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ON DYNAMICAL SYSTEMS METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS

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Abstract: A new modification of Continuous Newton's method for solving nonlinear operator equations in a Hilbert space is proposed. In the framework of this approach we consider a system of differential equations, the space component of which converges to a solution of the nonlinear equation. That allows us to avoid the inversion of a derivative operator. Under the assumption that the original problem is well-posed, i.e., the Fréchet derivative operator is continuously invertible, exponential convergence of the method is proven.

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1. Introduction

Consider a nonlinear operator equation on a pair of real Hilbert spaces H_1 and H_2:

$$F(x) = 0, \quad F : H_1 \to H_2. \tag{1.1}$$

We assume that equation (1.1) is solvable, not necessarily uniquely, and $\hat{x}$ is a

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solution. In this paper only well-posed equations are considered that is $F'(x)$ is continuously invertible in a neighborhood of $\hat{x}$. In order to calculate $\hat{x}$ approximately various quasi-Newton procedures have been suggested. A principal point in the numerical implementation for the most of them is the inversion of the Fréchet derivative operator $F'(x)$ or computation of the inverse by iterations with a high level of accuracy. Since the inversion of $F'(x)$ can be rather difficult for certain applied problems, several approaches were taken in order to reduce the cost associated with this inversion. Probably the best known among them is the algorithm introduced by C. Broyden [6] for finite dimensional Hilbert spaces:

$$x_{n+1} = x_n - J_n^{-1}F(x_n), \tag{1.2}$$

$$J_{n+1}^{-1} = J_n^{-1} - \frac{(s_n, J_n^{-1}\cdot\ )}{(s_n, J_n^{-1}(F(x_{n+1}) - F(x_n)))} J_n^{-1}F(x_{n+1}). \tag{1.3}$$

The iterations run as long as $F(x_n)$ and therefore $s_n := x_{n+1} - x_n$ do not vanish. Here x_0 is an initial guess, which is assumed to be sufficiently close to $\hat{x}$, and J_0^{-1} is some regular and sufficiently close initial approximation to $[F'(\hat{x})]^{-1}$. As it follows from the Sherman-Morrison formula the matrix J_n itself satisfies the recursion

$$J_{n+1} = J_n + \frac{(s_n, \cdot\ )}{||s_n||^2}F(x_{n+1}). \tag{1.4}$$

Hence for J_{n+1} the secant condition holds:

$$J_{n+1}s_n = F(x_{n+1}) - F(x_n) = \int_0^1 F'(x_n + ts_n)\, dt\, s_n \approx F'(x_n)s_n,$$

i.e. it approximates the Jacobian $F'(x_n)$ in the direction s_n. In form (1.2)-(1.3) Broyden's method can also be defined in infinite dimensional Hilbert spaces (see [11], [14]) for continuously invertible $F'(\hat{x})$. In finite dimensional case, under standard assumptions, Broyden's method is superlinearly convergent. In infinite dimensional Hilbert space compactness of the operator $J_0 - F'(\hat{x})$ is additionally required for superlinear convergence.

In [11] the regularized version of Broyden's method for nonlinear unstable problems ($F'(\hat{x})$ is not boundedly invertible) is suggested with the regularization being done by mollifying the data and by stopping the iterative process at an appropriate index $n = N$. The convergence is analyzed under the following basic assumption:

$$F'(\hat{x}) = F'(x)R_x^{\hat{x}}, \quad \left\|R_x^{\hat{x}} - I\right\| \le C_R||\hat{x} - x||, \quad \hat{x}, x \in U(x_0). \tag{1.5}$$

A popular secant method (BFGS method) was found independently by Broyden, Fletcher, Goldfarb, and Shanno in 1970 (see [8]). The BFGS method gives the most successful secant-type update for J_n^{-1} in case when the Jacobian $F'(x)$ is symmetric positive definite. There are many other iterative and regularized iterative methods which can be found in the papers [3, 4, 5, 7, 10, 15] and in the books [9, 12, 13] where also bibliographies can be found.

A method based on the inversion of the derivative operator only at the initial approximation point was proposed in [1, 2]. In the framework of this method one considers a system of differential equations for an unknown vector-function $x(t) \in C^1(R_+, H)$ (H is a Hilbert space) and a linear operator-function $J(t) \in C^1(R_+, L(H))$ approximating $F'(x(t))^{-1}$.

One of the open questions related to Newton's method and all its modifications is this: why does one have to find the inverse operators $F'(x_n)^{-1}$ if only the values of these operators on the vectors $F(x_n)$ are needed.

