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ON EDGE-ROBUST $(1, \leq l)$-IDENTIFYING CODES IN BINARY HAMMING SPACES

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Abstract: The motivation for identifying codes comes, for instance, from fault diagnosis in multiprocessor systems. Another application is emergency sensor networks. In this paper we consider t-edge-robust $(1, \leq l)$-identifying codes, that is, identifying codes that remain identifying even if the underlying structure changes due to some failures. In particular, we give a classification of 1-edge-robust $(1, \leq 2)$-identifying codes in binary Hamming spaces. Using this we obtain an infinite family of optimal 1-edge-robust $(1, \leq 2)$-identifying codes.

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1. Introduction

Let $G = (V, E)$ be an undirected and connected graph. Denote by $d(u, v)$ the *(graphic) distance* between vertices u and v, i.e., the number of edges in any shortest path between u and v. For the *ball* of radius r centred at $u \in V$ we use the notation $B_r(u) = \{v \in V \mid d(u, v) \leq r\}$. We use the notation xy for an edge $\{x, y\} \in \binom{V}{2}$, where $\binom{V}{2}$ denotes the set of unordered pairs of V. A nonempty subset of V is called a *code* and its elements are *codewords*.

Karpovsky, Chakrabarty and Levitin [11] introduced the concept of identifying codes.

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Definition 1. Let $r, l \geq 0$. A code $C \subseteq V$ is $(r, \leq l)$-identifying if the sets

$$I_r(X) = I_r(G, C; X) := (\bigcup_{x \in X} B_r(x)) \cap C$$

are distinct for all $X \subseteq V$, where $|X| \leq l$. The set $I_r(X)$ is called the *I-set* of X.

Identifying codes can be applied (see [11]) to locating faulty processors in a multiprocessor system; the vertices of the underlying graph correspond to the processors and an edge is a communication link between two processors. An identifying code enables us to determine uniquely the set of faulty processors $X \subseteq V$ using solely the alarming set $I(X)$.

Identification has been investigated in many graphs: the binary Hamming spaces (that is, binary hypercubes), the square lattice, the triangular grid and the hexagonal mesh, see, e.g., [1], [2], [8]–[11], [14].

In [7], Honkala, Karpovsky and Levitin considered a stronger requirement for a code which can locate the faulty processors even if the I-sets are corrupted by changes in the set of edges, that is, codes remain identifying even if some communication links are deleted or some false links emerge (see [15] for the motivation of sensor networks). Denote the symmetric difference by $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Definition 2. A code $C \subseteq V$ is called a t-edge-robust $(r, \leq l)$-identifying code (in $G = (V, E)$) if the code C is $(r, \leq l)$-identifying in every graph $G' = (V, E')$, where in the set of edges E' at most t edges are deleted or added, that is, $E' = E \triangle E^1$ and the set $E^1 \subseteq \binom{V}{2}$ has size at most t.

Thus the underlying graph G' is obtained from G by adding and/or deleting together at most t edges. The I-sets change accordingly. For applications to these codes consult [7], [15]. Edge-robust identifying codes are considered also in recent articles [4], [5], [6], [13].

In this paper we concentrate on binary Hamming spaces. Let F^n be the n-fold Cartesian product of the binary field $F = \{0, 1\}$. The distance $d(x, y)$ between x and y in F^n is the number of coordinate positions in which they differ. The Hamming weight $w(x)$ of $x \in F^n$ is defined via $w(x) = d(0, x)$. Let G_n be the graph with vertex set F^n and there is an edge between two vertices if and only if their distance equals one. The edge set is denoted by E_n. The vertices are called *words* (of length n). We denote $S_i(x) = \{y \in F^n \mid d(x, y) = i\}$.

The so-called multiple coverings have been very useful for identification in G_n; we define them now. If $d(x,y) \leq 1$, we say that x and y *cover* each other. A code $C \subseteq F^n$ is called a *μ-fold 1-covering* if for every $x \in F^n$ we have $|I(x)| \geq \mu$, that is, every word is covered by at least μ codewords of C. An excellent account for these codes is [3, Chapter 14].

For the rest of the paper we assume that $r = 1$ and $t \geq 1$. We write $I_1(G, C; X) = I(G, C; X)$ and whenever convenient we also drop C from the notation.

In [7], t-edge-robust $(1, \leq l)$-identifying codes with $l = 1$ are considered, and the other cases except $t = 1$ and $l = 2$ are treated in [12]. In this paper, we focus on the remaining case $t = 1$ and $l = 2$. We show that such a code, i.e., a 1-edge-robust $(1, \leq 2)$-identifying code, is always a 4-fold 1-covering (in fact, we prove a more general result for any $l \geq 2$). We give in Theorem 2 a classification of 1-edge-robust $(1, \leq 2)$-identifying codes in G_n. Indeed, five local patterns are given such that avoiding them in a 4-fold 1-covering immediately gives a 1-edge-robust $(1, \leq 2)$-identifying code. Do we find small (i.e., good) codes that avoid these patterns? The classification yields, using a construction from [14], an infinite family of the smallest possible (i.e., *optimal*) 1-edge-robust $(1, \leq 2)$-identifying codes. So, for infinitely many G_n we completely settle the problem, see Corollary 1. The five patterns also give us insight what kind of patterns in general can be avoided using the construction in [14]. The patterns are also easily implemented to a computer program that checks whether or not a code is edge-robust identifying.

