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EMBEDDINGS OF PROJECTIVE CURVES IN QUADRICS

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Abstract: Let X be a smooth projective curve, $L \in \mathrm{Pic}(X)$ and a linear subspace $V \subseteq H^0(X, L)$. Here we study triples (X, L, V) such that the symmetric multiplication map $S^2(V) \to H^0(X, L^{\otimes 2})$ is not injective.

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For any smooth and connected projective curve, any $L \in \mathrm{Pic}(X)$ and any linear subspace $V \subseteq H^0(X, L)$, let $\sigma_L : S^2(H^0(X, L)) \to H^0(X, L^{\otimes 2})$ and $\sigma_{L,V} : S^2(V) \to H^0(X, L^{\otimes 2})$ denote the symmetric multiplication maps. If X is a general curve of genus $g \geq 3$, then σ_L is injective for all $L \in \mathrm{Pic}^d(X)$ with $d \leq g + 1$ and the inequality $d \leq g + 1$ is sharp when g is even (see [5], Theorem 1.1 and Example 3.1). Here we will consider the case in which X is not general, but we make some geometric assumption on the line bundle L (e.g. we assume L very ample). For any spanned $L \in \mathrm{Pic}(X)$ and any linear subspace $V \subseteq H^0(X, L)$ spanning L, let $u_L : X \to \mathbf{P}^n$, $n := h^0(X, L) - 1$, and $u_{L,V} : X \to \mathbf{P}^m$, $m := \dim(V) - 1$, denote the morphisms associated to respectively to the complete linear system $|L|$ and to the linear system $|V|$. Our starting point is the following classical example.

Example 1. Let X be a smooth curve of genus $g \geq 3$. There is a very ample $L \in \mathrm{Pic}^{g+3}(X)$ such that $h^1(X, L) = 0$ and σ_L is not injective (i.e. such

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that $u_L(X) \subset \mathbf{P}^3$ is contained in a quadric Q) if and only if X is hyperelliptic. If X is hyperelliptic, then $\dim(\mathrm{Ker}(\sigma_L)) = 1$ for every very ample $L \in \mathrm{Pic}^{g+3}(X)$ such that $h^1(X,L) = 0$. The associated quadric Q is smooth and $u_L(X)$ is a divisor of type $(2, g+1)$ or $(g+1, 2)$ on Q. Now fix an integer $d \geq g+3$, a very ample $M \in \mathrm{Pic}^d(X)$ such that $h^1(X,L) = 0$ and a linear subspace $V \subseteq H^0(X,L)$ such that $\dim(V) = 4$ and $u_{L,V}$ is not injective. Assume that $\sigma_{L,V}$ is not injective. The classifiaction of smooth curves on smooth quadric surfaces and on quadric cones (see [2], Example III.5.6, Remark IV.6.4.1 (d) and Example V.2.9) shows that $u_{M,V}(X)$ is linearly normal, i.e. $d = g+3$, $V = H^0(X,M)$. The first part of the example gives that X is hyperelliptic.

Here there is a sample of our results.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix integers n, d, g such that $g \geq 0$, $n \geq 3$ and either $d \geq g+n+1$ or $d \geq 2n+2$ and $(n+1)(d-n-1) \geq ng$. There is an irreducible component $T(d,g,n)$ of the Hilbert scheme* $\mathrm{Hilb}(Q_n)$ *of the smooth quadric hypersurface $Q_n \subset \mathbf{P}^{n+1}$ such that* $\dim(T(d,g,n)) = 3d + (3-n)(g-1)$ *and a general $C \in T(d,g,n)$ is a smooth, connected and non-degenerate such that $p_a(C) = g$,* $\deg(C) = d$, *$h^1(C, N_{C,Q_n}) = 0$ and $h^1(C, \mathcal{O}_C(2)) = 0$. If $d \geq g+n$, then $h^1(C, \mathcal{O}_C(1)) = 0$.*

Proposition 1. *Fix integers g, c such that $g \geq c+6 \geq 6$. Set $d := g+3-c$. Let $T[c]$ denote the set of all pairs (X, L), where X s a smooth genus g curve, $L \in$* $\mathrm{Pic}^d(X)$, *$h^1(X,L) = c$, L is very ample and σ_L is not injective. Let $T_c \subseteq \mathcal{M}_g$ the image of $T[c]$*

(i) If $c = 0$ then every $X \in T_c$ is hyperelliptic. $T[c]$ contains every pair (X, L) with X hyperelliptic, L very ample and and non-special. Hence the set T_c is an irreducible $(2g-1)$-dimensional variety and for any fixed $X \in T_c$ the set of all L's is an irreducible g-dimensional variety.

