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PARABOLIC HÖLDER NORMS AND MEASURES IN NON-SMOOTH DOMAINS

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Abstract: Let $u(x,t)$ be a solution to an initial/boundary value problem for a second order parabolic equation on a bounded domain Ω_T in R^{n+1}. Let $\omega = \omega^{(X_0,T_0)}$ denote the boundary measure of Ω_T, generated by the operator $\partial/\partial t - L$. Given some technical restrictions on the parabolic operator and on the domain Ω_T, conditions for μ, a Borel measure on Ω_T, and $\nu d\omega$, a measure on the parabolic boundary of Ω_T, are found and shown to be sufficient to control the $L^q(\Omega_T,\mu)$ norm of a local Hölder norm of $u(x,t)$ by the $L^p(\partial_p\Omega_T,\nu d\omega)$ norm of u's boundary values.

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1. Introduction

The intention of this paper is to prove comprehensive norm estimates of the form

$$\left(\int_{\Omega_T} \|u(x,t)\|^q \, d\mu(x,t)\right)^{1/q} \leq C \left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \, \nu(z,\tau) d\omega(z,\tau)\right)^{1/p}, \qquad (1)$$

for solutions to

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$$
\begin{aligned}
&(\partial/\partial t - L)\, u(x,t) = 0 \ (x,t) \in \Omega_T\,, \\
&u(z,\tau) = f(z,\tau) \ (z,\tau) \in \partial_p \Omega_T\,.
\end{aligned} \tag{2}
$$

$L = \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial}{\partial x_j})$ is a strictly parabolic operator and Ω_T is a bounded domain in $\mathbb{R}^{n+1}$. These results continue in the line of work set forth in Sweezy and Wilson [7] concerning the inequality (1) with $\|u(x,t)\| = |\nabla_x u(x,t)|$. We are interested in discovering the measures, μ and $\nu d\omega$, for which we can obtain such an inequality, with as large a range of exponents, p and q, as possible. If $\|u(x,t)\| = |\nabla_x u(x,t)|$ with $u(x,t)$ a solution to the heat equation in the upper half space, this question was answered in work of Wheeden and Wilson [8], who obtained conditions on the respective measures as part of their investigation of the situation in which $u(x)$ is a harmonic function in $\mathbb{R}^n_+$. Building on subsequent work of Wilson [9], Sweezy and Wilson extended their investigation of this problem to the case of rough boundary domains and to the gradient of a solution to a more general second order operator equation. In this paper, following a suggestion of Professor Wheeden, instead of the gradient of a solution, we consider the quantity

$$
\|u(x,t)\| = \|u(x,t)\|_{H^{\alpha}_{loc}} \equiv \sup_{(y,s)\in P_{\delta/100}(x,t)} \frac{|u(y,s)-u(x,t)|}{\left(|x-y| + |t-s|^{1/2}\right)^{\alpha}},
$$

where $P_{\delta/100}(x,t) = \{(y,s) : d_p(y,s;x,t) \equiv |x-y| + |t-s|^{1/2} < \delta(x,t)/100\}$, with $\delta(x,t) = d_p(x,t;\partial_p\Omega_T) \equiv \inf\{\left(|x-y| + |t-s|^{1/2}\right),\ (y,s) \in \partial_p\Omega_T\}$. The parabolic boundary, $\partial_p\Omega_T$, of the domain Ω_T consists of the lateral boundary and the "bottom" part of the boundary of the domain Ω_T. The precise definitions will be given in Section 2 of this paper.

There are several reasons for considering a local Hölder norm instead of working with the gradient, $\nabla_x u(x,t)$. One important reason is that with a Hölder norm one can gain control of the rate of change of the temperature function as it changes in time as well as its rate of change with respect to the space variable, x. Another reason is that, for the most general kind of operator whose solutions are amenable to our methods, namely a strictly parabolic, divergence form operator, $L = \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial}{\partial x_j})$, with the coefficients $a_{ij}(x,t)$ bounded and measurable, and with there being a constant $\lambda > 0$ so that $\lambda\,|\zeta|^2 \leq \sum \zeta_i a_{ij}(x,t)\zeta_j$, we were obliged to put additional restrictions on the range of the exponents, p and q, for which we could prove an inequality of the nature of (1) for $\|u(x,t)\| = |\nabla u(x,t)|$. We also had to assume an extra condition on the measure μ. Note that the additional restriction on p and q and the extra condition on μ are not necessary in dealing with solutions to the heat operator, $\partial/\partial t - \Delta$.

The results in this paper will be proved for boundary data $f(z,\tau)$ not assumed to be zero on the "bottom" part of the boundary, i.e. on $\tau = 0$.

In order to state the two main theorems, it is necessary to establish several technical definitions. Section 2 of this paper is devoted to describing precisely the setting in which the theorems will be shown to be valid, and stating Theorems A and B. Theorem B is the Littlewood-Paley type inequality which we need to prove Theorem A. In Section 3 of this paper we assume that Theorem B is valid and prove Theorem A. Section 4 is given to establishing the estimates that will allow us to prove Theorem B. The case of non-zero boundary data on $t = 0$ was not considered in [7]; to deal with $f(z, 0) \neq 0$, one must establish several key estimates across the bottom of the given domain Ω_T for the kernel function associated to the operator $\partial/\partial t - L$.

The overall scheme for proving Theorems A and B is basically the method employed in [7] for the gradient norm. One writes the integral to be estimated, $\int_{\Omega_T} \|u(x,t)\|^q_{H^\alpha_{loc}}\, d\mu(x,t)$, as a semi-discrete expression involving a dual operator, applies the appropriate Littlewood-Paley type inequality (Theorem B) to the dual operator integral, and derives sufficient conditions on the measures from there. To avoid unnecessary repetition we only prove the new results/estimates that are needed to establish the results of this paper; whenever possible we refer the reader to previously established facts.

2.

Points in $\mathbb{R}^{d+1}$ are denoted by (x', x_d, t) with $x' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}^1$ and $t \in \mathbb{R}^1$. The domain $\Omega_T \subset \mathbb{R}^{d+1}$ will be taken to be a bounded domain whose boundary, $\partial\Omega_T$, consists of three parts, a top part $T\Omega_T \equiv \{(x,t) \in \overline{\Omega_T} : t = T\}$, a bottom part $B\Omega_T = B = \{(x,t) \in \overline{\Omega_T} : t = 0\}$ and a lateral part $S\Omega_T = S_T \equiv \{(x,t) \in \partial\Omega_T : 0 < t < T\}$. The lateral boundary can be described locally as the graph of a Lipschitz function, which may have been rotated and/or translated. More precisely $\partial_p\Omega_T$ can be covered by finitely many cylinders $\Psi_R(z,\tau) \equiv \{(x,t) : |x - z| < R, |\tau - s| < R^2\}$, with $(z,\tau) \in \partial_p\Omega_T$; if $(z,\tau) \in S_T$, then $\Psi_R(z,\tau) \cap S_T = \{(w,\sigma) : w_d = \psi(w',\sigma)$ where $|\psi(y',s) - \psi(x',t)| \leq M(|y'-x'| + |t-s|^{1/2})\}$. The constant $M > 0$ is called the Lipschitz constant of the domain Ω_T.

It is best to think of the domain Ω_T as being a finite part of a larger domain Ω that is infinite in the time variable, so that $\Omega_T = \Omega \cap \{0 < t < T\}$.

Every point P on $\partial\Omega$ satisfies the condition that there is a polygonal curve γ lying completely inside Ω which starts at the point P and has a strictly increasing time coordinate. For the kinds of operators we will be considering, $\partial/\partial t - L$ as described above, Aronson [1] proved the existence and uniqueness of the fundamental solution $\Gamma(x,t;y,s)$ on $\mathbb{R}^{d+1}$. The Green's function for the operator on a given domain Ω_T can be taken to be $G(x,t;y,s) \equiv \Gamma(x,t;y,s) - \int_{\partial_p\Omega_T} \Gamma(w,\tau;y,s)d\omega^{(x,t)}(w,\tau)$; it is not hard to see that the Green's function of Ω, when it is restricted to Ω_T, will be identical to the Green's function on Ω_T (see [5]). We exploit this fact several times.