Before addressing the classification, we recall a well-known lemma whose proof is easy and omitted.

Lemma 1. *Consider the graph G_n.*

(i) For $a, b \in F^n$ we have

$$|B_1(a) \cap B_1(b)| = \begin{cases} n+1 & \text{if } a = b\,, \\ 2 & \text{if } d(a,b) = 1 \text{ or } 2\,, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The intersection of three Hamming spheres of radius one consists of a unique point if the intersection is nonempty.

(iii) Let $x \in F^n$. If $a, b \in S_i(x)$, $a \neq b$, for some i, $0 < i < n-1$, then $B_1(a) \cap B_1(b)$ (if nonempty) contains a unique point in $S_{i-1}(x)$ and in $S_{i+1}(x)$.

(iv) Let $x \in F^n$ and $0 < i < n-1$. A word $a \in S_i(x)$ covers i words in

$S_{i-1}(x)$, only itself in $S_i(x)$, $n-i$ words in $S_{i+1}(x)$ and none elsewhere.

2. A Classification

From now on we make the following convention. By writing $I(X)$ we mean the I-set of X with respect to the *original* graph G_n, that is, $I(X) = I(G_n; X)$ and for the I-set of X with respect to some other graph G'_n we always write $I(G'_n; X)$ mentioning also the graph. It is convenient to write $I(\{x\}) = I(x)$ for $x \in F^n$. The following result was stated but not proved in [12]. It shows that when $t = 1$ and $l = 2$, all 1-edge-robust $(1, \leq 2)$-identifying codes must be 4-fold 1-coverings.

Theorem 1. *Let $l \geq 2$, $t \geq 1$ and $2l + t - 1 \leq n + 1$. Any t-edge-robust $(1, \leq l)$-identifying code is a $(2l + t - 1)$-fold 1-covering.*

Proof. Let $C \subseteq F^n$ be t-edge-robust $(1, \leq l)$-identifying.

Suppose first that there exists $x \notin C$ such that $|I(x)| \leq 2l + t - 2$. We show that this is impossible. Let $p_i \in S_1(x)$ $(i = 1, \ldots, 2l + t - 2)$ be such that $I(x) \subseteq \{p_i \mid i = 1, \ldots, 2l + t - 2\}$; the set of p_i's just helps us to handle simultaneously all sizes of $I(x)$ from 0 to $2l + t - 2$. For every pair $\{p_{2i-1}, p_{2i}\}$, $i = 1, \ldots, l - 1$, we have a unique word y_i, $y_i \neq x$, such that it covers in G_n the pair. Indeed, $d(p_{2i-1}, p_{2i}) = 2$ and the words covering both of them are the two words obtained by changing in p_{2i} either of the two coordinates in which they differ; x is one of the two words. Now $I(\{y_1, \ldots, y_{l-1}\})$ and $I(\{y_1, \ldots, y_{l-1}, x\})$ differ in G_n if and only if $p_i \in C$ for any $i = 2l - 1, \ldots, 2l + t - 2$. There are at most t such $p_i's$ and thus in the graph G'_n, where the t edges xp_i $(i = 2l - 1, \ldots, 2l + t - 2)$ are deleted we have

$$I(G'_n; \{y_1, \ldots, y_{l-1}\}) = I(G'_n; \{y_1, \ldots, y_{l-1}, x\}),$$

which is a contradiction.

Assume next that $x \in C$ and again $|I(x)| \leq 2l + t - 2$. Let $p_i \in S_1(x)$ for $i = 1, \ldots, 2l + t - 3$ be such that $I(x) \setminus \{x\} \subseteq \{p_i \mid i = 1, \ldots, 2l + t - 3\}$. As above, define y_i for all the pairs $\{p_{2i-1}, p_{2i}\}$, where $i = 1, \ldots, l - 2$ (the set of y_i's can be empty). Then the sets $I(G'_n; \{y_1, \ldots, y_{l-2}, p_{2l-3}\})$ and $I(G'_n; \{y_1, \ldots, y_{l-2}, p_{2l-3}, x\})$ are the same in the graph G'_n, where the t edges $xp_{2l-2}, \ldots, xp_{2l+t-3}$ are removed.

Thus for every $x \in F^n$ we must have $|I(x)| \geq 2l + t - 1$. □

From now on we consider the case $t = 1$ and $l = 2$. As mentioned, the previous theorem says that a 1-edge-robust $(1, \leq 2)$-identifying code is a 4-fold 1-covering. Next we work in the opposite direction and classify the 4-fold 1-coverings that are 1-edge-robust $(1, \leq 2)$-identifying. It turns out that avoiding five local patterns is enough. Intuitively, it is clear that the patterns must be local (this is a common phenomenon in identification in general because only the faulty processors close to each other can have same alarming I-set), moreover, as will be seen in the proof, only edge additions between vertices that are very close to each other matter, although we allow any additions.

