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RANK 2 STABLE VECTOR BUNDLES WITH CANONICAL DETERMINANT ON SPECIAL CURVES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we study rank 2 stable vector bundles with canonical determinant on low gonality curves, e.g. their symmetric and their skew-symmetric multiplication maps.

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Let X be a smooth projective curve of genus $g \ge 3$. Let $B(X)$ (resp. $B(X)'$) denote the set of all rank 2 stable (resp. semistable) vector bundles E on X such that $\det(E) \cong \omega_X$. For every integer $n > 0$ set $B(X,n)' := \{E \in B(X)' : h^0(X,E) = n\}$, $B(X,n)'_+ := \{E \in B(X)' : h^0(X,E) \ge n\}$, $B(X,n) := B(X,n)' \cap B(X)$ and $B(X,n)_+ := B(X,n)'_+ \cap B(X)$. For every vector bundlee E on X let $\sigma_F : S^2(X,E) \to H^0(X, S^2(E)))$ denote the symmetric multiplication map and $\eta_E : \bigwedge^2(H^0(X,E) \to H^0(X, \wedge^2(E)) = H^0(X, \det(E))$ denote the skew-symmetric multiplication map. For the importance of the injectivity of the map σ_E for all $E \in B(X,n)$ for some $E \in B(X,n)$, see [3], [5] or [2]. For the geometric significance of the injectivity of η_E, see [5]. Of course, if $\det(E) \cong \omega_X$ and η_E is injective, then $\binom{n}{2} \le g$, where $n := h^0(X,E)$.

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Remark 1. Let E be a rank 2 vector bundle on X. The skew-symmetric multiplication map η_E is injective if and only if $h^0(X, A) \leq 1$ for all rank 1 subsheaves A of E.

Let E be a rank two vector bundle on a smooth curve X. Take a maximal degree line subbundle L of X and set $e(E) = \deg(E) - 2 \cdot \deg(L)$. Hence $e(E) \equiv 0 \pmod 2$. Hence $e(E)$ is even if $\det(E) \cong \omega_X$. E is stable (resp. semistable) if and only if $e(E) > 0$ (resp. $e(E) \geq 0$). A theorem of C. Segre and M. Nagata says that $0 < e(E) \leq g$ ([4]).

Remark 2. Fix $E \in B(X, n)'$ such that η_E is injective. Take an exact sequence

$$0 \to M \to E \to \omega_X \otimes M^* \to 0 \tag{1}$$

with $\operatorname{rank}(M) = 1$ and $\deg(M)$ maximal. Thus $\deg(M) = g - 1 - e(E)/2$. Remark 1 implies $h^0(X, M) \leq 1$. Hence $h^0(X, \omega_X \otimes M^*) \leq e(E)/2 + 1$. Hence $n \leq e(E)/2 + 2$. Notice that $n \leq e(E)/2$ if $h^1(X, M) = 0$.

Proposition 1. *Let X be a smooth curve of genus $g \geq 3$. There is $E \in B(X, n)' \backslash B(X, n)$ such that η_E is injective if and only if $n \leq 2$.*

Proof. Notice that $E \in B(X)' \backslash B(X)$ if and only if $e(E) = 0$. Hence the “only if” part follows from Remark 2. Now we will check the “if” part. Take any $M \in \operatorname{Pic}^{g-1}(X)$ such that $h^0(X, M) = 1$. The vector bundle $M \oplus \omega_X \otimes M^*$ gives the “if” part for $n = 2$. A general extrension of $\omega_X \otimes M^*$ by M gives the “if” part for $n = 1$. □

Proposition 2. *Let X be a smooth and connected projective curve with gonality $\leq k$. Let E be a rank 2 vector bundle on X such that $h^0(X, E) \geq 2k+1$. Then η_E is not injective.*

Proof. By assumption there is a zero-dimensional scheme $D \subset X$ such that $\operatorname{length}(D) \leq k$ and $h^0(X, \mathcal{O}_X(D)) \geq 2$. Since $h^0(X, E) > \operatorname{length}(D) \cdot \operatorname{rank}(E)$, we have $h^0(X; E(-D)) > 0$. Thus $\mathcal{O}_X(D)$ is a subsheaf of E. Apply Remark 1. □

Proposition 3. *Let X be a smooth and connected projective curve with gonality $\leq k$. Let E be a rank 2 vector bundle on X such that $h^0(X, E) \geq 2k-1$ and $h^1(X, \det(E)) > 0$. Then η_E is not injective.*