The solvability of the Dirichlet problem on Ω_T was proved by N. Eklund [2]. K. Nystrom proved the existence of a kernel function $K(x,t;y,s)$ associated to the operator $\partial/\partial t - L$ on Ω_T in addition to several crucial properties of K. He also established a representation for any solution to (2), see [5]. We will be utilizing this representation in the form $u(x,t) = \int_{\partial_p\Omega_T} f(z,\tau)K(x,t;z,\tau)d\omega(z,\tau)$.

There is a collection of regions in $\partial_p\Omega_T$ that will be important to the proofs of both theorems. These regions will be called "parabolic dyadic cubes" and denoted by $\mathcal{D}$. At each level these "cubes" have the essential properties of being nested or disjoint, (all cubes at a given level are disjoint), they have comparable dimension, and they cover the entire parabolic boundary. To create this family of "cubes", we can take the images under ψ (the local Lipschitz map that defines S_T) of actual nonisotropic dyadic cubes that cover the Euclidean unit cube in $\mathbb{R}^d$. So a cube in $\mathcal{D}$ will be written as $Q_b = \psi(Q)$, where the side length of Q, $l(Q) = 2^{-k}$ for some $k = -1, 0, 1, 2, ..., n, n+1, ...$ $Q = \{(x', t) : \left| x'_j - x'_{0,j} \right| < 2^{-k}, |t - t_0| < 2^{-2k}\}$. The definition of ψ implies that $\operatorname{diam}(Q_b) \sim 2^{-k}$. The cubes that lie completely on $B\Omega$ will be Euclidean cubes. A cube that intersects both S_T and $B\Omega$ can be considered to be a lateral boundary cube. It is possible to choose the "cubes" that lie in the overlap of two Lipschitz cylinders' intersection with $\partial_p\Omega_T$ to be disjoint; at worst they have bounded overlap at each level. Most calculations are done locally within a single cylinder, so allowing such overlap will only increase bounding constants by a fixed amount. We will need to utilize regions that are the analogue of 2^j dilates of a dyadic cube and the annular regions $2^jQ\backslash 2^{(j-1)}Q$. As long as the dilate in $\mathbb{R}^d$, 2^jQ, lies in the unit cube containing Q, 2^jQ_b will be $\psi(2^jQ)$. Once the region intersects the part of $\partial_p\Omega_T$ that lies outside the coordinate cylinder containing Q_b, one can align the cubes in $\mathbb{R}^d$ that are mapped to neighboring coordinate cylinders in $\mathbb{R}^{d+1}$ and take 2^jQ_b to be the image of 2^jQ on $\partial_p\Omega_T$ under the various coordinate maps. Or more simply one can define $2^jQ_b \equiv \{(z,\tau) : d_p((z,\tau);Q_b) \lesssim 2^j\}$. Either way, once j is large enough, we define $2^j Q_b$ to be $\partial_p \Omega_T$. For such j, $R_{j+1}(Q_b) \equiv 2^{j+1} Q_b \backslash 2^j Q_b = \emptyset$.

To each boundary cube Q_b we associate the upper "half" of a Carleson box, $T(Q_b) \subset \Omega_T$. This region can be defined to be the image under the map $\Phi(x', x_d, t) = (x', x_d + \psi(x', t), t)$ of the upper half of the Carleson rectangle in $\mathbb{R}^{d+1}$ whose base is the parabolic cube Q, and whose height is $l(Q)$. Φ is one-to-one, so the $T(Q_b)$ within a single coordinate cylinder are disjoint. In this way, $T(Q_b)$ will be a Whitney-type region in Ω_T, its diameter comparing with is distance from $\partial_p \Omega_T$, which in turn compares with its distance from Q_b. Again, on the overlap of the cylinders $\Psi_R \cap \Omega_T$ the $T(Q_b)$'s have at worst bounded overlap. $\delta > 0$ is chosen so that the region in Theorem A, $\Omega_{T,\delta}$, lies inside $\bigcup_{Q_b \in \mathcal{D}} T(Q_b)$. Notice that a fixed dilate of any $T(Q_b)$ will remain inside Ω_T; indeed, up to a fixed constant, such a dilate will retain its Whitney-type character.

Remember that distance, dimension and diameter refer to the parabolic metric, $d_p(x, t; y, s) \equiv |x - y| + |t - s|^{1/2}$. Now some standard definitions that will be used throughout the paper are given.

A boundary disk is defined to be the intersection of a local cylinder with the parabolic boundary of the domain, $\Delta_r(z, \tau) \equiv \Psi_r(z, \tau) \cap \partial_p \Omega_T$. For any parabolic dyadic cube, Q_b, there is an $r \sim l(Q)$ so that for fixed constants c_1 and c_2, $\Delta_{c_1 r}(z, \tau) \subset Q_b \subset \Delta_{c_2 r}(z, \tau)$, where (z, τ) can be taken to be $\psi(x_0, t_0)$ if (x_0, t_0) is the center of Q in $\mathbb{R}^d$. If $\tau = 0$, and $r \leq \delta(z, \tau; S_T)$, then $\Delta_r(z, \tau)$ will be a Euclidean disk in $B\Omega$.

There are the "Harnack" points, $\overline{A_r}(z, \tau)$ and $\underline{A_r}(z, \tau)$, which are defined as follows. If $(z, \tau) \in S_T$, then $\overline{A_r}(z, \tau) = (z', z_d + Cr, \tau + 4r^2)$ and $\underline{A_r}(z, \tau) = (z', z_d + Cr, \tau - 4r^2)$. If $\tau = 0$, and $r \leq \delta(z, \tau; S_T)$, then $\overline{A_r}(z, 0) = (z, 4r^2)$ and $\underline{A_r}(z, 0) = (z, -4r^2)$. But for $\tau = 0$, and $r > \delta(z, \tau; S_T)$, then $\overline{A_r}(z, 0) = \overline{A_r}(z_1, 0)$ where $(z_1, 0) \in \overline{S_T} \cap \{t = 0\}$ and $|z - z_1|$ is minimal for $z_1 \in \overline{S_T}$. Similarly $\underline{A_r}(z, 0) = \underline{A_r}(z_1, 0)$ in this case.

We will be assuming that a certain measure, call it ν, is a measure that is A^∞ with respect to ω. We write this in symbols as "$\nu \in A^\infty(\omega)$" (or $d\nu \in A^\infty(d\omega)$); it simply means that there are positive constants a and b so that, for all cubes Q and measurable sets $E \subset Q$,

$$\frac{\nu(E)}{\nu(Q)} \leq a \left(\frac{\omega(E)}{\omega(Q)} \right)^b .$$

In the context of this paper the cubes Q will be the previously defined non-isotropic cubes. The fact that the parabolic measure ω satisfies a doubling condition means that this definition is valid putting boundary cubes Q_b or boundary disks Δ_r in place of Q.

The conjugate Hölder index to p, $1 \le p \le \infty$, will be designated by p', that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Now we can state the two main theorems of the paper.