Theorem 2. *A 4-fold 1-covering $C \subseteq F^n$ for which none of the five patterns of Figure 1 exists is 1-edge-robust $(1, \leq 2)$-identifying. The reverse statement is also true.*

Proof. Suppose $C \subseteq F^n$ is a 4-fold 1-covering and none of the five patterns appears. We show first that C is 1-edge-robust $(1, \leq 2)$-identifying. Let $X, \Gamma \subseteq F^n$, where $|X| \leq 2$, $|\Gamma| \leq 2$ and $\Gamma \neq X$. We have to check that always $I(G; X) \neq I(G; \Gamma)$, where G is G_n or any G'_n where one edge has been added or deleted. Note that, while going from G_n to G'_n, one edge deletion or addition can remove from $I(X) \setminus I(\Gamma)$ at most one codeword — this also holds for the other part of symmetric difference $I(\Gamma) \setminus I(X)$.

First we show that if $I(G; X) = I(G; \Gamma)$ then it implies that X and Γ both are *pairs*, i.e., consist of two words. We can assume that $|X| \geq |\Gamma|$ (just change the roles if this is not the case) and so that there exists $x \in X$ such that $x \notin \Gamma$. Now $|\Gamma| = 0$ or 1, i.e., it is not a pair and $\Gamma = \emptyset$ or $\Gamma = \{\alpha\}$. In these two cases we obtain easily that $I(G; X) \neq I(G; \Gamma)$: indeed, since C is 4-fold 1-covering, we have $|I(x)| \geq 4$ and α can cover by Lemma 1(i) at most two words of $I(x)$ in G_n (trivially none is covered if $\Gamma = \emptyset$) and at most one more of them can, by an edge change, be either removed from $I(x)$ or added to $I(\Gamma)$ if $G = G'_n$ (and if $G = G_n$ nothing is removed or added). Therefore, there exists at least one word in $I(G; X)$ such that it does not belong to $I(G; \Gamma)$. This implies that it suffices to consider only the I-sets of two *pairs* $X = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$, where $x \in X \setminus \Gamma$ and $\alpha \in \Gamma \setminus X$. Furthermore, if $\alpha \in \Gamma$ alone covers (in G_n) some of the words of $I(x)$, that is, $\beta \in \Gamma$ does not help it (meaning $I(\beta) \cap I(x) = \emptyset$), then the above argument is still true. This yields that it suffices to work only with two pairs, where $1 \leq d(x, \alpha) \leq 2$, $1 \leq d(x, \beta) \leq 2$ (here the distance is with respect to G_n).