(ii) Assume $c > 0$. Then $d \geq 6$ and either there is an integer a such that $\lceil d/2 \rceil \leq a \leq d-3$, $g = a(d-a) - d + 1$ and $c = (a-2)(d-a-2)$ or d is odd, say $d = 2m+1$, $g = m^2 - m$ and $c = m(m-1)$. T_c is a disjoint union of $a+1$ irreducible varieties A_a, $\lfloor d/2 \rfloor$, with $\dim(A_a) = a(d-a) + d$. *For every $X \in T_c$ there is a unique $L \in$* $\mathrm{Pic}^d(X)$ *such that $(X, L) \in T[c]$.*

Proof. Notice that $h^0(X,L) = 4$. Since the case $c = 0$ is covered by Example 1, we may assume $c > 0$. Set $C := u_L(X)$. Since $d > 4$, C is contained in a unique quadric surface Q. First assume that Q is smooth. Call (a,b) the type of C with, say, $a \geq b$. Hence $d = a+b$, $g = ab - a - b + 1$, and $\omega_C = \mathcal{O}_C(a-2, b-2)$. Since $c > 0$, we get $b \geq 3$. Hence $c = h^0(C, \mathcal{O}_C(a-3, b-3)) = (a-2)(b-2)$. Now assume that Q is a quadric cone. If d is even, then C is the complete intersection of Q with a degree $d/2$ surface (see [2], Example III.5.6). Hence $g = d^2/4 - d + 1$. If d is odd, then C is linked to a line inside Q by a surface of degree $(d+1)/2$ (see [2], Remark IV.6.4.1 (d) and Example V.2.9). Set $m := (d-1)/2$. In this case the strict transform D of C in the blowing-up F_2 of the vertex of Q is an elemento of $|mh + (2m+1)f|$. Since $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h - 4f)$, we get $g = m^2 - m$. In this case we have $c = h^0(D, \mathcal{O}_D((m-2)h + (2m-3)f) = \sum_{i=0}^{m-2}(2m-2-2i) = m(m-1)$. □

Remark 1. Fix an integer $g \geq 4$ and a general $X \in \mathcal{M}_g$. Set $a := \lfloor (g+3)/2 \rfloor$ and $b := g + 3 - a$. There are spanned $M \in \mathrm{Pic}^a(X)$, $R \in \mathrm{Pic}^b(X)$ such that $h^0(X, M) = h^0(X, R) = 2$, $A \ncong B$ if g is odd, and $h^1(X, M \otimes R) = 0$. Thus $h^0(X, M \otimes R)$ is spanned. We may find M, R with the additional property that $u_{M\otimes R}$ is birational onto its image and $u_{M\otimes R}(X)$ has only ordinary nodes as singularities. By construction $u_{M\otimes R}(X)$ is contained in a smooth quadric surface.

For all integers n, s such that $0 \leq s \leq n-2$ let $Q_n \subset \mathbf{P}^{n+1}$ denote a smooth quadric hypersurface and $Q_{n,s} \subset \mathbf{P}^{n+1}$ a quadric cone with s-dimensional vertex. Q_n and $Q_{n,s}$ are unique, up to a projective transformation.

Remark 2. Let $D \subset Q_n \subset \mathbf{P}^{n+1}$ be a rational normal curve. Hence $\deg(D) = n+1$. $T\mathbf{P}^{n+1}|D$ is isomorphic to the direct sum of $n+1$ line bundles of degree $n+2$ (see [1], Lemma 1.2), while the normal bundle $N_{D,\mathbf{P}^{n+1}}$ is isomorphic to the direct sum of n line bundles of degree $n+3$. Consider the normal bundle exact sequences induced by the inclusion $Q_n \subset \mathbf{P}^{n+1}$:

$$0 \to TQ_n \to T\mathbf{P}^{n+1}|Q_n \to \mathcal{O}_{Q_n}(2) \to 0\,, \tag{1}$$

$$0 \to N_{D,Q_n} \to N_{D,\mathbf{P}^{n+1}} \to \mathcal{O}_D(2) \to 0\,. \tag{2}$$

Now assume $n = 3$. From (1) we easily get that $TQ_3|D$ is isomorphic to a direct sum of either a line bundle of degree 3, a line bundles of degree 4 and a line bundle of degree 5 or or of 3 line bundles of degree 4. From (2) we easily get N_{D,Q_3} is isomorphic to a direct sum either of a line bundle of degree 4 and a line bundle of degree 6 or of 2 line bundles of degree 5.