Theorem A. *For Ω_T and $\partial/\partial t - L$ as described above, assume that $u(x,t)$ is a weak solution of*

$$\begin{aligned}(\partial/\partial t - L)\, u(x,t) &= 0 \ (x,t) \in \Omega T \\ u(z,\tau) &= f(z,\tau) \ (z,\tau) \in \partial_p \Omega_T\,,\end{aligned}$$

with $f(z,\tau) \in L^\infty(\partial_p\Omega_T, d\omega)$, and $\omega = \omega^{(X_0,T)}$ being the parabolic measure on $\partial_p\Omega_T$ generated by the operator $\partial/\partial t - L$, measured from the fixed point (X_0, T). Let μ be a Borel measure defined on Ω_T, and let ν be a non-negative weight defined on $\partial_p\Omega_T$ so that $\nu \in L^1_{loc}(\partial_p\Omega_T, d\omega)$. Further assume that for $\sigma(z,\tau) \equiv (\nu(z,\tau))^{1-p'}$, then $\sigma d\omega \in A^\infty(d\omega)$. Suppose for all parabolic cubes Q_b on $\partial_p\Omega_T$, with $T(Q_b)$ denoting the top half of the Carleson-type region associated to Q_b as described above,

$$\mu(T(Q_b))^{1/q} \times \left(\int_{sl\partial_p\Omega_T} \left(\omega(Q_b) \sum_{j=0}^{\infty} \frac{2^{-j(2\beta-\eta)}}{\omega(2^j Q_b)\}} \chi_{R_j(Q_b)}(z,\tau) \right)^{p'/2} \sigma(z,\tau) d\omega(z,\tau) \right)^{1/p'} \le \omega(Q_b) l(Q_b)^{2\alpha}.$$

Then there is a constant $C = C(d, \lambda, \alpha, \beta, \delta, \eta, \Omega_T, r_0, p, q)$ so that for $1 < p \le q < \infty$, $q \ge 2$, and $\Omega_{T,\delta} \equiv \{(x,t) \in \Omega_T : \delta(x,t) < \delta\}$, the following inequality is valid:

$$\left(\int_{\Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} \, d\mu(x,t) \right)^{1/q} \le C \left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \, \nu(z,\tau) d\omega(z,\tau) \right)^{1/p}.$$

Remark. It will be shown below that an analogous condition on $\Omega_T \backslash \Omega_{T,\delta}$ is sufficient to prove that

$$\left(\int_{\Omega_T \backslash \Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} \, d\mu(x,t) \right)^{1/q} \le C \left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \, \nu(z,\tau) d\omega(z,\tau) \right)^{1/p},$$

adding, we obtain the result of (1) for the entire domain:

$$\left(\int_{\Omega_T} \|u(x,t)\|^q_{H^\alpha_{loc}} \, d\mu(x,t) \right)^{1/q} \le C \left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \, \nu(z,\tau) d\omega(z,\tau) \right)^{1/p}.$$

As mentioned above we need the result of Theorem B to prove Theorem A. The conditions that the family of functions $\{\phi_{(Q_b)}\}$ in Theorem B must satisfy are geometric decay, smoothness and cancellation. The precise form of these conditions are given in (i), (ii), and (iii).

(i) $\left|\phi_{(Q_b)}(z,\tau)\right| \lesssim (\omega(Q_b))^{1/2}(2^{-j\beta}/\omega(2^jQ_b))$ for $(z,\tau)\in R_j(Q_b)$.

(ii) For (z,τ) and (w,s) both in $sl\partial_p\Omega\cap R_k(Q_b)$ for some $k\geq 0$,

$$\left|\phi_{(Q_b)}(z,\tau)-\phi_{(Q_b)}(w,s)\right| \lesssim \left(\frac{d_p(z,\tau;w,s)}{l(Q_b)}\right)^{\alpha}\omega(Q_b)^{1/2}\,,$$
$$\sum_{j=0}^{\infty}((2^{-j\beta}/\omega(2^jQ_b))(\chi_{R_j(Q_b)}(z,\tau)+\chi_{R_j(Q_b)}(w,s))\,.$$

(iii) For any $\{\lambda_{Q_b}\}_{Q_b\in\mathcal{D}}$, $\displaystyle\int_{\partial_p\Omega_T}\left|\sum_{Q_b\in\mathcal{F}}\lambda_{Q_b}\phi_{(Q_b)}(z,\tau)\right|^2 d\omega(z,\tau)$

$$\lesssim\sum_{Q_b\in\mathcal{F}}\left|\lambda_{Q_b}\right|^2,\ \mathcal{F}\text{ is a finite family of cubes.}$$

Theorem B. *Let $\{\varphi_{(Q_b)}\}_{Q_b\in\mathcal{D}}$ be a family of functions defined on $\partial_p\Omega_T = S\Omega_T\cup B\Omega_T$, the parabolic part of $\partial\Omega_T$, so that the $\varphi_{(Q_b)}$ satisfy conditions (i), (ii), and (iii) given above. Let $\sigma d\omega\in A^\infty(\omega,\partial_p\Omega_T)$. For $0<p<\infty$ there is a constant $C=C(\alpha,\beta,\lambda,d,\eta,\Omega_T,r_0,p)$ such that, for any finite sum $f(z,\tau)=\sum_{Q_b\in\mathcal{F}}\lambda_{Q_b}\phi_{(Q_b)}(z,\tau)$, then*

$$\int_{sl\partial_p\Omega_T}|f(z,\tau)|^p\,\sigma(z,\tau)d\omega(z,\tau)\leq C\int_{sl\partial_p\Omega}|g^*(f)(z,\tau)|^p\,\sigma(z,\tau)d\omega(z,\tau),$$

where, for functions of the form $f(z,\tau)=\sum_{Q_b\in\mathcal{F}}\lambda_{Q_b}\phi_{(Q_b)}(z,\tau)$, as given above, $g^*(f)(z,\tau)$ is defined by

$$g^*(f)(z,\tau)=\left(\sum\left|\lambda_{Q_b}\right|^2\cdot\sum_{j=0}^{\infty}(2^{-j(2\beta-\eta)}/\omega(2^jQ_b))\cdot\chi_{R_j(Q_b)}(z,\tau)\right)^{1/2}.$$

3.

In this section we prove Theorem A assuming that Theorem B is valid.

Proof. To see that $\left(\int_{\Omega_T} \|u(x,t)\|^q_{H^\alpha_{loc}}\, d\mu(x,t)\right)^{1/q}$ is bounded by

$$C\left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p\, \nu(z,\tau) d\omega(z,\tau)\right)^{1/p},$$

we divide Ω_T into two regions $\Omega_{T,\delta}$ and $\Omega_T \backslash \Omega_{T,\delta}$. $\Omega_{T,\delta}$ is further subdivided into the Whitney-type regions $T(Q_b)$. In fact the second region $\Omega_T \backslash \Omega_{T,\delta}$ can also be thought of as a Whitney-type region since it consists of points in Ω_T whose distance from $\partial_p\Omega_T$ is greater than or equal to $\delta > 0$, a fixed constant. We start by considering the integral over this region.

$$\int_{\Omega_T\backslash\Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}}\, d\mu(x,t) = \int_{\Omega_T\backslash\Omega_{T,\delta}} \left(\sup_{(y,s)\in P_{\delta(x,t)/100}} \frac{|u(y,s)-u(x,t)|}{(d_p(x,t;y,s))^\alpha}\right)^q d\mu(x,t)\,.$$

Now, for $(x,t) \in \Omega_T\backslash\Omega_{T,\delta}$,

$$\begin{aligned}\left(\sup_{(y,s)\in P_{\delta(x,t)/100}} \frac{|u(y,s)-u(x,t)|}{(d_p(x,t;y,s))^\alpha}\right) &\le \left(\sup_{(y,s)\in P_{\delta(x,t)/50}} \int_{\partial_p\Omega_T} |f(z,\tau)|\, \frac{C\,(d_p(x,t;y,s))^\alpha\, K(y,s;z,\tau)}{(\delta)^\alpha\,(d_p(x,t;y,s))^\alpha} d\omega(z,\tau)\right)\\ &\le \left(\int_{\partial_p\Omega_T} |f(z,\tau)|\, \frac{CK(x_\delta,t_\delta;z,\tau)}{(\delta)^\alpha} d\omega(z,\tau)\right).\end{aligned}$$

So we obtain

$$\begin{aligned}\int_{\Omega_T\backslash\Omega_{T,\delta}} &\left(\sup_{(y,s)\in P_{\delta(x,t)/100}} \frac{|u(y,s)-u(x,t)|}{(d_p(x,t;y,s))^\alpha}\right)^q d\mu(x,t)\\ &\le \int_{\Omega_T\backslash\Omega_{T,\delta}} \left(\int_{\partial_p\Omega_T} |f(z,\tau)|\, \frac{CK(x_\delta,t_\delta;z,\tau)}{(\delta)^\alpha} d\omega(z,\tau)\right)^q d\mu(x,t).\end{aligned}$$

The first inequality above follows from substituting the representation for the solution u in terms of the integral of the boundary function f with respect to the kernel function K, and then from applying interior Hölder continuity to the kernel function in the forward variable; the second inequality is obtained using Harnack's inequality. Notice that $K(x,t;z,\tau)$ is a non-negative solution to $(\partial/\partial t - L)\, w(x,t) = 0$ $(x,t) \in \Omega_T$ for each (z,τ) in $\partial_p\Omega_T$. (x_δ, t_δ) is a point in $\Omega_T \backslash \Omega_{T,\delta}$.