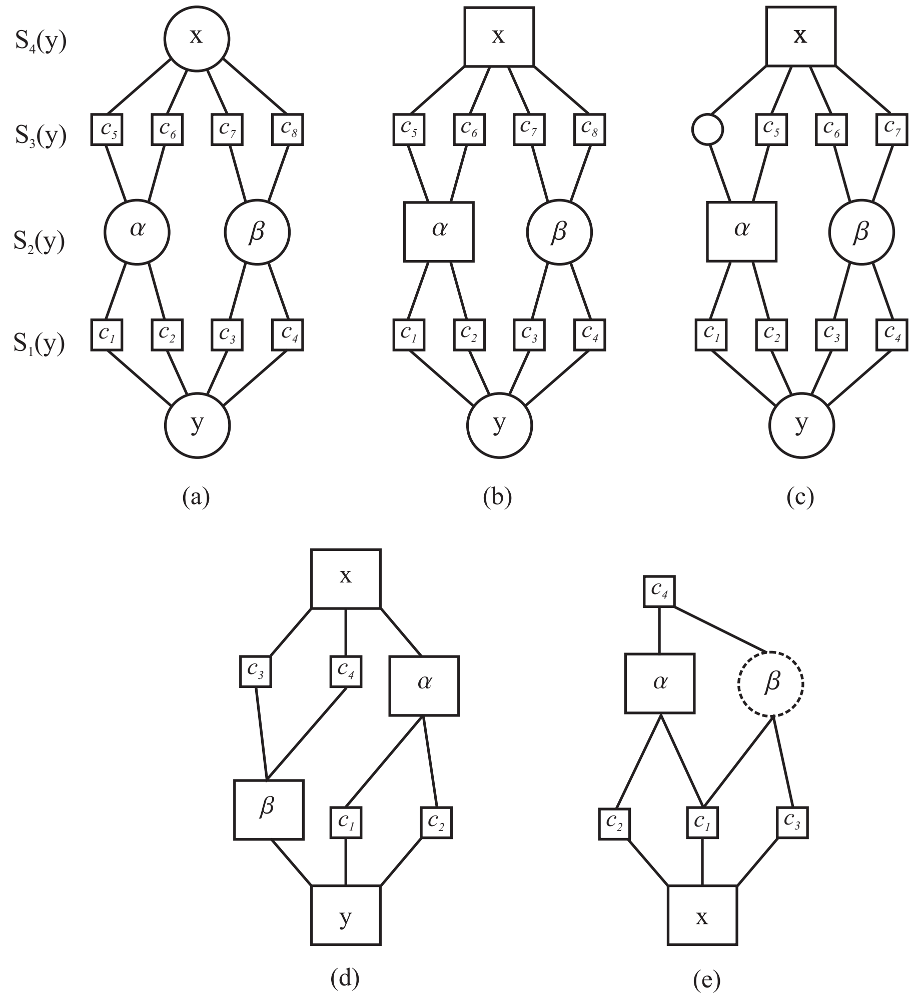

Figure 1: The forbidden configurations in the graph G_n, where a vertex (i.e., a word) marked by a square denotes a codeword and a vertex marked by a circle is a non-codeword. (a) Pattern 1 (b) Pattern 2, (c) Pattern 3, (d) Pattern 4 and (e) Pattern 5, where the dashed circle means that it can be a codeword or not. All the codeword-neighbours of x, y, α and β are given, except in Pattern 1 and in Pattern 4 there can be one more codeword-neighbour of x, which is not adjacent to any y, α or β. Moreover, in Pattern 5 β may have other codeword-neighbours not adjacent to x or α.

We show next that $I(G;X) \neq I(G;\Gamma)$ for such pairs. Assume to the contrary that

$$I(G;X) = I(G;\Gamma)$$

for some pairs X, Γ and for some graph G. Note that throughout the proof we write $S_i = S_i(x)$ with respect to G_n.

Case 1. Assume first G is the original graph G_n. Roughly speaking, what we do is the following: $x \in X \setminus \Gamma$ covers some codewords in $S_0 \cup S_1$ and α and β try to cover them (to have at least $I(x) \subseteq I(\Gamma)$), and in turn, there will be some codewords covered by α and β such that x cannot cover them and hence we seek for y which could take care of those, leading to $I(X) = I(\Gamma)$. So, staring with x we check what kind of configuration of α, β and γ and codewords around x would give $I(X) = I(\Gamma)$. It turns out that only the five patterns will lead to this situation.

Note that trivially x cannot cover anything in S_i for $i \geq 2$.

If $|I(x)| \geq 5$, then $I(X) = I(\Gamma)$ is not possible (by Lemma 1(i) the two elements of Γ can cover at most four words of $I(x)$). Thus it suffices to assume $|I(x)| = 4$. We separate the examination into two cases $x \notin C$ and $x \in C$.

<u>$x \notin C$</u>: Assume first that $x \notin C$ and $I(x) = \{c_5, c_6, c_7, c_8\}$ (see Figure 1), in other words, $c_i = x + e_i$ $(i = 5, 6, 7, 8)$, where e_i's are distinct words of weight one. Because α and β can each cover, by Lemma 1(i), at most two codewords of $I(x)$, they both need to cover two, say, α covers c_5 and c_6 and β covers c_7 and c_8. Consequently, $\alpha = x + e_5 + e_6$ and $\beta = x + e_7 + e_8$. This implies that $\alpha, \beta \in S_2$ and $d(\alpha, \beta) = 4$. Now α (resp. β) must cover a codeword in S_3, say c (resp, a), because by Lemma 1(iv), α (resp. β) covers in $S_1 \cup S_2$ at most three codewords and $|I(\alpha)| \geq 4$.

Assume first that $\alpha \in C$ (resp. $\beta \in C$). The assumption $I(X) = I(\Gamma)$ requires that we find y such that it covers both α and a (resp. β and c) because x cannot cover either of them. But now, by triangular inequality, $d(\alpha, a) \geq d(\alpha, \beta) - 1 = 3$, so such y cannot exist, and either α or a belongs to $I(\Gamma)$ but not to $I(X)$, a contradiction.

Thus we may assume that $\alpha, \beta \notin C$ (Figure 1). If $|I(\alpha) \cap S_3| \geq 3$, we are done since y must cover these (at least) three codewords and, by Lemma 1(ii), this means that $y = \alpha$, but $\alpha \notin X$. If $|I(\beta) \cap S_3| \geq 3$ then similarly $y = \beta$ (which is allowed), but now c (defined as above) cannot be covered by y and hence $c \in I(\Gamma) \setminus I(X)$ giving $I(X) \neq I(\Gamma)$. Therefore, without loss of generality we can assume $I(\alpha) = \{c_1, c_2, c_5, c_6\}$ and $I(\beta) = \{c_3, c_4, c_7, c_8\}$ for some $c_1, c_2, c_3, c_4 \in S_3$. Now y must cover c_1, c_2, c_3 and c_4. Because it covers $c_1, c_2 \in S_3 \cap I(\alpha)$, by Lemma 1(iii), we know that $y \in S_4$. If $|I(y)| \geq 5$, then we are done, since $|I(y) \cap S_3| \leq 4$, by Lemma 1(iv), and α and β cannot cover words from S_i, where $i \geq 4$. Consequently, $I(y) = \{c_1, c_2, c_3, c_4\}$ but now we have the forbidden pattern 1 (where $|I(x)| = 4$), a contradiction.