The goal of this paper is to develop a continuous method, the discretization of which deals with the iterations $\xi_n \approx F'(x_n)^{-1}F(x_n)$ instead of $J_n \approx F'(x_n)^{-1}$. Thus instead of iterations in $L(H,H)$ only iterations in H are required, and that is much more efficient from the computational point of view.

The following dynamical system in $H \times H$ is considered:

$$\dot{x}(t) = -\xi(t), \tag{1.6}$$

$$\dot{\xi}(t) = -\xi(t) - \rho F'^{*}(x(t))\big\{F'(x(t))\xi(t) - F(x(t))\big\}, \tag{1.7}$$

$$x(0) = x_0 \in H_1, \quad \xi(0) = \xi_0 \in H_1, \quad \rho > 0.$$

Our approach generates an algorithm which requires iterations of two vectors (x, ξ) instead of iterations of a vector and a matrix. This method avoids the inversion of $F'(x)$ but, unlike (1.2)-(1.3), it does require its computation. Since only well-posed problems are considered we are able to prove an exponential convergence of the method ($||x(t) - \hat{x}|| \leq Ce^{-ct}$). However since usually $c \ll 1$ the proposed method converges much slower than Newton's method for which $c = 1$.

2. Nonlinear Problems with Non-Monotone Operators

To study the convergence of (1.6)-(1.7) we need the following lemma.

Lemma 2.1. *Let a, b, and c be positive constants such that $a < 1$ and*

$$v_0 + \frac{cu_0}{1-a} < \frac{a(1-a)}{cb}. \tag{2.1}$$

Let $u(t)$ and $v(t)$ be continuous in $[0,+\infty)$ and differentiable in $(0,+\infty))$ functions satisfying the following system of differential inequalities:

$$\begin{aligned} \dot{u}(t) &\le -au + bv^2(t), \quad u(0) = u_0\,, \\ \dot{v}(t) &\le cu(t) - v(t), \quad v(0) = v_0. \end{aligned} \tag{2.2}$$

Then

$$u(t) < \left(u_0 + \frac{b\gamma^2}{a}\right) e^{-at}, \quad v(t) < \gamma e^{-at} \quad \textit{for all} \quad t > 0, \tag{2.3}$$

where

$$\gamma = \left(\frac{1-a}{cu_0 + (1-a)v_0} - \frac{cb}{a(1-a)}\right)^{-1}. \tag{2.4}$$

Proof. From (2.2) one has

$$\frac{d}{dt}(e^{at}u) \le be^{at}v^2, \quad \frac{d}{dt}(e^t v) \le ce^t u.$$

Therefore

$$e^t v(t) \le v_0 + c\int_0^t e^{(1-a)s}\,(e^{as}u(s))\,ds\,,$$

$$e^{at}u(t) \le u_0 + b\int_0^t e^{a\tau}v^2(\tau)d\tau\,. \tag{2.5}$$

Combining this two inequalities one gets

$$e^t v(t) \le v_0 + cu_0\int_0^t e^{(1-a)s}ds + cb\int_0^t e^{(1-a)s}\int_0^s e^{a\tau}v^2(\tau)d\tau\,ds$$

$$\le v_0 + \frac{cu_0}{1-a}\left(e^{(1-a)t} - 1\right) + cb\int_0^t e^{a\tau}v^2(\tau)\int_\tau^t e^{(1-a)s}ds\,d\tau\,.$$

Thus,

$$v(t) \le v_0e^{-t} + \frac{cu_0}{1-a}\left(e^{-at} - e^{-t}\right) + \frac{cb}{1-a}\int_0^t \left(e^{-a(t-s)} - e^{-(t-s)}\right)v^2(s)ds. \tag{2.6}$$

From this inequality one has

$$v(t) < v_0 e^{-t} + \frac{cu_0}{1-a} e^{-at} + \frac{cb}{1-a} \int_0^t e^{-a(t-s)} v^2(s) ds. \tag{2.7}$$

Since $0 < a < 1$ one gets

$$e^{at} v(t) < v_0 + \frac{cu_0}{1-a} + \frac{cb}{1-a} \int_0^t e^{-as} \left(e^{as} v(s)\right)^2 ds. \tag{2.8}$$

Then it follows from the Bihari inequality that

$$e^{at} v(t) < \left[\left(v_0 + \frac{cu_0}{1-a} \right)^{-1} - \frac{cb}{a(1-a)} + \frac{cb}{a(1-a)} e^{-at} \right]^{-1} = \gamma^{-1} + \frac{cb}{a(1-a)} e^{-at}, \tag{2.9}$$

where γ is defined in (2.4). Therefore

$$e^{at} v(t) < \gamma. \tag{2.10}$$

Thus the second estimate in (2.3) is proved. Then, from (2.5) and the second inequality in (2.3) one gets

$$e^{at} u(t) \leq u_0 + \frac{b\gamma^2}{a} \left(1 - e^{-at}\right) < u_0 + \frac{b\gamma^2}{a}. \qquad \square$$

