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MIRROR SYMMETRY AND QUANTUM COHOMOLOGY FOR PROJECTIVE BUNDLES

Artur Elezi
Department of Mathematics and Statistics
American University
4400 Massachusetts Ave. NW, Washington D.C., 20016, USA
e-mail: aelezi@american.edu

Abstract: In Elezi [6] we conjectured a relation between the quantum $\mathcal{D}$-modules of a smooth variety X and the projectivisation of a direct sum of line bundles over it. In this paper we prove the conjecture when X is a semi-ample complete intersection in a toric variety. We use the conjecture to show that the relations of the small quantum cohomology ring of X that come from differential operators lift to the projective bundle. The basic cohomology relation of the projective bundle deforms to a relation in the small quantum cohomology.

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1. Introduction

Let Y be a smooth, projective variety. Denote by $Y_{k,\beta}$ the moduli stack of rational stable maps of class $\beta \in H_2(Y, \mathbb{Z})$ with k-markings Fulton et al [7] and $[Y_{k,\beta}]$ its virtual fundamental class (Behrend et al [3], Li et al [12]). Genus zero Gromov-Witten invariants are defined as appropriate integrals over $[Y_{k,\beta}]$. Throughout this paper we will be interested mainly in $k = 1$. The moduli space $Y_{1,\beta}$ is equipped with the following features

— $e : Y_{1,\beta} \to Y$ - the evaluation map.

— ψ - the first chern class of the cotangent line bundle on $Y_{1,\beta}$.

— ft : $Y_{1,\beta} \to Y_{0,\beta}$ - the forgetful morphism.

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For any ring $\mathcal{A}$, the formal completion of $\mathcal{A}$ along the semigroup MY of the rational curves of Y is defined to be

$$\mathcal{A}[[q^\beta]] := \{ \sum_{\beta \in \mathrm{MY}} a_\beta q^\beta, \quad a_\beta \in \mathcal{A}, \quad \beta - \text{effective} \}, \tag{1}$$

where $\beta \in H_2(Y, \mathbb{Z})$ is *effective* if it is a positive linear combination of rational curves. This new ring behaves like a power series since for each β, the set of α such that α and $\beta - \alpha$ are both effective is finite. Alternatively, we may define

$$q^\beta := q_1^{d_1} \cdot ... \cdot q_k^{d_k} = \exp(t_1 d_1 + ... + t_k d_k),$$

where $\{d_1, d_2, ..., d_k\}$ are the coordinates of β relative to the dual basis of $\{p_1, ..., p_k\}$.

Let $*$ denote the small quantum product of Y. The small quantum cohomology ring

$$(QH_s^* Y, *)$$

is a deformation of the cohomology ring $(H^*(Y, \mathbb{Q}[q^\beta]), \cup)$. Its structural constants are three point Gromov-Witten invariants of genus zero.

Let $\hbar$ be a formal variable and

$$J_\beta(Y) := e_* \left(\frac{[Y_{1,\beta}]}{\hbar(\hbar - \psi)} \right) = \sum_{k=0}^{\infty} \frac{1}{\hbar^{2+k}} e_*(\psi^k \cap [Y_{1,\beta}]).$$

The sum is finite for dimension reasons. Let

$$p = \{p_1, p_2, ..., p_k\}$$

be a nef basis of $H^2(Y, \mathbb{Q})$. For $t = (t_0, t_1, ..., t_k)$ let

$$tp := t_0 + \sum_{i=1}^{k} t_i p_i.$$

The $\mathcal{D}$-module for the quantum differential equation of Y

$$1 \leq i \leq k, \ \hbar \partial / \partial t_i = p_i *,$$

is generated by (Givental [9])

$$J(Y) = \exp\left(\frac{tp}{\hbar}\right) \sum_{\beta \in H_2(Y,\mathbb{Z})} q^\beta J_\beta(Y),$$

where we use the convention $J_0 = 1$. The generator $J(Y)$ encodes *all* of the genus zero, one marking Gromov-Witten invariants and gravitational descendants of Y.