$x \in C$: Assume that $x \in C$ (recall that $|I(x)| = 4$). Now either α or β covers x, otherwise $x \in I(X)$ but $x \notin I(\Gamma)$. Assume first that α does. By Lemma 1(ii) β ($\neq x$) cannot cover the three words of $I(x) \setminus \{x\}$ and thus $\alpha \in I(x)$. Therefore, $I(x) = \{x, \alpha, c_3, c_4\}$ and β must cover c_3 and c_4. Consequently, $\beta = x + c_3 + c_4 \in S_2$. It suffices to check the case where $|I(\alpha) \cap S_2| = 2$ because otherwise $y = \alpha$ (y must cover all in $I(\alpha) \cap S_2$). We can write $I(\alpha) = \{x, \alpha, c_1, c_2\}$. This implies $y \in S_3$ since $y \neq \alpha$ and it has to cover c_1 and c_2. Moreover, it is enough to assume $I(y) \cap S_4 = \emptyset$ and since $|B_1(y) \cap S_2| = 3$, by Lemma 1(iv), we know that $y \in C$ and, furthermore, $\beta \in I(y)$ (or we are done). Obviously, $I(\beta) = \{y, \beta, c_3, c_4\}$ or $I(X) \neq I(\Gamma)$. However, now the pattern 4 shows that this is not possible.

Suppose β covers x. We may assume that $y = \beta$ because if $\beta \neq y$, then we may change the roles of α and β and the previous argument works (notice that the only difference between α and β is that we always assume $\alpha \notin X$ and β may or may not belong to X — and if it does then $\beta = y$). Clearly, α must cover c_3 and c_4 from $I(x) = \{x, \beta, c_3, c_4\}$ and thus $\alpha \in S_2$. But now there exists a word $c \in I(\alpha) \cap S_3$ which cannot belong to $I(X)$ as $y = \beta \in S_1$.

Case 2. Assume now that $G = G'_n$ with one edge change. This case has the same kind of flavour as Case 1, but it is technically more demanding, and can be found in details in Appendix A.

The reverse statement. By the previous theorem a 1-robust $(1, \leq 2)$-identifying code is also a 4-fold 1-covering. Moreover, it is easy to see that there cannot be patterns like 1 to 5 or we have two pairs $\{x, y\}$ and $\{\alpha, \beta\}$ (for the pattern 5 $y = \beta$) whose I-sets are identical in G_n or in a suitable G'_n. □

3. An Infinite Sequence of Optimal Codes

In the next theorem we give a construction (also well know in other contexts [3, 14]) which yields from a 1-edge-robust $(1, \leq 2)$-identifying code of length n

another 1-edge-robust $(1, \leq 2)$-identifying code of length $2n+1$. Let us denote by $\pi(u) = \sum_{i=1}^{n} u_i \bmod 2$, where $u = (u_1, u_2, \ldots, u_n) \in F^n$. With a suitable initial code (given in Theorem 4) we get by applying the construction again and again optimal 1-edge-robust $(1, \leq 2)$-identifying codes for infinitely many lengths.

Theorem 3. *If $C \subseteq F^n$ is a 1-edge-robust $(1, \leq 2)$-identifying code, then*

$$D = \{(\pi(u), u, u+c) \mid u \in F^n, c \in C\} \subseteq F^{2n+1}$$

is also 1-edge-robust $(1, \leq 2)$-identifying.

Let us first state some preliminary observations from [14] of the construction D and the relations between C and D.

Let us examine the I-set of an arbitrary word $w \in F^{2n+1}$. The word can be written as $w = (a_w, u_w, u_w + v_w)$, where $a_w \in F$ and $u_w, v_w \in F^n$.

(i) If $\pi(u_w) = a_w$, then (in G_{2n+1})

$$I(D; w) = \{(\pi(u_w), u_w, u_w + c) \mid c \in C,\ d(c, v_w) \leq 1\}.$$

(ii) If $\pi(u_w) \neq a_w$ then (in G_{2n+1})

$$\begin{aligned} I(D; w) &= \{(a_w, u', u_w + v_w) \mid c \in C,\ d(c, v_w) = 1,\ u' + c = u_w + v_w\} \\ &\cup \ (\{(a_w + 1, u_w, u_w + v_w)\} \cap D). \end{aligned}$$

The word $(a_w + 1, u_w, u_w + v_w)$ is in D if and only if $v_w \in C$.

By [14] (in both cases)

$$I(D; w) = \{(a_i, u_i, s_i) \mid i = 1, \ldots, k\} \\ \Rightarrow I(C; v_w) = \{u_i + s_i \mid i = 1, \ldots, k\}. \quad (1)$$

Observe also, that if w is a codeword of D, then w belongs to the case (i).

In the case (i) the first $n+1$ bits (resp. in the case (ii) the last n bits) are the same for all words in $I(D; w)$. Suppose

$$|I(D; w) \cap I(D; z)| = 2$$

for some $w \neq z$. If $\pi(u_w) = a_w$ as in the case (i) (resp. $\pi(u_w) \neq a_w$ as in the case (ii)), then $\pi(u_z) = a_z$ (resp. $\pi(u_z) \neq a_z$) and $u_w = u_z$ (resp. $u_w + v_w = u_z + v_z$).