We also have the standard estimate $K(x_\delta, t_\delta; z,\tau) \le \frac{C(d,\lambda,\Omega_T,\delta)}{\omega(Q_{b,\delta})}$ (see Theorem 4.4 in [5]), where $l(Q_{b,\delta}) \sim \delta$. Using this it is easy to see that if

$$(\mu(\Omega_T \backslash \Omega_{T,\delta}))^{1/q} \int_{\partial_p\Omega_T} |f(z,\tau)| \frac{C}{\delta^\alpha \omega(Q_{b,\delta})} d\omega(z,\tau) \le C \left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \nu(z,\tau) d\omega(z,\tau) \right)^{1/p},$$

the desired result will be valid. Employing Hölder's inequality on the integral $\left(\int_{\partial_p\Omega_T} |f(z,\tau)| \frac{C}{\delta^\alpha \omega(Q_{b,\delta})} d\omega(z,\tau) \right)$, we see it is bounded by

$$\left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \nu(z,\tau) d\omega(z,\tau) \right)^{1/p} \frac{C}{\delta^\alpha \omega(Q_{b,\delta})} \left(\int_{\partial_p\Omega_T} \nu(z,\tau)^{-p'/p} d\omega(z,\tau) \right)^{1/p'},$$

so the condition

$$(\mu(\Omega_T \backslash \Omega_{T,\delta}))^{1/q} \left(\int_{\partial_p\Omega_T} (\nu(z,\tau))^{(1-p')} d\omega(z,\tau) \right)^{1/p'} \le C \left(l(Q_{b,\delta}) \right)^\alpha \omega(Q_{b,\delta}), \quad (4)$$

is sufficient to give

$$\left(\int_{\Omega_T \backslash \Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t) \right)^{1/q} \le C \left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \nu(z,\tau) d\omega(z,\tau) \right)^{1/p}.$$

But condition (4) is simply a version of the condition given in Theorem A for the interior Whitney-type region $\Omega_T \backslash \Omega_{T,\delta}$.

To prove Theorem A for $\Omega_{T,\delta}$, we first write $\int_{\Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t)$ as $\sum_{i=0}^{m} \int_{\Psi_{r_0}(z_i,\tau_i) \cap \Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t)$, where the $\Psi_{r_0}(z_i,\tau_i)$ are the Lipschitz coordinate cylinders that cover $\Omega_{T,\delta}$, i.e. $\bigcup_{i=0}^{m} \Psi_{r_0}(z_i,\tau_i) \cap \Omega_T \supseteq \Omega_{T,\delta}$. We will treat one region at a time; so we can assume that μ is supported in the particular cylinder we are focusing on. Let $\mathcal{R} \equiv \{x', x_d, t) \in \Omega_{T,\delta} : (x', t) \in Q_1 \subseteq \mathbb{R}^1, \psi(x', t) < x_d < \psi(x', t) + \delta\}$. For $t_i \geq r_0^2$ this region describes part of a cylinder $\Psi_{r_0}(z_i, \tau_i) \cap \Omega_T$ such that $\Psi_{r_0}(z_i, \tau_i) \cap \partial_p \Omega_T \subset S_T$. For $\Psi_{r_0}(z_i, \tau_i) \cap \partial_p \Omega_T \subset B\Omega$ the region $\mathcal{R}$ will be of the form $\{(x, t) : x \in Q_2, 0 < t < \delta^2\}$. $Q_1 \equiv \{(x', t) : |x_i| < R,\ i = 1, 2, ..., d-1, \text{ and } |t| < R^2\}$, $Q_2 \equiv \{x : |x_i| < R,\ i = 1, 2, ..., d\}$. If $\Psi_{r_0}(z_i, \tau_i)$ contains parts of both the side and bottom boundary, we can choose to treat $\Psi_{r_0}(z_i, \tau_i)$ as a "side" cylinder. The argument is the same for both kinds of regions, so we can assume that $\mathcal{R} = \Psi_{r_0}(z_i, \tau_i) \cap \Omega_{T,\delta}$ is a region on the lateral boundary. Now write $\int_{\Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t)$ as $\sum_{Q_b \in \mathcal{D}} \int_{T(Q_b)} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t)$, taking the support of $\mu \subseteq \mathcal{R}$, and proceed as above with

$$\int_{T(Q_b)} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t) \leq \mu(T(Q_b) \left(\sup_{(x,t) \in T(Q_b), (y,s) \in P_{\delta/100}(x,t)} \left(\frac{|u(y,s) - u(x,t)|}{(d_p(x,t;y,s))^\alpha} \right) \right)^q .$$

And since

$$\begin{aligned} &|u(y,s) - u(x,t)| \\ &= \left| \int_{\partial_p \Omega_T} f(z,\tau) K(y,s;z,\tau) d\omega(z,\tau) - \int_{\partial_p \Omega_T} f(z,\tau) K(x,t;z,\tau) d\omega(z,\tau) \right| \\ &\leq C \int_{\partial_p \Omega_T} |f(z,\tau)| \frac{d_p(y,s;x,t)^\alpha}{l(Q_b)^\alpha} K(x_{T(Q_b)}, t_{T(Q_b)}; z, \tau) d\omega(z,\tau) , \end{aligned}$$

from Hölder continuity of the kernel function and from Harnack's inequality for non-negative solutions to (2), one has that

$$\sup_{(x,t) \in T(Q_b), (y,s) \in P_{\delta/100}(x,t)} \left(\frac{|u(y,s) - u(x,t)|}{(d_p(x,t;y,s))^\alpha} \right) \leq \frac{C}{l(Q_b)^\alpha} \int_{\partial_p \Omega_T} |f(z,\tau)| K(x_{T(Q_b)}, t_{T(Q_b)}; z, \tau) d\omega(z,\tau).$$