We regard $J(Y)$ as an element of $H^*(Y, \mathbb{Q})[t][[q^\beta]]$. It may be used to generate relations in QH^*_sY in the following way: let

$$\mathcal{P}(\hbar, \hbar\partial/\partial t_i, q_i)$$

be a polynomial differential operator, where q_i and $\hbar$ act via multiplication and $q_i = e^{t_i}$ are on the left of derivatives. If

$$\mathcal{P}(\hbar, \hbar\partial/\partial t_i, q_i)J(Y) = 0\,,$$

then

$$\mathcal{P}(0, p_i, q_i) = 0$$

is a relation in the small quantum cohomology ring QH^*_sY.

If Y is a toric variety, $J(Y)$ is related to an explicit hypergeometric series $I(Y)$ via a change of variables (Givental [8], Lian et al [11]). Furthermore, if Y is Fano then the change of variables is trivial, i.e.

$$J(Y) = I(Y).$$

Since $I(Y)$ is known explicitly, this yields two immediate benefits. On one hand, the one point Gromov-Witten invariants and gravitational descendants of Y are determined completely. On the other hand, differential operators that annihilate $I(Y)$ are easy to find, hence producing relations in the small quantum cohomology ring of Y.

The special case

$$J(\mathbb{P}^n) = I(\mathbb{P}^n)$$

was known even earlier (Astashkevich et al [1]). In Elezi [6] we conjectured a relativized extension of this result which we recall. Let X be a projective manifold. Following Grothendieck's notation, let

$$\pi : \mathbb{P}(V) = \mathbb{P}(\oplus_{j=0}^{n} L_j) \to X$$

be the projective bundle of hyperplanes of a vector bundle V. Assume that $L_0 = \mathcal{O}_X$. The H^*X-module $H^*\mathbb{P}(V)$ is generated by $z := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ with the relation

$$\prod_{i=0}^{n}(z - c_1(L_i)) = 0.$$

Let $s_i : X \to \mathbb{P}(V)$ be the section of π determined by the i-th summand of V and $X_i := s_i(X)$. Then $\mathcal{O}_{\mathbb{P}(V)}(1)|_{X_i} \simeq L_i$. Let $\{p_1, ..., p_k\}$ be a nef basis of $H^2(X, \mathbb{Q})$. In Elezi [6] we showed that

Lemma 1. *If the line bundles $L_i, i = 1, ..., n$ are nef then:*

(a) $\{p_1, ..., p_k, z\}$ is a nef basis of $H^2(\mathbb{P}(V), \mathbb{Q})$.

(b) The Mori cones of X and $\mathbb{P}(V)$ are related via

$$M\mathbb{P}(V) = MX \oplus \mathbb{Z}_{\geq 0} \cdot [l]\,,$$

where $[l]$ is the class of a line in the fiber of π.

Here MX is embedded in $M\mathbb{P}(V)$ via the section s_0. If $C \in \mathbb{P}(V)$ is a rational curve, there exists a unique pair $(\nu \geq 0, \beta \in MX)$ such that:

$$[C] = \nu[l] + \beta.$$

We will identify the homology class $[C]$ with (ν, β). The generator $J_{\mathbb{P}(V)}$ is an element of $H^*(\mathbb{P}(V), \mathbb{Q})[t, t_{k+1}][[q_1^\nu, q_2^\beta]]$.

For a line bundle L and a curve α we denote

$$L(\alpha) := c_1(L) \cdot \alpha.$$

Define the "twisting" factor

$$\mathcal{T}_{\nu,\beta} := \prod_{i=0}^{n} \frac{\prod_{m=-\infty}^{0}(z - c_1(L_i) + m\hbar)}{\prod_{m=-\infty}^{\nu - L_i(\beta)}(z - c_1(L_i) + m\hbar)}.$$

Let $I_{\nu,\beta} := \mathcal{T}_{\nu,\beta} \cdot \pi^* J_\beta$ where π^* is the flat pull back and define a "twisted" hypergeometric series for the projective bundle $\mathbb{P}(V)$:

$$I(\mathbb{P}(V)) := \exp\left(\frac{tp + t_{k+1}z}{\hbar}\right) \cdot \sum_{\nu,\beta} {q_1}^\nu {q_2}^\beta I_{\nu,\beta}.$$

In Elezi [6] we proposed the following conjecture.