Proof of Theorem 3. Since C is a 4-fold 1-covering, then by virtue of (1), the code D is as well. Thus Theorem 2 says that we only need to check that the patterns 1 to 5 (where the I-sets are with respect to D and G_{2n+1}) cannot exist in F^{2n+1}. Notice that in the patterns 1 to 3 of Theorem 2 we have

$$|I(D;y) \cap I(D;\alpha)| = |I(D;y) \cap I(D;\beta)| = |I(D;\beta) \cap I(D;x)| = 2$$

and hence we obtain (as explained above)

$$u_x = u_\alpha = u_\beta = u_y \, , \tag{2}$$

if $\pi(u_y) = a_y$ and

$$u_\alpha + v_\alpha = u_\beta + v_\beta = u_x + v_x = u_y + v_y \, , \tag{3}$$

otherwise. Note that in the pattern 2 and 3 the word x is a codeword of D and thus we only need (2).

In the pattern 4 all the words x, y, α, β are codewords and belong to the case (i), so we obtain (2) immediately. The words x and α of the pattern 5 are codewords and belong to the case (i). Since $|I(D;x) \cap I(D;\beta)| = 2$ we get $u_x = u_\alpha = u_\beta$.

Pattern 1. Consider the pattern 1 of Theorem 2. If such four words x, y, α and β satisfying the pattern 1 in F^{2n+1} existed, then by (1) we obtain $\{v_x, v_y\} = \{v_\alpha, v_\beta\}$. This is shown next. Let first $|I(x)| = 4$, then by (1) we immediately see that $I(C; \{v_x, v_y\}) = I(C; \{v_\alpha, v_\beta\})$. Since C is a $(1, \leq 2)$-identifying code we get $\{v_x, v_y\} = \{v_\alpha, v_\beta\}$. If $|I(x)| = 5$, we write $I(x) = \{c_5, c_6, c_7, c_8, c_9\}$ (notice that c_9 can be equal to x also). We get $I(G'_n, C; \{v_x, v_y\}) = I(G'_n, C; \{v_\alpha, v_\beta\})$, where G'_n is obtained by adding the edge $v_\alpha c_9$ to E_n. However this is yields $\{v_x, v_y\} = \{v_\alpha, v_\beta\}$ because C is 1-edge-robust $(1, \leq 2)$-identifying.

The fact $\{v_x, v_y\} = \{v_\alpha, v_\beta\}$ combined with (2) for $\pi(u_y) = a_y$ (resp. combined with (3) for $\pi(u_y) \neq a_y$) implies that $x = \alpha$ (if $v_x = v_\alpha$) or $x = \beta$ (if $v_x = v_\beta$). This is clearly in contradiction with the pattern 1.

Pattern 2 *and* 3. Assume now that we had four words x, y, α and β satisfying the pattern 2 (resp. the pattern 3). By (1), we obtain $I(G'_n, C; \{v_x, v_y\}) = I(G'_n, C; \{v_\alpha, v_\beta\})$ by adding the edge $v_\alpha v_x$ to E_n. This gives $\{v_x, v_y\} = \{v_\alpha, v_\beta\}$. As above, we obtain, using (2), the contradiction $x = \alpha$ or $x = \beta$ with the pattern 2 (resp. the pattern 3).

Pattern 4. Suppose there are four words x, y, α and β satisfying the pattern 4. Let first $|I(x)| = 4$. Then again $I(C; \{v_x, v_y\}) = I(C; \{v_\alpha, v_\beta\})$ and so we can conclude $\{v_x, v_y\} = \{v_\alpha, v_\beta\}$. Now (2) yields the contradiction $x = \alpha$ or $x = \beta$. If $|I(x)| = 5$ we write $I(x) = \{x, \alpha, c_3, c_4, c_5\}$ and again $I(G'_n, C; \{v_x, v_y\}) =$

$I(G'_n, C; \{v_\alpha, v_\beta\})$ by adding the edge $v_\alpha c_5$ to E_n. This leads similarly with the aid of (2) to the contradiction.

Pattern 5. Assume that there are three words x, α and β giving the pattern 5. By (1) we get $I(G'_n, C; \{v_x, v_\beta\}) = I(G'_n, C; \{v_\alpha, v_\beta\})$ adding the edge $v_\alpha v_x$ to E_n. Thus $\{v_x, v_\beta\} = \{v_\alpha, v_\beta\}$ and so $v_x = v_\alpha$ which is not possible ($x \neq \alpha$) by the fact $u_x = u_\alpha = u_\beta$. □

Although we only needed to consider the five forbidden patterns, our approach gives also insight what sort of patterns *in general* can be avoided using the construction D.

With the aid of the previous theorem, the following result gives an infinite sequence of optimal 1-edge-robust $(1, \leq 2)$-identifying codes.

Theorem 4. *The smallest possible cardinality of a 1-edge-robust* $(1, \leq 2)$*-identifying code of length seven equals* 64.

Proof. The cardinality of the code is at least 64 since it is a 4-fold 1-covering in F^7. It can be checked (by computer) that none of the patterns 1 to 5 is satisfied by the 4-fold 1-covering

$$C = \mathcal{H}_3 \cup (1000000 + \mathcal{H}_3) \cup (0100000 + \mathcal{H}_3) \cup (0010000 + \mathcal{H}_3)\,,$$

where $\mathcal{H}_3$ is the [7], [4], [3] – Hamming code. □

Using this code as a starting point for the construction D and using D over and over again to the resulting code, we get the following corollary — *the optimality is guaranteed by the lower bound of Theorem 1.*

Corollary 1. *For* $r \geq 3$*, the smallest possible cardinality of a 1-edge-robust* $(1, \leq 2)$*-identifying code of length* $2^r - 1$ *equals* $2^{2^r - r + 1}$.