To bound

$$\left(\sum_{Q_b \in \mathcal{D}} \mu(T(Q_b)) \left(\frac{C}{l(Q_b)^\alpha} \int_{\partial_p \Omega_T} |f(z,\tau)| K(x_{T(Q_b)}, t_{T(Q_b)}; z, \tau) d\omega(z,\tau) \right)^q \right)^{1/q}$$

by $C\left(\int_{\partial_p\Omega_T}|f(z,\tau)|^p\,\nu(z,\tau)d\omega(z,\tau)\right)^{1/p}$, it is enough to bound, for all $\{g(Q_b)\}$ with only finitely many nonzero terms, such that

$$\|\{g(Q_b)\}\|_{l^{q'}(\mu)}=\left(\sum_{Q_b\in\mathcal{D}}g(Q_b)^{q'}\mu(T(Q_b))\right)^{1/q'}\leq 1,$$

either

$$\sum_{Q_b\in\mathcal{D}}g(Q_b)\left(\frac{C}{l(Q_b)^\alpha}\int_{\partial_p\Omega_T}|f(z,\tau)|\,K(x_{T(Q_b)},t_{T(Q_b)};z,\tau)d\omega(z,\tau)\right)\mu(T(Q_b)),$$

or

$$\int_{\partial_p\Omega_T}\sum_{Q_b\in\mathcal{D}}\frac{g(Q_b)\mu(T(Q_b))}{l(Q_b)^{2\alpha}}\times\sqrt{\omega(Q_b)}l(Q_b)^\alpha\sqrt{\omega(Q_b)}K(x_{T(Q_b)},t_{T(Q_b)};z,\tau)\,|f(z,\tau)|\,d\omega(z,\tau)\,,$$

by $C\left(\int_{\partial_p\Omega_T}|f(z,\tau)|^p\,\nu(z,\tau)d\omega(z,\tau)\right)^{1/p}$ (we may assume that $g(Q_b)\geq 0$). This is equivalent to bounding

$$\sup_{\|\{g(Q_b)\}\|_{l^{q'}(\mu)}\leq 1}\int_{\partial_p\Omega_T}T(g(Q_b))(z,\tau)\,|f(z,\tau)|\,d\omega(z,\tau)$$

given that $T(g(Q_b))(z,\tau)=\sum_{Q_b\in\mathcal{D}}\lambda_{Q_b}\varphi_{(Q_b)}(z,\tau)$, where $\lambda_{Q_b}=\frac{g(Q_b)\mu(T(Q_b))}{l(Q_b)^{2\alpha}\sqrt{\omega(Q_b)}}$ and $\varphi_{(Q_b)}(z,\tau)=l(Q_b)^\alpha\sqrt{\omega(Q_b)}K(x_{T(Q_b)},t_{T(Q_b)};z,\tau)$. Now Hölder's inequality implies

$$\int_{\partial_p\Omega_T}T(g(Q_b))(z,\tau)\,|f(z,\tau)|\,d\omega(z,\tau)\leq\left(\int_{\partial_p\Omega_T}(T(g(Q_b))(z,\tau))^{p'}\,\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'}\times\left(\int_{\partial_p\Omega_T}|f(z,\tau)|^p\,\nu(z,\tau)d\omega(z,\tau)\right)^{1/p};$$

so we want to show that

$$\left(\int_{\partial_p\Omega_T}(T(g(Q_b))(z,\tau))^{p'}\,\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'}\leq\left(\sum_{Q_b\in\mathcal{D}}g(Q_b)^{q'}\mu(T(Q_b))\right)^{1/q'}.$$

Theorem B can be used to deal with the integral of $T(g(Q_b))^{p'}$ on the part of $\partial_p\Omega_T$ that lies close to a single coordinate cylinder, $\mathcal{R} = \Psi_{r_0}(z_0,\tau_0)\cap\Omega_{T,\delta}$ as above. Let $\mathcal{N} \equiv \{(z,\tau)\in\partial_p\Omega_T : d_p(z,\tau;\mathcal{R}) \geq \kappa_0\}$ for some fixed constant $\kappa_0 > 0$. As in [7] we can bound $\int_{\mathcal{N}} T(g(Q_b))(z,\tau)^{p'}\sigma(z,\tau)d\omega(z,\tau)$ directly. For $(z,\tau)\in\mathcal{N}$, by (i),

$$|T(g(Q_b))(z,\tau)| = \sum_{Q_b\in\mathcal{D}\cap cl(\mathcal{R})} |\lambda_{Q_b}|\,\left|\varphi_{(Q_b)}(z,\tau)\right|$$

$$\leq \sum |\lambda_{Q_b}|\, l(Q_b)^{\alpha}\sqrt{\omega(Q_b)}\sum_{j=0}^{N_0}\frac{2^{-j(\beta)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(z,\tau).$$

The fact that (z,τ) is far from the cubes Q_b for which $\mu(T(Q_b))\neq 0$ means that for $(z,\tau)\in R_j(Q_b)$, we have $l(2^jQ_b)\sim l(\partial_p\Omega_T)$, $2^{-j}\sim l(Q_b)$ and there are only a fixed number (independent of (z,τ)) of $R_j(Q_b)$ for which $\chi_{R_j(Q_b)}(z,\tau)\neq 0$. The result is that

$$|T(g(Q_b))(z,\tau)| \leq C\sum |\lambda_{Q_b}|\, l(Q_b)^{(\alpha+\beta)}\sqrt{\omega(Q_b)} =$$
$$C\sum\frac{g(Q_b)\mu(T(Q_b))}{l(Q_b)^{2\alpha}\sqrt{\omega(Q_b)}}l(Q_b)^{(\alpha+\beta)}\sqrt{\omega(Q_b)} = C\sum g(Q_b)\mu(T(Q_b))l(Q_b)^{(\beta-\alpha)}.$$

By the condition in Theorem A we see that

$$\mu(T(Q_b))^{1/q} \leq l(Q_b)^{2\alpha-\beta+\eta/2}\sqrt{\omega(Q_b)}\left(\int_{\mathcal{N}}\sigma d\omega\right)^{-1/p'}.$$

This estimate is valid because $\mathcal{N}$ is a smaller region than $\partial_p\Omega_T$, and we can see that for $(z,\tau)\in\mathcal{N}$,

$$\left(\omega(Q_b)\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(z,\tau)\right)^{p'/2} \leq \left(\omega(Q_b)l(Q_b)^{(2\beta-\eta)}\right)^{p'/2}.$$

Writing $\mu(T(Q_b)) = \mu(T(Q_b))^{(1/q'+1/q)}$ and using the estimate on $\mu(T(Q_b))^{1/q}$, this means that

$$T(g(Q_b))(z,\tau)$$
$$\leq C\sum g(Q_b)\mu(T(Q_b))^{1/q'}l(Q_b)^{(\alpha+\eta/2)}\sqrt{\omega(Q_b)}\left(\int_{\mathcal{N}}\sigma d\omega\right)^{-1/p'}.$$

Then Hölder's inequality implies

$$\sum g(Q_b)\mu(T(Q_b))^{1/q'} l(Q_b)^{(\alpha+\eta/2)}\sqrt{\omega(Q_b)} \leq \left(\sum g(Q_b)^{q'}\mu(T(Q_b))\right)^{1/q'} \left(\sum \left(l(Q_b)^{(\alpha+\eta/2)}\sqrt{\omega(Q_b)}\right)^q\right)^{1/q}.$$

Using $q \geq 2$ which gives $2/q \leq 1$, it is easy to see that

$$\left(\sum \left(l(Q_b)^{(\alpha+\eta/2)}\sqrt{\omega(Q_b)}\right)^q\right)^{1/q} \leq \left(\sum_{Q_b\in\mathcal{D}\cap cl(\mathcal{R})} \left(l(Q_b)^{(2\alpha+\eta)}\omega(Q_b)\right)\right)^{1/2} \leq C_0.$$

The last inequality follows as $2\alpha+\eta>0$.

Since $\left(\sum g(Q_b)^{q'}\mu(T(Q_b))\right)^{1/q'} \leq 1$, when $(z,\tau)\in\mathcal{N}$, we have

$$T(g(Q_b))(z,\tau) \leq C\left(\int_{\mathcal{N}}\sigma d\omega\right)^{-1/p'}.$$

Thus

$$\left(\int_{\mathcal{N}}(T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'} \leq C.$$

To control

$$\left(\int_{\partial_p\Omega_T\setminus\mathcal{N}}(T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'},$$

we apply Theorem B to the function $T(g(Q_b))(z,\tau)$; this yields

$$\left(\int_{\partial_p\Omega_T\setminus\mathcal{N}}(T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'} \leq C\left(\int_{\partial_p\Omega_T}(g^*(T(g(Q_b)))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'}.$$

Therefore, if we can find conditions on μ and ν so that

$$\left(\int_{\partial_p\Omega_T} g^*(T(g(Q_b)))^{p'}\sigma d\omega\right)^{1/p'} \leq C\left(\sum_{Q_b\in\mathcal{D}} g(Q_b)^{q'}\mu(T(Q_b))\right)^{1/q'},$$

we will be done.