Theorem 2.2. *Let H_1 and H_2 be real Hilbert spaces, $F : H_1 \to H_2$.*

1. Assume that there exists the Fréchet derivative $F'(x) \in L(H_1)$ of the operator F in H_1, for some $r > 0$ $F'(x)$ is boundedly invertible in $U(r, x_0) := \{x \in H_1,\ ||x - x_0|| \leq r\}$, and

$$||[F'(x)]^{-1}|| \leq M_1, \quad ||F'(x)|| \leq M_2, \quad \forall x \in U(r, x_0). \tag{2.11}$$

2. F is twice Fréchet differentiable in $U(r, x_0)$, and

$$||F''(x)|| \leq M_3 \quad \forall x \in U(r, x_0). \tag{2.12}$$

3. ξ_0 satisfies the following condition

$$||\xi(0)|| < (M_1^2 M_2 M_3)^{-1}. \tag{2.13}$$

4.

$$r \geq M_1^2 \frac{\gamma(\rho)}{\rho}, \tag{2.14}$$

where $\rho > 0$ and $\gamma(\rho) > 0$ are defined by (2.28)-(2.29) below. Then:

1. There exists a unique solution $(x(t), \xi(t))$, $t \in [0, +\infty)$ to problem (1.6)-(1.7).

2. There exists

$$\lim_{t \to +\infty} x(t) = \hat{x}, \tag{2.15}$$

and $\hat{x}$ is a solution to (3.1).

3. The following estimates hold

$$||x(t) - \hat{x}|| \leq M_1^2 \frac{\gamma(\rho)}{\rho} e^{-\rho M_1^{-2} t}, \tag{2.16}$$

$$||F(x(t))|| \leq e^{-t} ||F(x_0)|| + \left(||F'(x_0)\xi(0) - F(x_0)|| + M_1^2 M_3 \frac{\gamma^2(\rho)}{\rho} \right) \times (1 - \rho M_1^{-2})^{-1} e^{-\rho M_1^{-2} t}, \tag{2.17}$$

$$||F'(x(t))\xi(t) - F(x(t))|| \leq \left(||F'(x_0)\xi(0) - F(x_0)|| + M_1^2 M_3 \frac{\gamma^2(\rho)}{\rho} \right) e^{-\rho M_1^{-2} t}. \tag{2.18}$$

Proof. Under the assumptions of Theorem 2.2 there exists a unique solution $(x(t), \xi(t))$ of (1.6)-(1.7) on some interval $[0, \tau]$, and, at least for sufficiently small $t > 0$, $x(t) \in U(r, x_0)$.

Denote $w(t) := F(x(t))$, $W(t) := F'(x(t))\xi(t) - F(x(t))$. One has

$$F'(x(t))\xi(t) = F(x(t)) + W(t) = w(t) + W(t). \tag{2.19}$$

Therefore

$$\dot{w}(t) = F'(x(t))\dot{x}(t) = -F'(x(t))\xi(t) = -w(t) - W(t). \tag{2.20}$$

Then

$$\begin{aligned} \dot{W}(t) &= F''(x(t))\dot{x}(t)\xi(t) + F'(x(t))\dot{\xi}(t) - F'(x(t))\dot{x}(t) \\ &= F'(x(t))(\dot{\xi}(t) + \xi) - F''(x(t))\xi(t)\xi(t). \end{aligned}$$

Thus,

$$\dot{W}(t) = -\rho F'(x(t)) F'^*(x(t)) W(t) - F''(x(t))\xi(t)\xi(t). \tag{2.21}$$

(2.20), (2.21), and (1.7) imply the following differential inequalities:

$$\frac{1}{2}\frac{d}{dt}||w(t)||^2 = -||w(t)||^2 - (W(t), w(t)), \tag{2.22}$$

$$\frac{1}{2}\frac{d}{dt}||W(t)||^2 = -\rho||F'^*(x(t))W(t)||^2 - (F''(x(t))\xi(t)\xi(t), W(t)), \tag{2.23}$$

$$\frac{1}{2}\frac{d}{dt}||\xi(t)||^2 = -||\xi(t)||^2 - \rho(F'^*(x(t))W(t), \xi(t)). \tag{2.24}$$