Conjecture 1. *Let $L_i,\ i = 1, ..., n$ be nef line bundles such that $-K_X - c_1(V)$ is ample. Then $J(\mathbb{P}(V)) = I(\mathbb{P}(V))$.*

In the next section we prove this conjecture when X is a semi-ample complete intersection in a toric variety.

In the last section we study the consequences of the proposed conjecture in the relation between QH_s^*X and $QH_s^*\mathbb{P}(V)$. Recall that $H^*\mathbb{P}(V)$ is an H^*X-module generated by $z := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ with the relation

$$\prod_{i=0}^{n}(z - c_1(L_i)) = 0. \tag{2}$$

We show that the relations of QH_s^*X that come from the quantum differential equations lift to relations in $QH_s^*\mathbb{P}(V)$. We also show that (2) deforms into the relation

$$z\prod_{i=1}^{n}(z-c_1(L_i))=q_1$$

in $QH_s^*\mathbb{P}(V)$. These yield a complete description of $QH_s^*\mathbb{P}(V)$ when all the relations in QH_s^*X come from quantum differential operators. This, for example, is the case when X is a product of projective spaces.

2. Toric Case Proof

2.1. Toric Varieties and Torus Actions

Assume Y is a toric variety determined by a fan $\Sigma\subset\mathbb{Z}^m$. Denote by $b_1,...,b_{r=m+k}$ its one dimensional cones. Let $Z(\Sigma)\subset\mathbb{C}^r$ be the variety whose ideal is generated by the products of those variables which do *not* generate a cone in Σ. The toric variety Y is the geometric quotient of

$$\mathbb{C}^r-Z(\Sigma)$$

by a torus of dimension k (see Audin [2], Cox [5]).

Let $\tilde{L}_i,\ i=0,1,...,n$ be toric line bundles and $\tilde{V}=\oplus_{i=0}^{n}\tilde{L}_i$. The projective bundle

$$\pi:\mathbb{P}(\tilde{V})\to Y$$

is also a toric variety and there is a canonical way to obtain its fan (see Oda [14]). Let $\mathbb{Z}^n$ be a new lattice with basis $\{f_1,...,f_n\}$. The edges $b_1,...,b_r$ of Σ are lifted to new edges $B_1,B_2,...,B_r$ in $\mathbb{Z}^m\oplus\mathbb{Z}^n$ and subsequently Σ is lifted in a new fan Σ_1 in the obvious way. Let $\Sigma_2\subset 0\oplus\mathbb{Z}^n$ be the fan of $\mathbb{P}^n$ with edges

$$F_0=-\sum_{i=1}^{n}f_i, F_1=f_1,...,F_n=f_n.$$

The canonical fan associated to $\mathbb{P}(V)$ consists of the cones

$$\sigma_1+\sigma_2\,,$$

where σ_1,σ_2 are cones in Σ_1,Σ_2.

Let $N=r+n+1$. The torus $\mathbb{T}=(\mathbb{C}^*)^N$ acts on both Y and $\mathbb{P}(V)$ by scaling of coordinates in respectively $\mathbb{C}^r$ and $\mathbb{C}^N$. The one dimensional cones correspond to $\mathbb{T}$ invariant divisors. The edges F_i in the canonical fan of the projective bundle, correspond to the divisors $z-c_1(L_i)$, where $z:=c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$

while B_i correspond to the pullback of the base divisors associated with b_i (see Mavlutov [13]).