There are basically two different cases that must be considered, $1 < p < 2 \leq q < \infty$ and $2 \leq p \leq q < \infty$. The second case is a little simpler than the first so we begin with it. For $2 \leq p \leq q < \infty$, we use the fact that $p'/2 \leq 1$ and $q'/p' \leq 1$. We need to show that

$$\left(\int_{\partial_p \Omega_T} g^*(T(g(Q_b)))^{p'} \sigma d\omega\right)^{q'/p'} \leq C \left(\sum_{Q_b \in \mathcal{D}} g(Q_b)^{q'} \mu(T(Q_b))\right),$$

and, taking the sequence $\{g(Q_b)\}_{Q_b \in \mathcal{D}}$ to have only finitely many non-zero terms, as we may (see [8]), gives

$$\begin{aligned}
&\left(\int_{\partial_p \Omega_T} g^*(T(g(Q_b)))^{p'} \sigma d\omega\right)^{q'/p'} \\
&= \left(\int_{\partial_p \Omega_T} \left(\sum |\lambda_{Q_b}|^2 \sum_{j=0}^{\infty} (2^{-j(2\beta-\eta)}/\omega(2^j Q_b)) \cdot \chi_{R_j(Q_b)}\right)^{p'/2} \sigma d\omega\right)^{q'/p'} \\
&\leq \left(\int_{\partial_p \Omega_T} \sum |\lambda_{Q_b}|^{p'} \left(\sum_{j=0}^{\infty} (2^{-j(2\beta-\eta)}/\omega(2^j Q_b)) \cdot \chi_{R_j(Q_b)}\right)^{p'/2} \sigma d\omega\right)^{q'/p'} \\
&\leq \left(\sum |\lambda_{Q_b}|^{p'} \int_{\partial_p \Omega_T} \left(\sum_{j=0}^{\infty} (2^{-j(2\beta-\eta)}/\omega(2^j Q_b)) \cdot \chi_{R_j(Q_b)}\right)^{p'/2} \sigma d\omega\right)^{q'/p'} \\
&\leq \sum_{Q_b \in \mathcal{D}} |\lambda_{Q_b}|^{q'} \left(\int_{\partial_p \Omega_T} \left(\sum_{j=0}^{\infty} (2^{-j(2\beta-\eta)}/\omega(2^j Q_b)) \cdot \chi_{R_j(Q_b)}\right)^{p'/2} \sigma d\omega\right)^{q'/p'}.
\end{aligned}$$

Considering each sum term-by-term, we need to have

$$|\lambda_{Q_b}|^{q'} \left(\int_{\partial_p \Omega_T} \left(\sum_{j=0}^{\infty} (2^{-j(2\beta-\eta)}/\omega(2^j Q_b)) \cdot \chi_{R_j(Q_b)}\right)^{p'/2} \sigma d\omega\right)^{q'/p'} \leq C g(Q_b)^{q'} \mu(T(Q_b)),$$

for each $Q_b \in \mathcal{D}$ and some generic constant C. Since $\lambda_{Q_b} = \frac{g(Q_b)\mu(T(Q_b))}{l(Q_b)^{2\alpha}\sqrt{\omega(Q_b)}}$, it suffices to require that

$$\frac{\mu(T(Q_b))}{l(Q_b)^{2\alpha}\sqrt{\omega(Q_b)}}\left(\int_{\partial_p\Omega_T}\left(\sum_{j=0}^{\infty}(2^{-j(2\beta-\eta)}/\omega(2^jQ_b))\cdot\chi_{R_j(Q_b)}\right)^{p'/2}\sigma d\omega\right)^{1/p'} \leq C\mu(T(Q_b))^{1/q'}.$$

But, multiplying both sides of this inequality by $\frac{l(Q_b)^{2\alpha}\sqrt{\omega(Q_b)}}{\mu(T(Q_b))^{1/q'}}$, we obtain exactly the condition given in Theorem A.

The case $1 < p < 2 \leq q < \infty$ is longer. Here $p'/2 > 1$, so we use duality and the strong maximal theorem to bound

$$\left(\int_{\partial_p\Omega_T\setminus\mathcal{N}}(T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'}$$

by $C\left\|\{g(Q_b)\}\right\|_{l^{q'}(\mu)}$. First we can find a function $h(z,\tau) \in L^s(\sigma d\omega)$ with $\frac{1}{s}+\frac{2}{p'}=1$, and $\|h\|_{L^s(\sigma d\omega)} \leq 1$, so that

$$\left(\int_{\partial_p\Omega_T\setminus\mathcal{N}}(T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)\right)^{1/p'} \leq 2\int_{\partial_p\Omega_T\setminus\mathcal{N}}\left|T(g(Q_b))(z,\tau)\right|^2 h(z,\tau)\sigma(z,\tau)d\omega(z,\tau).$$

Although $\sigma \in A^\infty(\omega)$, $h\sigma$ may not lie in $A^\infty(\omega)$. To get around this barrier we use the fact that $M_{r,\zeta}(h)d\zeta \in A^\infty(d\omega)$ whenever $d\varsigma \in A^\infty(d\omega)$. Here we must define a parabolic maximal function for any measure $d\zeta$ on $\partial_p\Omega_T$ that satisfies a center-doubling condition. Taking $1 < r < s < \infty$, we let

$$M_{r,\zeta}(h)(z,\tau) = \left(\sup_{Q,(z,\tau)\in Q}\frac{1}{\zeta(Q)}\int_Q\left|h(w,s)\right|^r d\zeta(w,s)\right)^{1/r},$$

where the supremum is taken over all parabolic boundary cubes Q that contain (z,τ). We take $\zeta(Q) = \int_Q d\zeta$. In Appendix in [7] it is shown that $M_{r,\zeta}(h)(z,\tau)d\zeta(z,\tau) \in A^\infty(d\zeta)$. In our situation we will be taking $d\zeta = \sigma d\omega$. We also have that $h(z,\tau) \leq M_{r,\zeta}(h)(z,\tau)$ ζ-a.e., and that the maximal function $M_{r,\zeta}(h)(z,\tau)$ obeys the strong L^p norm inequality if $p > r$, in other words

$\|M_{r,\zeta}(h)\|_{L^s(\sigma d\omega)} \leq C\|h\|_{L^s(\sigma d\omega)}$ whenever $s > r > 1$. Putting all this together and applying Theorem B to $\left(\int_{\partial_p\Omega_T\setminus\mathcal{N}} |Tg|^2 M_{r,\varsigma}(h)\sigma d\omega\right)^{p'/2}$ means that we have

$$\int_{\partial_p\Omega_T\setminus\mathcal{N}} (T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)$$

$$\leq \left(2\int_{\partial_p\Omega_T\setminus\mathcal{N}} |(T(g(Q_b))(z,\tau))|^2 h(z,\tau)\sigma(z,\tau)d\omega(z,\tau)\right)^{p'/2}$$

$$\leq \left(2\int_{\partial_p\Omega_T\setminus\mathcal{N}} |T(g(Q_b))|^2 M_{r,\varsigma}(h)\sigma d\omega\right)^{p'/2}$$

$$\leq C\left(\int_{\partial_p\Omega_T} |g^*(T(g(Q_b)))|^2 M_{r,\varsigma}(h)\sigma d\omega\right)^{p'/2}$$

$$= \left(\int_{\partial_p\Omega_T}\left(\sum_{Q\in\mathcal{G}}|\lambda_Q|^2\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right]\right) M_{\nu,r}(h)\sigma d\omega\right)^{p'/2}$$

$$\leq \left(\sum_{Q\in\mathcal{G}}|\lambda_Q|^2\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right] M_{\nu,r}(h)\sigma d\omega\right)^{p'/2}.$$