Finally, we give two more optimal codes.

Theorem 5. *The smallest possible size of a 1-edge-robust* $(1, \leq 2)$*-identifying code of length four (resp. five) equals* 14 *(resp. equals* 22*).*

Proof. The lower bounds are clear since codes needs to be 4-fold 1-coverings [3, p.383]. The fact that the code $C_4 = F^4 \setminus \{0000, 1111\}$ and $C_5 = F^5 \setminus \{11000, 01100, 00110, 00011, 10001, 11010, 01101, 10110, 01011, 10101\}$ do not satisfy the patterns 1 to 5 can be checked by Theorem 2. □

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A. Appendix: Case 2 of Theorem 2

Proof. (Continued) *Case 2.* Let then $G = G'_n$, where one edge, say e, has been added to E_n or deleted from E_n. Let us first consider what the assumption $I(G;X) = I(G;\Gamma)$ tells about the symmetric difference $I(X) \triangle I(\Gamma)$ in the original graph G_n.

We can assume by Case 1 that $I(X) \triangle I(\Gamma)$ is nonempty. A deletion or an addition of an edge removes from $I(X) \triangle I(\Gamma)$ at most one element – except in the case that an edge, say vu, is added, where $u \in X$ and $u \in I(X) \setminus I(\Gamma)$ and $v \in \Gamma$, where $v \in I(\Gamma) \setminus I(X)$. In this case at most two elements are removed. We can of course denote $u = x$ and $v = \alpha$ because $x \in X \setminus \Gamma$ and $\alpha \in \Gamma \setminus X$.

Consequently, $I(G;X) = I(G;\Gamma)$ implies that $I(X) \triangle I(\Gamma)$ consists either of one element, say g, or $I(X) \triangle I(\Gamma)$ consists of x and α. We can assume in the first case that x covers g and then we divide the investigation into two parts $x \neq g$ and $x = g$.

(i) Assume first that $I(X) \triangle I(\Gamma)$ consists of one element g, where $x \neq g$ and $g \in B_1(x)$.

$\underline{x \notin C:}$ It suffices to examine the cases $|I(x)| = 5$ and $|I(x)| = 4$, because again α and β can cover at most four words of $I(x)$ and g is not covered by them. Let first $I(x) = \{c_5, c_6, c_7, c_8, g\}$. Now Γ must cover $I(x) \setminus \{g\}$ and the argument goes exactly as in Case 1 where $x \notin C$ leading to contradiction with the nonexistence of the pattern 1 with $|I(x)| = 5$.

Assume next that $|I(x)| = 4$, that is, $I(x) = \{c_5, c_6, c_7, g\}$. Now α or β covers two of the three words in $I(x) \setminus \{g\}$.

Suppose α covers two, say c_5 and c_6. This implies that $\alpha \in S_2$. Moreover, $c_7 \in I(\beta)$ and $\beta \in S_1 \cup S_2$. Suppose first that $\beta \in S_1$. Then $|I(\beta) \cap S_2| \geq 3$ and $y = \beta$ by Lemma 1(ii) because y must cover all of them. But there exists a word $c \in I(\alpha) \cap S_3$ such that $c \notin I(X)$ which gives the contradiction.

$(*)$ Take next $\beta \in S_2$. If $\alpha \in C$, then $\alpha, c \in I(y)$ and therefore $y = c$. But now $I(y) \cap S_2$ contains at least one element not in $I(\Gamma)$ (there cannot be any elements of $I(y)$ in S_4 or we are immediately done and so there must be three in S_2 but α and β can cover at most two by Lemma 1(iv)). Let then $\alpha \notin C$. Then $|I(\alpha) \cap S_3| = 2$ (if there is more than two we are done because $y \neq \alpha$) and this implies that $y \in S_4$ and we can assume (as in the first part of Case 1) that $|I(y)| = |I(y) \cap S_3| = 4$. Now β must cover two of these words in $I(y)$ giving $d(\alpha, \beta) = 4$ and $|I(\beta) \cap S_3| = 2$ or we are done. But clearly $\beta \notin C$ (because x and y cannot cover it) and hence $I(\beta) \cap I(\alpha) \cap S_1$ should be nonempty (the fourth codeword must be in S_1 by Lemma 1(iv)) but then $d(\alpha, \beta) = 2$, a contradiction.

Assume next that β covers c_5 and c_6. It is again enough to assume $\beta = y$. So $\beta \in S_2$ and $c_7 \in I(\alpha)$. Clearly, $I(\alpha) \cap S_2 = \emptyset$ or we are done since y cannot cover such codewords. Thus $\alpha \in S_2$ and $\alpha \notin C$. This gives $|I(\alpha) \cap S_3| \geq 2$ and X cannot cover these.