Let $L'_a : a = 1, 2, ..., M$ be semi-ample line bundles, i.e. generated by global sections. Let X be the zero locus of a generic section s of

$$E = \oplus_{a=1}^{M} L'_a.$$

Such an X is called a semi-ample complete intersection. Let $L_i,\ i = 0, 1, ..., n$ and V be the restriction of $\tilde{L}_i$ and $\tilde{V}$ to X. The total space of $\mathbb{P}(V)$ is the zero locus of the section $\pi^*(s)$ of the pull back bundle $\pi^*(\tilde{V})$. To assure that the conditions of the conjecture are met for the bundle $\mathbb{P}(V)$ over X we assume that $\tilde{L}_i, i = 1, 1, ..., n$ are nef and $-K_Y - \sum_{a=1}^{M} c_1(L'_a) - \sum_{i=0}^{n} c_1(\tilde{L}_i)$ is ample.

Let E_d be the bundle on $Y_{1,d}$ whose fiber over the moduli point $(C, x_1, f : C \to Y)$ is $\oplus_a H^0(f^*(L'_a))$. Denote by s_E its canonical section induced by s, i.e.

$$s_E((C, x_1, f)) = f^*(s).$$

The stack theoretic zero section of s_E is the disjoint union

$$Z(s_E) = \coprod_{i_*(\beta)=d} X_{1,\beta}. \tag{3}$$

The map $i_* : H_2X \to H_2Y$ is not injective in general, hence the zero locus $Z(s_E)$ may have more then one connected component. An example is the quadric surface in $\mathbb{P}^3$. The sum of the virtual fundamental classes $[X_{1,\beta}]$ is the refined top Chern class of E_d with respect to s_E.

There is a stack morphism

$$\mathbb{P}(\tilde{V})_{1,(\nu,d)} \to Y_{1,d}.$$

Let $\tilde{E}_{\nu,d}$ and $\tilde{s}_E$ be the pull backs of E_d and s_E. The zero section of $\tilde{s}_E$ is the disjoint union

$$z(\tilde{s}_E) = \coprod_{i_*(\beta)=d} \mathbb{P}(V)_{1,(\nu,\beta)}.$$

It follows that

$$\sum_{i_*(\beta)=d} [\mathbb{P}(V)_{1,(\nu,\beta)}] = c_{\mathrm{top}}(\tilde{E}_{\nu,d}) \cap [\mathbb{P}(\tilde{V})_{1,(\nu,d)}].$$

Consider the following generating functions

$$J(\mathbb{P}(\tilde{V}), E) = \exp\left(\frac{tp + t_{k+1}z}{\hbar}\right) \sum {q_1}^{\nu} {q_2}^{d} e_* \left(\frac{c_{\mathrm{top}}(\tilde{E}_{\nu,d}) \cap [\mathbb{P}(\tilde{V})_{1,(\nu,d)}]}{\hbar(\hbar - c)}\right)$$

and

$$\tilde{I}(\mathbb{P}(\tilde{V}),E)=\exp\left(\frac{tp+t_{k+1}z}{\hbar}\right)\sum q_1^{\nu}{q_2}^d\ \mathcal{T}_{\nu,d}\ \pi^*e_*\left(\frac{c_{\mathrm{top}}(E_d)\cap[Y_{1,d}]}{\hbar(\hbar-c)}\right).$$

Proposition 1. *If* $-K_Y-\sum_{a=1}^M c_1(L'_a)-\sum_{i=0}^n c_1(\tilde{L}_i)$ *is ample then*

$$J(\mathbb{P}(\tilde{V}),E)=\tilde{I}(\mathbb{P}(\tilde{V}),E)\,.$$

Proof. Let

$$I_d(Y,E)=\prod_a\frac{\prod_{m=-\infty}^{L'_a(d)}(L'_a+m\hbar)}{\prod_{m=-\infty}^{0}(L'_a+m\hbar)}\prod_i\frac{\prod_{m=-\infty}^{0}(B_i+m\hbar)}{\prod_{m=-\infty}^{B_i(d)}(B_i+m\hbar)}.$$

From Givental [8], Lian et al [10] and [11] we know that $J(\mathbb{P}(\tilde{V}),E)$ is related via a mirror transformation to

$$I(\mathbb{P}(\tilde{V}),E)=\exp\left(\frac{tp+t_{k+1}z}{\hbar}\right)\cdot\sum {q_1}^{\nu}{q_2}^d\mathcal{T}_{\nu,d}I_d(Y,E).$$