Now by Hölder's inequality and the Maximal Theorem

$$\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right] M_{\nu,r}(h)\sigma d\omega$$

$$\leq \left(\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right]^{p'/2}\sigma d\omega\right)^{2/p'}\left(\int_{\partial_p\Omega_T}(M_{\nu,r}(h))^s\sigma d\omega\right)^{1/s}$$

$$\leq C\left(\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right]^{p'/2}\sigma d\omega\right)^{2/p'}\|h\|_{L^s(\sigma d\omega)}.$$

And $\|h\|_{L^s(\sigma d\omega)} \leq 1$. We have established that

$$\int_{\partial_p\Omega_T\setminus\mathcal{N}} (T(g(Q_b))(z,\tau))^{p'}\sigma(z,\tau)d\omega(z,\tau)$$

$$\leq C\left(\sum_{Q\in\mathcal{D}}|\lambda_Q|^2\left(\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right]^{p'/2}\sigma d\omega\right)^{2/p'}\right)^{p'/2}.$$

Using the fact that $q'\leq 2$, so $q'/2\leq 1$, we have

$$\left(\int_{\partial_p\Omega\setminus\mathcal{N}}|Tg(Q_b)|^{p'}\sigma d\omega\right)^{1/p'}$$
$$\leq C\left(\sum_{Q\in\mathcal{D}}|\lambda_Q|^{q'}\left(\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right]^{p'/2}\sigma d\omega\right)^{q'/p'}\right)^{1/q'}.$$

We want the quantity on the right hand side of the inequality sign to be bounded by $\left(\sum_{Q\in\mathcal{D}}|g(Q)|^{q'}\mu(T(Q))\right)^{1/q'}$. If we substitute back for $\lambda_Q=C\frac{g(Q)\mu(T(Q))}{sll(Q)^{2\alpha}\sqrt{\omega(Q)}}$ and compare corresponding terms in both sums, we see we want to have the condition

$$\left(\frac{g(Q)\mu(T(Q))}{sll(Q)^{2\alpha}\sqrt{\omega(Q)}}\right)^{q'}\left(\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}\right]^{p'/2}\sigma d\omega\right)^{q'/p'}$$
$$\leq c\,|g(Q)|^{q'}\mu(T(Q)).$$

Simplifying this inequality, moving the $\mu(T(Q))$ to the left-hand side of the inequality and $l(Q)^{2\alpha q'}$ to the right hand side and taking q' roots gives

$$\mu(T(Q))^{1/q}\left(\int_{\partial_p\Omega_T}\left[\sum_{j=0}^{\infty}\frac{2^{-j(2\beta-\eta)}}{\omega(2^jQ)}\chi_{R_j}(x',t)\right]^{p'/2}\sigma d\omega\right)^{1/p'}$$
$$\leq cl(Q)^{2\alpha}\sqrt{\omega(Q)}.$$

Once again this is exactly the condition stated in Theorem B. ☐

4.

It is time to justify using Theorem B. To do this we need to verify that the family of functions $\varphi_{(Q_b)}(z,\tau)$ satisfy the conditions of geometric decay, smoothness and cancellation as they are given in (i), (ii), and (iii) stated above. Once one knows that these conditions hold for the $\{\varphi_{(Q_b)}\}_{Q_b\in\mathcal{D}}$, the proof of Theorem B proceeds in exactly the same way as the proof of Theorem 1 in [7]. In fact the $\varphi_{(Q_b)}(z,\tau)$ that appear in the present situation are not very different from the family of functions that are used in [7]. The factor of $l(Q_b)^\alpha$ does not cause any trouble with conditions (i) and (ii), because Ω_T being a bounded parabolic Lipschitz domain means that $l(Q_b)^\alpha \le C(M, r_0, \alpha, d)$. This factor, $l(Q_b)$, is needed to prove (iii); the proof of (iii) that will be presented in this paper after we discuss (i) and (ii), is different from the proof of almost orthogonality that appears in [7]. The other new ingredient is that conditions (i), (ii) and (iii) must hold on the entire $\partial_p\Omega_T$, i.e. on $\tau = 0$ as well as on the lateral part of the boundary, S_T. The fact that the kernel function can be described as a limit of the ratio of two Green's functions allows us to obtain the necessary estimates.

Proof. Recall that $\varphi_{(Q_b)}(z,\tau) = l(Q_b)^\alpha\sqrt{\omega(Q_b)}K(x_{T(Q_b)}, t_{T(Q_b)}; z,\tau)$ and that $K(x,t;z,\tau)$ is a non-negative solution to $(\partial/\partial t - L)\,v(x,t) = 0$ in the forward variables (x,t) that vanishes on $\partial_p\Omega_T\backslash(z,\tau)$. The functions $\varphi_{(Q_b)}$ satisfy (i) due to the geometric decay of the kernel function $K(x,t;z,\tau)$; this is proved in [5], Theorem 4.2, for points within a fixed distance of the projection of (x,t) (denoted by (x^*,t^*)), onto $\partial_p\Omega_T$ (i.e. the point on $\partial_p\Omega_T$ closest to (x,t)) for (z,τ) lying in the same coordinate cylinder that contains (x,t). If $\tau > t$, $K(x,t;z,\tau) = 0$; this case is discussed in [7]. For other points (z,τ) far away from (x^*,t^*), one can apply Hölder continuity, Harnack's inequality, and the doubling property of parabolic measure to obtain (**i**). Take r_0 to be the dimension of a single coordinate cylinder. Hölder continuity for non-negative solutions vanishing continuously on a boundary disk $\Delta_{2r_0}(Q_0,s_0)$ implies that, for $\delta(x,t) = r$ and $(x^*,t^*) \in \Delta_{r_0}(Q_0,s_0)$

$$K(x,t;z,\tau) \le C(\frac{r}{r_0})^\alpha K(\overline{A}_{r_0}(Q_0,s_0);z,\tau).$$

If $r_0 = 2^j r$, Harnack's inequality implies that

$$(\frac{r}{r_0})^\alpha K(\overline{A}_{r_0}(Q_0,s_0);z,\tau) \le C2^{-j\alpha}K(\overline{A}_{r_0}(z,\tau);z,\tau).$$

Now $\overline{A}_{r_0}(z,\tau)$ lies in the same coordinate cylinder that (z,τ) does so geometric decay for K gives

$$K(\overline{A}_{r_0}(z,\tau);z,\tau) \le \frac{C}{\omega(\Delta_{r_0}(z,\tau))}.$$

Finally, the fact that ω satisfies a doubling condition and that, for j_0 a fixed constant, $d_p((x^*,t^*),(z,\tau)) = 2^{j_0} r_0 = 2^{(j_0+j)} r$, means that

$$\frac{C}{\omega(\Delta_{r_0}(z,\tau))} \le \frac{C'}{\omega(\Delta_{2^{(j_0+j)}r}(z,\tau))}.$$

Altogether,

$$K(x,t;z,\tau) \le \frac{C 2^{-(j_0+j)\alpha}}{\omega(\Delta_{2^{(j_0+j)}r}(z,\tau))}.$$