Proof. By Proposition 2 we may assume that X has gonality k. By assumption there is a zero-dimensional scheme $D \subset X$ such that $\operatorname{length}(D) = k$

and $h^0(X, \mathcal{O}_X(D)) = 2$. Fix $D' \subset D$ such that $\deg(D') = k-1$. Since X has gonality k, Serre duality gives $h^1(X, \mathcal{O}_X(D)) = h^1(X, \mathcal{O}_X(D)$. Since $h^0(X, E) > \mathrm{length}(D') \cdot \mathrm{rank}(E)$, we have $h^0(X; E(-D')) > 0$. Thus $\mathcal{O}_X(D')$ is a subsheaf of E. Let F be the subheaf of E spanned by $H^0(X, E)$. If $\mathrm{rank}(F) = 1$, then we may apply Remark 1. Assume $\mathrm{rank}(F) = 2$. Hence F is spanned, $h^0(X, F) = h^0(X, E) \geq 2k-1$ and $h^1(X, \det(F)) > 0$. Let $u_F : X \to G(2, n)$, $n := h^0(X, F)$, be the morphism induced by $H^0(X, F)$. Since $h^1(X, \det(F)) > 0$, $\mathcal{O}_{G(2,n)}(1)$ is the determinant of the universal rank 2 quotient bundle of the Grassmannian $G(2, n)$, and $h^1(X\mathcal{O}_X(D)) = h^1(X, \mathcal{O}_X(D)$, the schemes $u_E(D')$ and $u_E(D)$ have the same linear span. Since $h^0(X, E(-D')) > 0$, we get $h^0(X, E(-D)) > 0$. Apply Remark 1. □

Proposition 4. *Let X be a smooth and connected projective curve such that there is $R \in \mathrm{Pic}^k(X)$ with $h^0(X, R) \geq 2$. Let E be a rank 2 vector bundle on X such that $h^0(X, E) \geq 2k+2$. Then σ_E is not injective.*

Proof. Since $\deg(R) = k$ and $h^0(X, E) > k \cdot \mathrm{rank}(E)$, we have $h^0(X, E \otimes R^*) > 0$. Let A be the subsheaf of $E \otimes R^*$ spanned by $H^0(X, E \otimes R^*)$. If $\mathrm{rank}(A) = 2$, then we get an injective map $R \oplus R \to E$. Since $\sigma_{R \oplus R}$ is not injective, σ_E is not injective. Now assume $\mathrm{rank}(A) = 1$. Since $\deg(R) = k$ and $h^0(X, E) \geq 2k+2$, we get $h^0(X, A) \geq 2$. Hence $\sigma_{A \otimes R}$ is not injective. Since $A \otimes R$ is a subsheaf of E, σ_E is not injective. □

Remark 3. Let X be a smooth and connected projective curve. Fix $P \in X$ and E a rank 2 vector bundle on X. Let G be the general vector bundle obtained from E making a positive elementary transformation supported by P. If E has infinitely many maximal degree line subbundles, then $e(G) = e(E) - 1$. If E has only finitely many maximal degree line subbundles, then $e(G) = e(E) + 1$.

Theorem 1. *Let X be a smooth hyperelliptic curve of genus $g \geq 3$. Let $R \in \mathrm{Pic}^2(X)$ denote its hyperelliptic line bundle.*

(i) Fix an integer n such that $3 \leq n \leq 2g-4$. For every integer a such that $\lceil (n-2)/2 \rceil \leq a \leq n-2$ let T_a be the set of all $E \in B(X, n)$ which are obtained from $R^{\otimes a} \oplus R^{\otimes (n-2-a)}$ making $2g-2n+2$ positive elementary transformations such that the union of their supports is an element of $|R^{\otimes (g-n+1)}|$. T_a is non-empty, irreducible and of dimension at most $4g-4n-1-\epsilon$, where $\epsilon = 0$ if $2a \neq n-2$ and $\epsilon = 1$ if $2a = n-2$.