Likewise

$$J(Y,E)=\exp\left(\frac{tp}{\hbar}\right)\sum {q_2}^d e_*\left(\frac{c_{\mathrm{top}}(E_d)\cap[Y_{1,d}]}{\hbar(\hbar-c)}\right)$$

is related to

$$I(Y,E)=\exp\left(\frac{tp}{\hbar}\right)\sum {q_2}^d I_d(Y,E).$$

Since $-K_{\mathbb{P}(\tilde{V})}-\sum_a c_1(L'_a)$ and $-K_Y-\sum_a c_1(L'_a)$ are ample, the mirror transformations are particularly simple. Indeed, both series can be written as power series of $\hbar^{-1}$ as follows:

$$I(\mathbb{P}(\tilde{V}),E)=1+\frac{P_1(q_1,q_2)}{\hbar}+o(\hbar^{-1}),\ \ I(Y,E)=1+\frac{P_2(q_2)}{\hbar}+o(\hbar^{-1}),$$

where $P_1(q_1,q_2),P_2(q_2)$ are both polynomials supported respectively in

$$\Lambda_1:=\{(\nu,d)\mid(-K_{\mathbb{P}(\tilde{V})}-\sum c_1(L'_a))=1;$$

$$z-c_1(\tilde{L}_j)\geq 0\,,\ \forall j=1,2,...,n;\ \ B_i\geq 0\,,\forall i=1,2,...,r\}$$

and

$$\Lambda_2:=\{d\mid(-K_Y-\sum c_1(L'_a))=1;B_i\geq 0\,,\ \forall i=1,2,...,r\}.$$

Then

$$J(\mathbb{P}(\tilde{V}),E)=\exp\left(\frac{-P_1(q_1,q_2)}{\hbar}\right)I(\mathbb{P}(\tilde{V}),E)$$

and

$$J(Y,E) = \exp\left(\frac{-P_2(q_2)}{\hbar}\right) I(Y,E).$$

Let us examine the relation between Λ_1 and Λ_2. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \pi^* V^*(1) \to T_{\mathbb{P}(V)} \to \pi^* T_X \to 0$$

we find that

$$-K_{\mathbb{P}(\tilde{V})} - \sum c_1(L'_a) = -K_Y - \sum_i (\tilde{L}_i) - \sum c_1(L'_a) + (n+1)z.$$

Assume $(\nu, d) \in \Lambda_1$. Since $-K_Y - \sum_{a=1}^M c_1(L'_a) - \sum_{i=0}^n c_1(\tilde{L}_i)$ is ample then $\nu = 0$. Now

$$(z - c_1(\tilde{L}_j)) \cdot (0,d) \geq 0 \ \forall j = 1, 2, ..., n$$

and $\tilde{L}_j$ are semipositive $\forall j = 1, 2, ..., n$. So

$$c_1(\tilde{L}_j) \cdot d = 0 \ \forall j = 1, 2, ..., n.$$

It follows that

$$(-K_Y - \sum c_1(L'_a)) \cdot d = (-K_{\mathbb{P}(\tilde{V})} - \sum c_1(L'_a)) \cdot d = 1,$$

so $d \in \Lambda_2$.

Conversely, let $d \in \Lambda_2$. Then

$$c_1(\tilde{L}_j) \cdot d = 0\,, \ \forall j = 1, 2, ..., n\,,$$

since $-K_Y - \sum_{a=1}^M c_1(L'_a) - \sum_{i=0}^n c_1(\tilde{L}_i)$ is ample. It follows that

$$(-K_{\mathbb{P}(\tilde{V})} - \sum c_1(L'_a)) \cdot d = (-K_Y - \sum c_1(L'_a)) \cdot d = 1\,,$$

and

$$(z - c_1(\tilde{L}_j)) \cdot d = 0, \ \forall j = 1, 2, ..., n\,,$$

so $(0, d) \in \Lambda_1$.

