,1)}{n^2\pi^2}-\frac{2(-1)^n}{n^2\pi^2}(v_1,\cos n\pi x)=0,\\ \frac{2(-1)^n}{n\pi}(v_2,\cos n\pi x)-\frac{4k+2}{n\pi}(v_1,1)-\frac{2}{n\pi}(v_2,1)=0\quad\text{for } k,n\in N.\end{cases}$$

In the similar way as in the first case we can prove that $v_1=0$, $v_2=0$. It means that for all $\tau\geq 2$, for almost all $s\in[0,\tau]$ $\ker ß^*(s)=\{0\}$, and by Theorem 4 the system (22) is approximately controllable.

Example 2. The system (2b) can be reformulated in the form (1) by defining $Z=L_2(0,1)\otimes L_2(0,1)$, $U=C$, $\Lambda=\begin{bmatrix}0 & I\\ \frac{d^2}{dx^2} & 0\end{bmatrix}$ with domain

$$D(\Lambda)=\left\{\begin{matrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}\in Z,\ z_1,\frac{dz_1}{dx}\ \text{are absolutely continuous},\\ \frac{d^2z_1}{dx^2}\in L_2(0,1),\quad z_1(0)=0,\quad z_2\in L_2(0,1).\end{matrix}\right\}$$

and the boundary operator $\beta: D(\beta)\subset Z\mapsto C$ by $\beta\left(\begin{bmatrix}z_1\\ z_2\end{bmatrix}\right)=z_1(1)$ with $D(\beta)=D(\Lambda)$.

$$\begin{cases}\frac{d\begin{bmatrix}z_1\\ z_2\end{bmatrix}}{dt}=\Lambda\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix},\quad t\geq 0,\ \begin{bmatrix}z_1\\ z_2\end{bmatrix}(0)=\begin{bmatrix}w_1\\ w_2\end{bmatrix},\\ \beta\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}(t)=u(t)\,.\end{cases}\tag{23}$$

The system (2b) is a boundary control system, since $A=\begin{bmatrix}0 & I\\ \frac{d^2}{dx^2} & 0\end{bmatrix}$ with domain $D(A)=D(\Lambda)\cap\ker(\beta)=D(A_0)\otimes L_2(0,1)$ is the generator of once integrated semigroups on $L_2(0,1)\otimes L_2(0,1)$ and Bu defined by $Bu=b(x)u$, where $b(x)$ is choosen $\begin{bmatrix} \frac{1}{2}x^2+\frac{1}{2}x \\ 0 \end{bmatrix}$, is contained in the domain of Λ with $\beta Bu = u$, and $\Lambda b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Theorem 9. *The system* (22) *is approximately controllable.*

Proof. In the similar way as in Neumann case using (14) and (16) it can be proved the adjoint operator is equal

$$ß_t^*\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(s) = \sum_{n=1}^{\infty} 2(-1)^{n+1}\frac{1}{n\pi}(v_1, \sin n\pi x)$$

$$+\sum_{n=1}^{\infty} 2(-1)^{n+1}\Big[-\frac{1}{n\pi}(v_1, \sin n\pi x)\cos n\pi(t-s) + (v_2, \sin n\pi x)\sin n\pi(t-s)\Big].$$

The series $\sum_{n=1}^{\infty} 2(-1)^{n+1}\frac{1}{n\pi}(v_1, \sin n\pi x)$ is convergent for $v_1 \in L_2(0,1)$ because

$$\sum_{n=1}^{\infty}\Big|2(-1)^{n+1}\frac{1}{n\pi}(v_1, \sin n\pi x)\Big| \le \Big[\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2} + \sum_{n=1}^{\infty}(v_1, \sin n\pi x)^2\Big] < \infty.$$

Besides the series $\sum_{n=1}^{\infty} 2(-1)^{n+1}(v_2, \sin n\pi x)\sin n\pi(t-s)$ for $v_2 \in \mathrm{L}_2(0,1)$ is convergent in $L_2(0,t)$.

It can be also proven that $ß_t^* : L_2(0,1)\otimes L_2(0,1) \mapsto L_2(0,t)$ for $t>0$. Let

$$\Phi(\tau) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n \cos n\pi\tau + \sum_{n=1}^{\infty} b_n \sin n\pi\tau,$$

$\tau = t-s$, where

$$a_n = 2(-1)^{n+1}\frac{1}{n\pi}(v_1, \sin n\pi x), \quad b_n = 2(-1)^{n+1}(v_2, \sin n\pi x).$$

The function Φ is periodic with the period 2 and $\Phi \in \mathrm{L}_2(0,2)$, so it can be expended in the Fourier series and

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2}\int_0^2 \Phi(s)ds, \ \ -a_n = \int_0^2 \Phi(s)\cos n\pi s ds, \ \ b_n = \int_0^2 \Phi(s)\sin n\pi s ds$$

for $n \in N$.

Thus if $\tau \ge 2$ and $\Phi(s) = 0$ for $s \in [0,\tau]$ then

$$\begin{cases} \sum_{n=1}^{\infty} 2(-1)^{n+1}\frac{1}{n\pi}(v_1, \sin n\pi x) = 0, \\ 2(-1)^{n+1}\frac{1}{n\pi}(v_1, \sin n\pi x) = 0, \\ 2(-1)^{n+1}(v_2, \sin n\pi x) = 0 \quad \text{for } n \in N. \end{cases}$$

It implies that $(v_1, \sin n\pi x) = 0$, $(v_2, \sin n\pi x) = 0$ for $n \in N$. The functions $\{\sin n\pi x\}$ form basis of $L_2(0,1)$ and it implies that $v_1 = 0$ and $v_2 = 0$. It means that for all $\tau \geq 2$, for almost all $s \in [0,\tau]$ $\ker ß^*(s) = \{0\}$, and by Theorem 4 the system (23) is approximately controllable.

Remark 2. The approximate controllability of the systems (22) (or (23)) on any interval $[0,\tau]$, $\tau \geq 2$ implies the approximate controllability of the problems (2a) (resp. (2b)) on $[0,\tau]$, $\tau \geq 2$ i.e., for all $w_1, w_2, \widetilde{w} \in L_2(\Omega)$ and all $\varepsilon > 0$ there exists $u, \dot{u} \in L_2[0,\tau]$ such that $||\widetilde{w} - w(t, w_1, w_2, u)||_{L_2(0,1)} < \varepsilon$, where $w(t, w_1, w_2, u)$ is the weak solution of (2a) (resp. (2b)).

Remark 3. The systems (22), (23) are not exact controllable on interval $[0,\tau]$, $\tau \geq 2$ since values of control operators $ß_\tau u$ belong to the space $H^1(0,1) \times L_2(0,1)$.

References

[1] R.F. Curtain, H. Zwart, *An Introduction to Infinite Dimensional Linear System Theory*, Springer-Verlag, New York (1995).

[2] K.J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer Verlag, New York (2000).

[3] T. Kowalski, W. Sadkowski, Applications of integrated semigroups for control theory, *International Journal of Differential Equations and Applications*, **2**, No. 2 (2003), 123-139.

[4] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, New York (1983).

[5] H. Tanabe, *Equations of Evolution*, Pitman (1979).

[6] H.R. Thieme, "Integrated semigroups" and integrated solutions to abstract Cauchy problems, *Journal of Mathematical Analysis and Applications*, **152** (1990), 416-447.