(ii) If $n \leq \lfloor (g-2)/2 \rfloor$, and d, c are integers such that $0 \leq c \leq d$ and $2d - c < n - g$, let $D_{n,d,c}$ be the set of all $E \in X(X, n)$ such that there is an effective divisor D, $d := \deg(D)$, $h^0(X, \mathcal{O}_X(2D)) = c+1$, $h^0(X, \mathcal{O}_X(D)) = 1$,

and E is an extension of $R^{\otimes(g-n)}(D)$ by $R^{\otimes(n-1)}(-D)$. If $n \le \lfloor (g-2)/2 \rfloor$, and d, c are integers such that $0 \le c \le d$ and $2d - c < n - g$, then $D_{n,d,c}$ is non-empty, equidimensional and of dimension $n - g - 2d + c$. If $\mathrm{char}(\mathbb{K}) \neq 2$ *(resp.* $\mathrm{char}(\mathbb{K}) = 2$*), then $D_{n,d,c}$ has at least $\binom{2g+2}{c}$ (resp. $\binom{g+1}{c}$) irreducible components.*

(iii) If $\lfloor (g-2)/2 \rfloor < n \le 2g - 2$, then $B(X,n) = \bigsqcup_{a=\lceil (n-2)/2 \rceil}^{n-2} T_a$ (disjoint union). If $3 \le n \le \lfloor (g-2)/2 \rfloor$, then $B(X,n) = \bigsqcup_{d,c} D_{n,d,c} \sqcup \bigsqcup_a T_a$ (disjoint union).

(iv) For all integers a, b such that $\lceil (n-2)/2 \rceil \le a < b \le n-2$, T_b is contained in the closure inside $B(X,n)$ of the algebraic set T_a.

(v) Fix integers d, d', c such that $D_{n,d,c}$ is defined and $0 \le d' \le d$. Then $D_{n,d,c}$ is contained in the closure inside $B(X,n)$ of the union Γ of all $D_{n,d',c'}$ such that $0 \le c' \le d'$ and $2d' - c' < n - g$

Proof. Fix $E \in A(X,n)$ and let F be the subsheaf of E spanned by $H^0(X,F)$. Hence F is spanned and $h^0(X,F) = n$. If $\mathrm{rank}(F) = 1$, then $\deg(F) \le g-2$ by the stability of E. Thus if $\mathrm{rank}(F) = 1$ the inequality $n \ge 2$ implies $F \cong R^{\otimes(n-1)}$. If $\mathrm{rank}(F) = 1$, let $F(D)$ be the saturation of F in E. Since $H^0(X,E) = h^0(X,F)$, we have $h^0(X, \mathcal{O}_X(D)) = 1$. Since E is stable, then $\deg(D) \le g - 2 - \deg(F) = g - 2n$. D is uniquely determined by F and hence it is uniquely determined by E. Thus $\mathrm{rank}(F) = 1$ if and only if $E \in D_{n,d,c}$ for some d, c, where $d := \deg(D)$. Since d and c are uniquely determined by F, $D_{n,d,c} \cap D_{n,d',c'} = \emptyset$ id $(d',c') \neq (d,c)$. If $\mathrm{rank}(F) = 2$, then $F \cong R^{\otimes a} \oplus R^{\otimes b}$ for some integers $a \ge b \ge 0$. Since $h^0(X,F) = h^0(E,E) = n$, we get $b = n - 2 - a$. Hence $E \in T_a$. We just saw all equalities and that the unions are disjoint unions. To conclude the proof it is sufficient to prove all the assertions concerning the fixed sets T_a and $D_{n,d,c}$. Fix any $P \in X$ and any rank 2 vector bundle on X. The set of all isomorphic classes of vector bundles obtained from G making a positive elementary transformation supported by P is parametrized (perhaps, not one-to-one) by $\mathbf{P}^1$. Hence T_a, if non-empty, is parametrized (perhaps not one-to-one) by an integral $(4g - 4n - 1)$-dimensional variety. The proof of [1], proof of Proposition 2.3 and Corollary 2.4 (see Remark 3), gives that a general $E \in T_a$ is stable. If $2a = n - 2$, notice that for any $P \in X$ every positive elementari transformation of $R^{\otimes a} \oplus R^{\otimes a}$ is isomorphic to $R^{\otimes a} \oplus R^{\otimes a}(P)$. Hence $\dim(T_a) \le 4g - 4n - 2$ if $2a = n - 2$. In the case $\mathrm{rank}(F) = 1$ the divisor D is uniquely determined by E. Since $h^0(X, \mathcal{O}_X(D)) = 1$, each Weierstrass point of X appears with multiplicity at most one in the divisor D. If $\mathrm{char}(\mathbb{K}) \neq 2$ (resp. $\mathrm{char}(\mathbb{K}) = 2$), then X has exactly $2g+2$ (resp. $g+1$) Weierstrass points. Fix n, d, c and an effective divisor such that $h^0(X, \mathcal{O}_X(D)) = 1$ and c contains exactly c Weierstrass points of X. Let D' be the difference between D and these c Weierstrass points contained in D. Notice that $h^0(X, R^{\otimes(g-n-c)}(-2D')) = g-n-c+1-2d+c$ if and only if R is not a subsheaf of $\mathcal{O}_X(2D')$. We get $\dim(\mathrm{Ext}^1(R^{g-n}(-D), R^{n-1}(D))) = g-2n+2-c-2(d-c) = g-2n-2d+c$. These observations explain the upper bound for the number of irreducible components of the algebraic set $D_{n,d,c}$. Now we fix D such that $h^0(X, \mathcal{O}_X(D)) = 1$ and $\deg(D) = d$. Let G be a general extension of $R^{g-n}(-D)$ by $R^{n-1}(D)$. We need to check that $h^0(X, G) = n$ and that G is stable. By semicontinuity and the openess of stability it is sufficient to check these properties for one extension G' of $R^{g-n}(-D)$ by $R^{n-1}(D)$. Fix $Q \in X$ and set $G'' := F(D) \oplus \mathcal{O}_X(-Q)$. Hence $h^0(X, G'') = n$. Take as G' a general vector bundle obtained from G'' making $2g-2+1-2(n-1)-d$ positive elementary transformations supported by points making a divisor of $|R^{\otimes(g-n)}(-D+Q)|$. Apply [1], Lemma 2.2, and Remark 3.