We have thus shown that

$$c_1(\tilde{L}_j) \cdot d = 0, \forall d \in \Lambda_2, \forall j = 1, 2, ..., n$$

and

$$\Lambda_1 = \{(0,d) \mid d \in \Lambda_2\}.$$

It follows that $\mathcal{T}_{0,d} = 1, \forall d \in \Lambda_2$ hence $P_1(q_1, q_2) = P_2(q_2)$.

Notice also that if we expand

$$\exp\left(\frac{-P_2(q_2)}{\hbar}\right) = \sum_{\alpha} c_\alpha q_2^\alpha\,,$$

then

$$c_1(\tilde{L}_j)\cdot\alpha = 0,\ \forall j = 1, 2, ..., n.$$

Hence for each $(\nu, d) \in M\mathbb{P}(\tilde{V})$ we have

$$\mathcal{T}_{\nu,d} = \mathcal{T}_{\nu,d+\alpha}.$$

Now the proposition follows easily. □

Recall from the discussion after equation (3) that the map

$$i_* : H_2(X) \to H_2(Y) \tag{4}$$

is not necessarily injective in general. If it is, then

$$[X_{1,\beta}] = c_{\mathrm{top}}(E_{i_*(\beta)}) \cap [Y_{1,i_*(\beta)}]$$

and

$$[\mathbb{P}(V)_{1,(\nu,\beta)}] = c_{\mathrm{top}}(\tilde{E}_{\nu,i_*(\beta)}) \cap [\mathbb{P}(\tilde{V})_{1,(\nu,i_*(\beta))}].$$

In that case one can easily show that

$$i_*(J_{\nu,\beta}(\mathbb{P}(V))) = J_{\nu,i_*(\beta)}(\mathbb{P}(\tilde{V}), E)$$

and

$$i_*(I_{\nu,\beta}(\mathbb{P}(V))) = \tilde{I}_{\nu,i_*(\beta)}(\mathbb{P}(\tilde{V}), E).$$

The proposition shows that the conjecture holds for complete intersection in toric varieties for which the map (4) is injective.

3. Relations in the Small Quantum Cohomology Ring

In this section we use the proposed conjecture to study small quantum deformations of the cohomological relation

$$H^*(\mathbb{P}(V) = H^*X/\langle\prod_{i=0}^{n}(z - c_1(L_i))\rangle = 0.$$

As explained in the introduction, some of the relations in the small quantum cohomology ring come from differential operators. Let

$$c_1(L_i) = \sum_{j=1}^{k} a_{ij} p_j, \ \ i = 0, 1, ..., n.$$

If the conjecture is true, we obtain two kinds of relations in $QH_s^*\mathbb{P}(V)$. First, the relations in QH_s^*X that come from differential operators may be lifted to relations in $QH_s^*\mathbb{P}(V)$. Indeed, consider a polynomial differential operator

$$\mathcal{P}(\hbar, \hbar\partial/\partial t_1, ..., \hbar\partial/\partial t_k, q_2) = \sum_{\alpha\in\Lambda} q_2^\alpha \mathcal{P}_\alpha\,,$$

where $\Lambda \subset MX$ is a finite set. Suppose that

$$0 = \mathcal{P}J(X) = \sum_{\alpha\in\Lambda} q_2^\alpha \sum_\beta \mathcal{P}_\alpha \left(\exp(\frac{pt}{\hbar}) q_2^\beta\right) J_\beta(X)$$

$$= \sum_{\alpha\in\Lambda} q_2^\alpha \sum_\beta c_{\alpha,\beta} \exp(\frac{pt}{\hbar}) q_2^\beta J_\beta(X) = \exp(\frac{pt}{\hbar}) \sum_{\alpha\in\Lambda,\beta} q_2^{\alpha+\beta} c_{\alpha,\beta} J_\beta(X).$$