Lemma 1. *Fix an integer $n \geq 3$ and a general $S \subset Q_n$ such that $\sharp(S) = n+2$. Then there exists a rational normal curve $D \subset Q_n \subset \mathbf{P}^{n+1}$ such that $S \subset D$.*

Proof. First assume $n = 3$. Fix $P \in Q_3$ and take a general hyperplane $H \subset \mathbf{P}^4$. Thus $Q \cap H$ is a smooth quadric surface Q_2. Fix a general $A \subset Q \cap H$ such that $\sharp(A) = 5$. There is a unique smooth curve C of type $(2, 1)$ on Q_2

such that $A \subset C$. The union of all lines through P and contained in Q_3 is the quadric cone $T_PQ_3 \cap Q$. There is one such line E such that $E \cap C \neq \emptyset$. $E \cup C$ is connected, nodal and $p_a(E \cup C) = 0$. Since N_{C,Q_3} is a direct sum of a degree 3 line bundle and a degree 4 line bundle, while N_{E,Q_3} is the direct sum of a degree 1 line bundle and a degree 0 line bundle, we see that $E \cup C$ is smoothable inside Q_3 (see [3] or [4]). Hence $E \cup C$ is a flat limit of rational normal curves. Now assume $n \geq 4$ and that the result is true for the integer $n' := n - 1$. Fix $P \in Q_n$ and a general hyperplane H. Repeat the proof of the case $n = 3$ using the inductive assumption. □

Lemma 2. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix an integer* $n \geq 3$. *Let* $D \subset Q_n$ *be a general rational normal curve. Then* N_{D,Q_n} *is a direct sum of* $n-1$ *line bundles of degree* $n+2$.

Proof. Let $a_1 \geq \cdots \geq a_{n-1}$ be the splitting type of N_{D,Q_n}. Since $\deg(N_{D,Q_n}) = (n-1)(n+2)$, we need to check that $a_{n-1} \geq n+2$. Fix a general $B \subset Q_n$ such that $\sharp(B) = n+1$. Let $f : W \to Q_n$ be the blowing-up of B. Let T be the set of all rational normal curves containing B and contained in Q_n3. Let Z the set of all strict transforms of elements of T. For any $C \in T$ let $C' \subset W$ denote its strict transform. Fix a general $E \in T$. $N_{E',W}$ has splitting type $a_1 - n - 2 \geq \cdots \geq a_{n-1} - n - 2$. Since Z is a covering family of W (Lemma 1), E' is a general element of Z and $\mathrm{char}(\mathbb{K}) = 0$, $N_{E',W}$ is spanned, i.e. $a_{n-1} - n - 2 \geq 0$. Thus $a_i = n+2$ for all i. □

Proof of Theorem 1. Since all cases with $d \leq 2n+1$ are easy, we assume $d \geq 2n+2$. If $d \leq g+n$, we may assume defined $T(d-n, g')$ with $g' := \min\{0, g-n\}$. Fix a general $E \in T(d-n, g')$ and a general $S \subset E$ such that $\sharp(S) = g - g' + 1$. Hence $\sharp(S) \leq n+1$. By Lemmas 1 and 2 there is a rational normal curve $D \subset Q_n$ such that $D \cap E = S$ and D intersects transversally E. Set $Y := E \cup C$. Since N_{D,Q_n} is a direct sum of lines bundles of degree at least $n+1$ and $\sharp(S) \leq n+2$, we easily get that Y is smoothable and $h^1(Y, N_{Y,Q_n}) = 0$ (see [3] or [4]). Hence there is a unique irreducible component Γ of $\mathrm{Hilb}(Q_n)$ containing Y. Set $T(d,g,n) = \Gamma$. Since $h^1(Y, N_{Y,Q_n}) = 0$, the same is true for a general $C \in T(d,g,n)$. Now assume $d < g+n$. By the inductive assumption the component $T(d-n, g-n-1)$ is defined. Fix a general $E' \in T(d-n, g-n-1)$ and a general $S' \subset E$ such that $\sharp(S) = n+3$. By Lemmas 1 and 2 there is a rational normal curve $D' \subset Q_n$ such that $D' \cap E' = S'$ and D intersects transversally E'. Set $Y' := E' \cup C'$. Since N_{D,Q_n} is a direct sum of lines bundles of degree $n+2$ and $\sharp(S) = n+3$, we conclude as in the previous case. □

Remark 3. Let P be the vertex of $Q_{3,0}$ and $D \subset Q_{3,0}$ a rational normal curve such that $P \notin D$. if we take $Q_{3,0}$ instead of Q_n in (2) the corresponding complex is exact outside P. Since $P \notin D$, we get that $N_{D,Q_{3,0}}$ is a direct sum of 2 line bundles of degree at least 4.