Now we will prove part (iv). It is sufficient to prove that a general $E \in T_b$ is contained in the closure of T_a. Since $n-2 \geq b > a \geq (n-2)/2$, the vector bundle $\mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(n-2-b)$ is contained in the closure of a family of vector bundles on $\mathbf{P}^1$ all isomorphic to $\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(n-2-a)$. Pulling back this family by the degree 2 morphism $X \to \mathbf{P}^1$ we get that $R^{\otimes b} \oplus R^{\oplus(n-2-b)}$ is in the closure of a family of vector bundles on X all isomorphic to $R^{\otimes a} \oplus R^{\oplus(n-2-a)}$. Use this flat family and that a general element of T_b (resp. T_a) is obtained from $R^{\otimes b} \oplus R^{\oplus(n-2-b)}$ (resp. $R^{\otimes a} \oplus R^{\oplus(n-2-a)}$) making $2g-2n+2$ general positive elementary transformations whose support is a general element of $|R^{\oplus(g-n+1)}|$.

Now we will prove part (v). Fix a degree d effective divisor D on X such that $h^0(X, \mathcal{O}_X(2D)) = c+1$ and take a divisor D' such that $D-D'$ is effective and $\deg(D') = d'$. Fix $Q \in X$ and set $A := R^{\otimes(n-1)}(D') \oplus \mathcal{O}_X(-Q)$. Hence $h^0(X, A) = n$. Let B a general vector bundle obtained from A making $2g-2n-d'+1$ positive elementary transformation supported by an element of $|R^{\otimes(g-n)}(-D'+Q)|$. An element of Γ is obtained from B making $d-d'$ general positive elementary transformations whose support is the divisor $D-D'$. Every $E \in D_{n,d,c}$ such that $R^{\otimes(n-1)}(D)$ is the saturation of the line bundle of E spanned by $H^0(X, E)$ is obtained from B making $d-d'$ very special positive elementary transformations whose support is the divisor $D-D'$: the only one for which the saturation in E line subbundle $R^{\otimes(n-1)}(D')$ of B is isomorphic to $R^{\otimes(n-1)}(D)$. □

Remark 4. Use the set-up of Theorem 1. Obviously, no T_a intersects the closure inside $B(X, n)$ of some $D_{n,d,c}$: use that in a family of vector bundles with constant cohomology the rank of the subsheaf spanned by the global sections is semicontinuos. Counting also the degree of this subsheaf we see that $D_{n,a,c}$ does not intersect the closure of any $D_{n,a',c'}$ with either $a' > a$ or $a' = a$ and $c' > c$. For the semicontinuity of the invariant $e(F)$ applied to the subsheaf F spanned by $H^0(X, E)$ we also see that T_a is disjoint from the closure of T_b if $b > a$.

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