For $\forall x \in U(r, x_0)$ and $\forall h \in H_1$

$$(F'(x)F'^*(x)h, h) \geq \frac{1}{||[F'(x)]^{-1}||^2}\ ||h||^2 \geq \frac{1}{M_1^2}||h||^2. \tag{2.25}$$

From (2.23), (2.24), (2.25), (2.11) and (2.12) one gets

$$\frac{d}{dt}\,||W(t)|| \leq -\frac{\rho}{M_1^2}||W(t)|| + M_3||\xi(t)||^2,$$

$$\frac{d}{dt}||\xi(t)|| \leq -||\xi(t)|| + \rho M_2||W(t)||. \tag{2.26}$$

Denote $u(t) := ||W(t)||$, $v(t) := ||\xi(t)||$,

$$a := \rho M_1^{-2}, \quad b = M_3, \quad \text{and} \quad c = \rho M_2. \tag{2.27}$$

In order to provide condition (2.1) of Lemma 2.1 one needs to choose ρ satisfying to the following inequality:

$$v_0 + \frac{\rho M_1^2 M_2}{M_1^2 - \rho}u_0 < \frac{M_1^2 - \rho}{M_1^4 M_2 M_3}. \tag{2.28}$$

For sufficiently small ρ this inequality follows from (2.13). Thus, there exist $M_1^2 > \rho^* > 0$ such that for every $\rho \in (0, \rho^*)$ (2.28) is satisfied.

Substituting (2.27) in (2.4) one represents γ as a function of ρ:

$$\gamma(\rho) = \left[\frac{1}{v_0 + \frac{\rho M_1^2 M_2 u_0}{M_1^2 - \rho}} - \frac{M_1^4 M_2 M_3}{M_1^2 - \rho}\right]^{-1}. \tag{2.29}$$

Therefore, (2.28) implies that $\gamma(\rho) > 0$ for $\rho \in (0, \rho^*)$. Then, from Lemma 2.1 one has:

$$||W(t)|| < \left(u_0 + M_1^2 M_3 \frac{\gamma^2(\rho)}{\rho}\right) e^{-\rho M_1^{-2} t}, \ ||\xi(t)|| < \gamma(\rho) e^{-\rho M_1^{-2} t}$$

$$\text{for all} \quad t > 0. \quad (2.30)$$

From (2.22) one obtains

$$\frac{d}{dt}||w(t)|| \le -||w(t)|| + ||W(t)||. \tag{2.31}$$

Hence

$$||w(t)|| \le e^{-t}\left[||w(0)|| + \int_0^t e^s ||W(s)|| ds\right]$$

$$< e^{-t}\left[||w(0)|| + \left(||W(0)|| + M_1^2 M_3 \frac{\gamma^2(\rho)}{\rho}\right) \int_0^t e^{(1-\rho M_1^{-2})s} ds\right]. \tag{2.32}$$

Since $0 < \rho M_1^{-2} < 1$, this estimate implies

$$||w(t)|| \le e^{-t}||w(0)|| + \left(||W(0)|| + M_1^2 M_3 \frac{\gamma^2(\rho)}{\rho}\right)(1 - \rho M_1^{-2})^{-1} e^{-\rho M_1^{-2} t}. \tag{2.33}$$

For all values of t_1 and t_2, such that $0 \le t_1 \le t_2$, $x(t_1), x(t_2) \in U(r, x_0)$ and $||\xi(t)|| \le R$ one obtains by (1.6)

$$||x(t_2) - x(t_1)|| \le \left\| \int_{t_1}^{t_2} \xi(s)\, ds \right\| \le \gamma(\rho) \int_{t_1}^{t_2} e^{-\rho M_1^{-2} s} ds$$

$$= M_1^2 \frac{\gamma(\rho)}{\rho} \left(e^{-\rho M_1^{-2} t_1} - e^{-\rho M_1^{-2} t_2}\right). \tag{2.34}$$

Estimate (2.34) implies that there exists a unique solution $(x(t), \xi(t))$ to (1.6)-(1.7) on $[0, +\infty)$. Setting $t_1 = t$ and $t_2 \to +\infty$ in (2.34) one gets (2.16). Inequality (2.17) now follows from (2.33). This completes the proof. □

3. Continuous Quasi-Newton's Procedure for Monotone Operator Equations

In this section we solve nonlinear operator equation

$$F(x) = 0, \quad F : H \to H, \tag{3.1}$$

on a real Hilbert space under the basic assumption that the Fréchet derivative of the operator F satisfies the following inequality for some positive constant c:

$$(F'(x)h, h) > c||h||^2 \quad \forall\, h, x \in H. \tag{3.2}$$

Let x_0 be an initial approximation for a solution to (3.1).