A version of Hölder continuity for ratios of positive solutions along with geometric decay allow one to prove the estimate in (ii) for the $\varphi_{(Q_b)}$'s. Hölder continuity for ratios of solutions that vanish on the lateral boundary was proved in [7] in Theorem 3; and then applied to

$$K(x,t;z,\tau) = \lim_{(y,s)\to(z.\tau)} \frac{G(x,t;y,s)}{G(X_0,T_0;y,s)}$$

to prove the form of (ii) that appears in [7]. The only difference here is the presence of the factor $(l(Q_b))^\alpha$. For points (z,τ) with $\tau = 0$, we can also prove Hölder continuity for a ratio of Green's functions. Using the maximum principle in the adjoint variable shows that for G_E being the Green's function of the larger domain, $\Omega_{(-1,T]}$, $G_E(x,t;y,s) = G(x,t;y,s)$ if (x,t) and (y,s) both lie in Ω_T, and if $0 < s < t < T$ (see the proof of Theorem 3.2 in [5]). Taking $d_p(x,t;x,0) = 2r$, a standard argument, based on Moser's proof of Hölder continuity for parabolic functions on the interior of a domain (see [4] and [3]), shows that,

$$\left|\frac{G_E(x,t;y,s)}{G_E(X_0,T_0;y,s)} - \frac{G_E(x,t;w,\sigma)}{G_E(X_0,T_0;w,\sigma)}\right| \le C\left(\frac{d_p(y,s;w,\sigma)}{r}\right)^\alpha \frac{G_E(x,t;\underline{A_r}(z,0))}{G_E(X_0,T_0;\overline{A_r}(z,0))}$$

uniformly for (y,s) and (w,σ) in $\Psi_r(z,0)$. So taking s and $\sigma > 0$, and letting them both $\to 0$, gives

$$|K(x,t;y,0) - K(x,t;w,0)| \le C\left(\frac{|y-w|}{r}\right)^\alpha \frac{G_E(x,t;\underline{A_r}(z,0))}{G_E(X_0,T_0;\overline{A_r}(z,0))}.$$

Using Harnack's inequality and reverse Harnack on the adjoint variable

$$\frac{G_E(x,t;\underline{A_r}(z,0))}{G_E(X_0,T_0;\overline{A_r}(z,0))} \le C\frac{G_E(x,t;\overline{A_{r/2}}(z,0))}{G_E(X_0,T_0;\overline{A_{r/2}}(z,0))}$$
$$= C\frac{G(x,t;\overline{A_{r/2}}(z,0))}{G(X_0,T_0;\overline{A_{r/2}}(z,0))} \le C\frac{\omega^{(x,t)}(\Delta_{r/2}(z,0))}{\omega^{(X_0,T_0)}(\Delta_{r/2}(z,0))}$$
$$\le C\frac{\omega^{(x,t)}(\Delta_\epsilon(z,0))}{\omega^{(X_0,T_0)}(\Delta_\epsilon(z,0))} \to CK(x,t;z,0) \text{ as } \epsilon \to 0.$$

The second line above is derived from using a well-known comparison of the Green's function with parabolic measure, see [5] Lemmas 2.8 and 2.9. The third follows from the lemma in Section 4 of [7] and the definition of the kernel function. This lemma is valid for boundary disks (or cubes) that lie on the bottom of Ω_T, as well as for cubes on the lateral boundary. A "corner" point can be treated as being in a lateral boundary cube of $\Omega_{(-1,T)}$.

To prove (iii) let $g(z,\tau) = \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}\varphi_{(Q_b)}(z,\tau)$, where $\mathcal{F}$ is a finite family of dyadic boundary cubes in $\partial_p\Omega_T$; the $\varphi_{(Q_b)}$ are as defined above. We can assume that the coefficients are non-negative, i.e. $\lambda_{Q_b} \ge 0$. The function $v(x,t) = \int_{\partial_p\Omega_T} g(z,\tau)K(x,t;z,\tau)d\omega(z,\tau)$ denotes a solution to $\left(\frac{\partial}{\partial t} - L\right)v = 0$ in Ω_T, $v(z,\tau) = g(z,\tau)$ on $\partial_p\Omega_T$. Then

$$\int_{\partial_p\Omega_T} (g(z,\tau))^2\, d\omega(z,\tau) = \int_{\partial_p\Omega_T} g(z,\tau) \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}\varphi_{(Q_b)}(z,\tau)d\omega(z,\tau)$$
$$= \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b} l(Q_b)^\alpha \sqrt{\omega(Q_b)} \int_{\partial_p\Omega_T} g(z,\tau)K(x_{T(Q_b)},t_{T(Q_b)};z,\tau)d\omega(z,\tau)$$
$$= \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b} l(Q_b)^\alpha \sqrt{\omega(Q_b)}\, v(x_{T(Q_b)},t_{T(Q_b)})$$
$$\le C\sum_{Q_b\in\mathcal{F}} \lambda_{Q_b} l(Q_b)^\alpha \sqrt{\omega(Q_b)} \left(\frac{1}{|T(Q_b)|}\int_{T(Q_b)} |v(x,t)|^2\, dxdt\right)^{1/2}$$
$$\le C\sum_{k=-N_0}^{\infty} \sum_{l(Q_b)=2^{-k}} \lambda_{Q_b} l(Q_b)^\alpha \sqrt{\omega(Q_b)} \left(\frac{1}{|T(Q_b)|}\int_{T(Q_b)} |v(x,t)|^2\, dxdt\right)^{1/2}$$
$$\le C\sum_{k=-N_0}^{\infty} 2^{-\alpha k} \sum_{l(Q_b)=2^{-k}} \lambda_{Q_b}\sqrt{\omega(Q_b)}\widetilde{N}(v)(z,\tau)\,,$$

for any $(z,\tau)\in 2Q_b$. We have used the averaged non-tangential maximal function $\widetilde{N}(v)(z,\tau) \equiv \sup_{T(Q_b)\cap\Gamma(z,\tau)\neq\emptyset} \left(\frac{1}{|T(Q_b)|}\int_{T(Q_b)} |v(x,t)|^2\, dxdt\right)^{1/2}$, with $\Gamma(z,\tau)$

being the non-tangential approach region to a boundary point (z,τ) that is the analogue of a cone for this kind of domain. $\Gamma(z,\tau) \equiv \{(y,s) : d_p(y,s : z,\tau) < C'\delta(y,s)\}$. Since $\widetilde{N}(v)(z,\tau) \leq C_0 M_\omega(g)(z,\tau)$ we also have $\left\|\widetilde{N}(v)\right\|_{L^2(\partial_p\Omega_T,d\omega)} \leq C_1 \|g\|_{L^2(\partial_p\Omega_T,d\omega)}$ by the Maximal Theorem, see [6] and [5]. Now by Cauchy-Schwarz, choosing (z,τ) strategically, and the fact that the boundary cubes Q_b are disjoint at any given size level, we have

$$\begin{aligned}
&C\sum_{k=-N_0}^{\infty} 2^{-\alpha k} \sum_{l(Q_b)=2^{-k}} \lambda_{Q_b}\sqrt{\omega(Q_b)}\widetilde{N}(v)(z,\tau)\\
&\leq C\sum_{k=-N_0}^{\infty} 2^{-\alpha k}\left(\sum_{l(Q_b)=2^{-k}} (\lambda_{Q_b})^2\right)^{1/2}\left(\sum_{l(Q_b)=2^{-k}} \omega(Q_b)\left(\widetilde{N}(v)(z,\tau)\right)^2\right)^{1/2}\\
&\leq C\sum_{k=-N_0}^{\infty} 2^{-\alpha k}\left(\sum_{l(Q_b)=2^{-k}} (\lambda_{Q_b})^2\right)^{1/2}\left\|\widetilde{N}(v)\right\|_{L^2(\partial_p\Omega_T,d\omega)}\\
&\leq C\left\|\widetilde{N}(v)\right\|_{L^2(\partial_p\Omega_T,d\omega)}\left(\sum_{Q_b\in\mathcal{F}} (\lambda_{Q_b})^2\right)^{1/2}\left(\sum_{k=-N_0}^{\infty} 2^{-2\alpha k}\right)^{1/2}\\
&\leq C\|g\|_{L^2(\partial_p\Omega_T,d\omega)}\left(\sum_{Q_b\in\mathcal{F}} (\lambda_{Q_b})^2\right)^{1/2}.
\end{aligned}$$

The constant C has changed several times, but remains independent from g. Dividing by $\|g\|_{L^2(\partial_p\Omega_T,d\omega)}$ gives the result,

$$\|g\|_{L^2(\partial_p\Omega_T,d\omega)} \leq C\left(\sum_{Q_b\in\mathcal{F}} (\lambda_{Q_b})^2\right)^{1/2}. \qquad \square$$

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