Let

$$\delta_\alpha = \prod_{i=1}^{n} \prod_{0}^{L_i\cdot\alpha-1} (\hbar\frac{\partial}{\partial t_{k+1}} - \sum_{j=1}^{k} a_{ij}\hbar\frac{\partial}{\partial t_j} - r_i\hbar), \ \tilde{\mathcal{P}} = \sum_{\alpha\in\Lambda} q_2^\alpha \delta_\alpha \mathcal{P}_\alpha,$$

with the convention that if

$$L_i(\alpha) = 0,$$

the factors of δ_α corresponding to L_i are missing. We compute

$$\tilde{\mathcal{P}}J(\mathbb{P}(V)) = \sum_{\alpha\in\Lambda} q_2^\alpha \delta_\alpha \sum_{\nu,\beta} \mathcal{P}_\alpha \left(q_2^\beta \exp(\frac{pt + zt_{k+1}}{\hbar})\right) q_1^\nu \mathcal{T}_{\nu,\beta} J_\beta$$

$$= \sum_{\alpha\in\Lambda} q_2^\alpha \delta_\alpha \sum_{\nu,\beta} c_{\alpha,\beta} \exp(\frac{pt + zt_{k+1}}{\hbar}) q_1^\nu q_2^\beta \mathcal{T}_{\nu,\beta} J_\beta.$$

A simple calculation shows that

$$\delta_\alpha \left(\exp(\frac{pt + zt_{k+1}}{\hbar}) q_1^\nu q_2^\beta \mathcal{T}_{\nu,\beta}\right) = \exp(\frac{pt + zt_{k+1}}{\hbar}) q_1^\nu q_2^\beta \mathcal{T}_{\nu,\alpha+\beta}.$$

It follows that

$$\tilde{\mathcal{P}}J(\mathbb{P}(V)) = \exp(\frac{pt + zt_{k+1}}{\hbar}) \sum_\nu q_1^\nu \sum_{\alpha\in\Lambda,\beta} c_{\alpha,\beta} q_2^{\alpha+\beta} \mathcal{T}_{\nu,\alpha+\beta} J_\beta(X) = 0.$$

Hence the relation $\mathcal{P}(0, p_1, ..., p_k, q_2) = 0$ in $QH^*_s X$ lifts into the relation

$$\mathcal{P}(0, p_1, ..., p_k, q_2 \prod_{i=1}^{n}(z - c_1(L_i)) = 0$$

in $QH^*_s\mathbb{P}(V)$, where

$$\left(\prod_{i=1}^{n}(z - c_1(L_i)\right)^{\alpha} := \prod_{i=1}^{n}(z - c_1(L_i))^{L_i(\alpha)}, \ \forall \alpha \in MX.$$

Second, we derive a q_1-deformation of the relation

$$\prod_{i}(z - c_1(L_i)) = 0.$$

Consider the following differential operator

$$\Delta(\hbar\frac{\partial}{\partial t_1}, ..., \hbar\frac{\partial}{\partial t_k}, \hbar\frac{\partial}{\partial t_{k+1}}, q_1) := \prod_{i=0}^{n}(\hbar\frac{\partial}{\partial t_{k+1}} - \sum_{j=1}^{k} a_{ij}\hbar\frac{\partial}{\partial t_j}) - q_1.$$

It is easy to show that it satisfies

$$\Delta J(\mathbb{P}(V)) = 0.$$

It follows that

$$\Delta(p_1, ..., p_k, z, q_1) = 0$$

in $QH^*_s\mathbb{P}(V)$. This produces the relation

$$\prod_{i=0}^{n}(z - c_1(L_i)) = q_1.$$

Much like $z^{n+1} = q$ is the deformation in $QH^*_s\mathbb{P}^n$ of $z^{n+1} = 0$, the above relation is the deformation of $\prod_{i=0}^{n}(z - c_1(L_i)) = 0$.

One can easily see that second order relations in $QH^*_s X$ come from quantum differential operators. In some cases, *all* the relations in $QH^*_s X$ come from quantum differential operators, hence may be lifted in $QH^*_s\mathbb{P}(V)$. This is the case when X is a product of projective spaces. The results of this section yield a complete description of $QH^*\mathbb{P}(V)$ which agrees with Costa et al [4] and Qin et al [15].

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