Proposition 2. *Fix integers d, g such that $d \geq g+4 \geq 4$. There is an irreducible component $A(d,g)$ of the Hilbert scheme* $\mathrm{Hilb}(Q_{3,0})$ *of $Q_{3,0}$ such that* $\dim(A(d,g)) = 3d$ *and a general $C \in A(d,g)$ is a smooth, connected and non-degenerate, $P \notin C$, $p_a(C) = g$,* $\deg(C) = d$, *$h^1(C, N_{C,Q_3}) = 0$ and* $h^1(C, \mathcal{O}_C(1)) = 0$.

Proof. Adapt the proof of the part "$n = 3$ and $d \geq g+4$" of the proof of Theorem 1 quoting Remark 3 instead of Lemma 2. □

Remark 4. Let P be the vertex of $Q_{3,0}$ and $D \subset Q_{3,0}$ a rational normal curve such that $P \in D$. Let $\pi : W \to Q_{3,0}$ be the blowing up of the vertex. Hence $W \cong \mathbf{P}(\mathcal{O}_{Q_2}(1) \oplus \mathcal{O}_{Q_2})$. Let $D' \subset W$ be the strict transform of D in W. It is easy to check that every direct summand of $N_{D',W}$ has degree at least 3.

Proposition 3. *Fix integers d, g such that $d \geq g+5 \geq 5$. There is an irreducible component $B(d,g)$ of the Hilbert scheme* $\mathrm{Hilb}(Q_{3,0})$ *of $Q_{3,0}$ such that* $\dim(B(d,g)) = 3d-1$ *and a general $C \in A(d,g)$ is a smooth, connected and non-degenerate, $P \in C$, $p_a(C) = g$,* $\deg(C) = d$, *$h^1(C, N_{C,Q_{3,0}}) = 0$ and* $h^1(C, \mathcal{O}_C(1)) = 0$.

Proof. Since the cases $d \leq 7$ are easy, we assume $d \geq 8$. Set $g' := \min\{g-3, 0\}$. Hence $A(d-4, g')$ is defined. Take a general $X \in A(d, g')$ and a general $S \subset X$ such that $\sharp(S) = 4$. Fix a smooth rational normal curve $D \subset Q_{3,0}$ such that $P \in D$, $S = D \cap X$ and D intersects transversally X. Let Y be the strict transform of $X \cup D$ in W. Remark 4 and the assumption $h^1(X, N_{X,Q_{3,0}}) = 0$ give that Y is a smooth point of $\mathrm{Hilb}(W)$. Take as $B(d,g)$ the images in $Q_{3,0}$ of general element of the unique ireducible component of $\mathrm{Hilb}(W)$ containing Y. □

Proposition 4. *Fix an odd integer $g \geq 5$ and a general $X \in \mathcal{M}_g$. For every $L \in$* $\mathrm{Pic}^{g+2}(X)$ *such that $h^1(X, L) \leq 2$ the map σ_L is injective.*

Proof. Fix $L \in \mathrm{Pic}^{g+2}(X)$. If L is not spanned, then σ_L is injective by [5], Theorem 1.1. Now assume L spanned. Since $g+2$ is odd and X has no g^1_t with $t \leq (g+1)/2$, it is easy to see that u_L is birational onto its image. This observation excludes the case $h^1(X, L) \leq 1$. Assume $h^1(X, L) = 2$. Hence

$u_L(X)$ is contained in an irreducible quadric surface. Since X has no g^1_u with $u \le \lfloor \deg(L)/2 \rfloor$, this is impossible. □

Remark 5. Fix integer $g \ge 4$ and $d \ge g+3$. Let X be any smooth genus g curve. The proof of [5], Example 3.1, gives the existence of infinitely many $L \in \mathrm{Pic}^d(X)$ such that σ_L is not injective. If X is general, then there are infinitely many spanned, non-special and birationally very ample $L \in \mathrm{Pic}^d(X)$ such that σ_L is not injective.