Consider the following problem:

$$\dot{x}(t) = -\xi(t), \tag{3.3}$$

$$\dot{\xi}(t) = -\xi(t) - F'(x(t))\xi(t) + F(x(t)), \tag{3.4}$$

$$x(0) = x_0 \in H, \quad \xi(0) = \xi_0 \in H.$$

Theorem 3.1 gives a sufficient condition for convergence of (3.3)-(3.4).

Theorem 3.1. *Let H and H be real Hilbert spaces, $F : H \to H$.*

1. Assume that there exists the Fréchet derivative $F'(x) \in L(H)$ of the operator F in H, and inequality (3.2) is fulfilled.

Then:

1. There exists a global solution $(x(t), \xi(t))$ to problem (3.3)-(3.4) and

$$\lim_{t\to+\infty} x(t) = \hat{x}, \tag{3.5}$$

where $\hat{x}$ is a solution to (3.1).

2. There exist constants c_1 and c_2 such that

$$||x(t) - \hat{x}|| \le c_1\, e^{-\min\{1,c\}\, t}, \quad ||F(x(t))|| \le c_2\, e^{-\min\{1,c\}\, t}. \tag{3.6}$$

Proof. Denote $w(t) := F(x(t))$. One has

$$\dot{w}(t) = F'(x(t))\dot{x}(t) = -F'(x(t))\xi(t).$$

Thus by (3.4) $\dot{\xi}(t) = -\xi(t) + \dot{w}(t) + w(t)$ and

$$\frac{d}{dt}(\xi(t) - w(t)) = -(\xi(t) - w(t)),$$

$$||\xi(t) - w(t)|| \le ||\xi_0 - w_0||\, e^{-t}. \tag{3.7}$$

Now since H is a real Hilbert space

$$\frac{1}{2}\,\frac{d}{dt}\,||\xi(t)||^2 = \left(\frac{d\xi}{dt}, \xi(t)\right)$$

$$= -||\xi(t)||^2 - (F'(x(t))\xi(t), \xi(t)) + (F(x(t)), \xi(t))$$

$$= -(F'(x(t))\xi(t), \xi(t)) + (F(x(t)) - \xi(t), \xi(t))$$

$$\leq -c||\xi(t)||^2 + ||w(t) - \xi(t)|| \, ||\xi(t)||. \tag{3.8}$$

Denote $v(t) := ||\xi(t)||$. Inequality (3.8) yields

$$v\dot{v} \leq -c\, v^2 + ||w(t) - \xi(t)|| \; v. \tag{3.9}$$

Divide this inequality by the nonnegative $v(t)$ and get a linear first-order differential inequality from which one gets according to (3.7)

$$||\xi(t)|| \leq e^{-ct} \left[\int_0^t ||w(s) - \xi(s)|| e^{cs} \, ds + ||\xi_0|| \right] \tag{3.10}$$

$$\leq \left[\frac{||w_0 - \xi_0||}{|1 - c|} + ||\xi_0|| \right] \; e^{-\min\{1,c\}\, t} := A\, e^{-\min\{1,c\}\, t}. \tag{3.11}$$

Without loss of generality we can assume that $c \neq 1$. Now one has

$$||x(t_2) - x(t_1)|| \leq \left\| \int_{t_1}^{t_2} \xi(s)\, ds \right\| \leq \frac{A}{\min\{1, c\}} \; \left(e^{-\min\{1,c\}\, t_1} - e^{-\min\{1,c\} t_2} \right). \tag{3.12}$$

Estimate (3.12) implies that there exists a unique solution $(x(t), \xi(t))$ to (3.3)-(3.4) on $[0, +\infty)$. Setting $t_1 = t$ and $t_2 \to +\infty$ in (3.12) one gets

$$||x(t) - \tilde{x}|| \leq c_1\, e^{-\min\{1,c\}\, t}, \quad c_1 = \frac{A}{\min\{1, c\}}, \tag{3.13}$$

where $\tilde{x} := \lim_{t \to +\infty} x(t)$. From (3.7) it follows that

$$\left| ||w(t)|| - ||\xi(t)|| \right| \leq ||\xi_0 - w_0||\, e^{-t}. \tag{3.14}$$

and by (3.11)

$$||w(t)|| \leq ||\xi(t)|| + ||\xi_0 - w_0||\, e^{-t} \leq (A + ||\xi_0 - w_0||) \; e^{-\min\{1,c\}\, t}. \tag{3.15}$$

Therefore $\hat{x} = \tilde{x}$, a solution to (3.1). The second inequality in (3.6) follows from (3.15), $c_2 = A + ||\xi_0 - w_0||$. □

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