Proposition 5. *Fix integers d, g such that $d \le g+4 \ge 8$ and a general $X \in \mathcal{M}_g$. There are infinitely many $L \in \mathrm{Pic}^d(X)$ and $V \subseteq H^0(X,L)$ such that $h^1(X,L) = 0$, $\dim(V) = 5$, $u_{L,V}$ is an embedding, $\sigma_{L,V}$ is not injective and $u_{L,V}(X)$ is contained in a quadric cone $Q_{3,0}$.*

Proof. By Remark 1 there is $A \in \mathrm{Pic}^{(g+3)}(X)$ such that A is spanned, $h^1(X,A) = 0$, $h^0(X,A) = 4$, u_A is birational onto its image, $u_A(X)$ has only ordinary nodes as its only singularities, and $u_A(X)$ is contained in a smooth quadric surface. Let $(P_i, E_i)_{i \in I}$ be the finitely many pairs of distinct points of X such that $u_A(P_i) = u_A(E_i)$. Fix a general $S \subset X$ such that $\sharp(S) = d-g-3$ and set $L := A(S)$. The effective divisor S induces an inclusion $j : H^0(X,A) \to H^0(X,L)$. Let V be a general 5-dimensional linear subspace of $H^0(X,L)$ containing $j(H^0(X,A))$. Since V contains $j(H^0(X,A))$, $\sigma_{L,V}$ is not injective. Since A is spanned, u_A is birational onto its image, $u_A(X)$ has only ordinary nodes as its only singularities, $u_{L,V}$ has the same properties and to check that $u_{L,V}$ is very ample it is sufficient to check that $u_{L,V}(P_i) \ne u_{L,V}(E_i)$ for all $i \in I$. Since I is finite and $u_A(P_i) = u_A(E_i)$ for all i, it is sufficient to check that $u_L(P_i) \ne u_L(E_i)$ for all $i \in I$. Since L is spanned, the latter property is equivalent to $h^0(X, A(S)(-P_i - E_i)) = h^0(X, A(S)) - 2$ for all $i \in I$. It is sufficient to do the case $\sharp(S) = 1$, say $S = \{P\}$. Fix $i \in I$. Since $h^1(X,A) = 0$, $h^0(X, A(B)) = h^0(X,A) + 1$ for all $B \in X$. Since A is spanned and $h^0(X, A(P_i)) = h^0(X,A) + 1$, we get $h^0(X, A(S)(-P_i - E_i)) = h^0(X, A(S)) - 2$ if $P = P_i$. By semicontinuity this is true for a general $P \in X$. Since I is finite, a general $P \in X$ will separates all pairs (P_i, E_i). Since $u_A(X)$ is contained in a rank 4 quadric, $u_{L,V}(X)$ is contained in some $Q_{3,0}$. □

Proposition 6. *Fix an even integer $g \ge 6$ and a general $X \in \mathcal{M}_g$. There is $L \in \mathrm{Pic}^{g+3}(X)$ such that L is spanned, u_L is birational onto, $h^1(X,L) = 1$ and $u_L(X) \subset Q_{3,0}$.*

Proof. There are $M, R \in \mathrm{Pic}^{g+2}(X)$ such that $h^0(X,M) = h^0(X,R) = 0$, M, R are spanned and $h^0(X, M \otimes R) = 4$. By Riemann-Roch we get $h^1(X, M \otimes$

$R) = 1$. Hence $h^0(X, \omega_X \otimes M^* \otimes R^*) = 1$. Thus $|\omega_X \otimes M^* \otimes R^*|$ is formed by a unique divisor, D. Since $g \geq 6$, $D_{red} \neq \emptyset$. Fix $P \in D_{red}$ and set $L := M \otimes R(P)$. By Rieman-Roch we get $h^1(X, L) = 1$ and $h^0(X, L) = 5$. Since M, R are spanned and $h^0(X, L) > h^0(X, M \otimes R)$, L is spanned. Since $u_{M \otimes R}(X)$ is contained in a rank 4 quadric, $u_L(X)$ is contained in a rank 4 quadric $Q_{3,0}$. The generality of X implies that $U_{M \otimes R}$ is birational onto its image for degree reasons. Hence u_L is birational onto its image. □

Remark 6. Fix integers g, n such that $g \geq 0$ and $n \geq 4$. If $d \gg 0$, say $d \geq 2g + n$, it is easy to adapt the proof of Proposition 2 to get the existence of $L \in \mathrm{Pic}^d(X)$ and a linear subspace $V \subset H^0(X, L)$ such that $\dim(V) = n + 1$, $u_{L,V}$ is an embedding and $u_{L,V}(X) \subset Q_{n-1,n-4}$.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) \neq 2$. In the proof of Lemma 2 (and hence the case $n = 3$ and $d < g + 4$ or $n \geq 4$ of Theorem 1) we used the assumption $\mathrm{char}(\mathbb{K}) = 0$. We do not know if Lemma 2 is true in positive characteristic.

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