<u>$x \in C$:</u> Again it suffices to check the cases $|I(x)| = 5$ and $|I(x)| = 4$. If $|I(x)| = 5$, then considering $I(x) \setminus \{g\}$ and utilizing the same argument as in Case 1, where $x \in C$, shows that $I(G; X) \neq I(G; \Gamma)$ by considering the pattern 4 with $|I(x)| = 5$. So we study the case $|I(x)| = 4$ and assume that $g \neq x$ in $I(x)$ is removed. Obviously, α or β covers x.

Let first α cover x. Evidently, $\alpha \in I(x) \setminus \{x, g\}$ because $\alpha \neq y$. We can assume $|I(\alpha) \cap S_2| = 2$ (denote these elements by c_1 and c_2) or we are done. It follows that $y \in S_3$ and it suffices to check the case where $I(y) = \{y, c_1, c_2, \beta\}$ (clearly, β belongs to S_2 in order to cover y, and $I(y)$ cannot contain nothing from S_4). The word β cannot cover other codewords than y and β in $S_2 \cup S_3$, so it must cover two in S_1 which, in turn, give $|I(\alpha) \cap S_2| = 3$ but we already assumed $|I(\alpha) \cap S_2| = 2$.

Let then $\beta = y$ cover x. Now α must cover the word, say c', which is left in $I(x)$. It is enough to consider $\alpha \in S_1$ (otherwise $I(x) \cap S_3$ is again nonempty and out of reach of X) and moreover $\alpha = c'$ or we are done. But then y cannot cover all the words of $S_2 \cap I(\alpha)$ by Lemma 1(iii).

(ii) Suppose now that $I(X) \triangle I(\Gamma)$ consists of x. It is enough again to consider $|I(x)| = 5$ and $|I(x)| = 4$. Obviously, $\alpha, \beta \in S_2$. The former case is treated again as the first part of Case 1. Let then $I(x) = \{x, c_1, c_2, c_3\}$. We get the contradiction as in $(*)$ of the case (i) above.

(iii) Let now $I(X) \triangle I(\Gamma)$ consist of x and α (so they both are codewords). Evidently, $\alpha, \beta \in S_2$ (neither of α or β can cover x in G_n).

It is enough to check the cases $|I(x)| = 5$ and $|I(x)| = 4$. Let first $|I(x)| = 5$ and $I(x) = \{x, c_5, c_6, c_7, c_8\}$. We can assume (like in the first part of Case 1) that $c_5, c_6 \in I(\alpha)$ and $c_7, c_8 \in I(\beta)$. Hence $d(\alpha, \beta) = 4$ in G_n. If $\beta \in C$ then there does not exist y such that it covers both β and an existing element $c_1 \in I(\alpha) \cap S_3$. Suppose $\beta \notin C$. It follows that $|I(\beta) \cap S_3| = 2$, say $I(\beta) \cap S_3 = \{c_3, c_4\}$, or otherwise $y = \beta$ and c_1 cannot be covered by y. This implies that $y \in S_4$ and $I(y) = \{c_1, c_2, c_3, c_4\}$ for some $c_2 \in S_3$. Evidently, $c_2 \in I(\alpha)$ or we are done. Hence $I(\alpha) = \{\alpha, c_1, c_2, c_5, c_6\}$ and $I(\beta) = \{c_3, c_4, c_7, c_8\}$. The pattern 2 gives the contradiction.

Assume next that

$$I(x) = \{x, c_5, c_6, c_7\}.$$

Now α must cover at least one of c_5, c_6 and c_7 (because β cannot cover all of them) and α can cover at most two of these words.

Suppose first that α covers two, say c_5 and c_6. Clearly, β must cover c_7. Either β covers one of c_5 and c_6 or not. Assume first that not. Considering $I(\beta) \cap (S_2 \cup S_3)$, which contains at least three codewords, it is easy to conclude that $y = \beta$ but then $I(\alpha) \cap S_3$ contains a codeword which is not in $I(X)$ (notice that again $d(\alpha, \beta) = 4$). Assume then that β covers c_5 or c_6, say c_5. There exists an element $c \in I(\alpha) \cap S_3$ and in order to cover it $y \in S_2 \cup S_4$ (the case $y \in S_3$ leads to the situation $y = c$ and we know that $\alpha \notin I(y)$ because α is in the symmetric difference). Suppose first $y \in S_4$. We can assume that $I(y) = I(y) \cap S_3$ and to cover these four codewords of $I(y)$ by Γ the word β should cover exactly two of them and α the other two, but then $d(\alpha, \beta) = 4$ and by our assumption this distance is two (c_5 is a common neighbour). Assume thus that $y \in S_2$. If $\beta \in C$, then clearly $y = \beta$. Also if $\beta \notin C$ by Lemma 1(iii) we obtain $y = \beta$. Therefore, $I(\alpha) = \{\alpha, c_5, c_6, c\}$ (if there are more codewords in this set we are immediately done) and $\{c_5, c_7, c\} \subset I(\beta)$ and $\beta = y$. This is not allowed by the pattern 5.

Assume now that α covers c_5 but not c_6 or c_7. It follows that β covers c_6 and c_7 and $\beta \in S_2$. There exists two elements in $I(\alpha) \cap S_3$, say c_1 and c_2 (there cannot be more than two codewords since $y \neq \alpha$). Thus $I(\alpha) = \{\alpha, c_1, c_2, c_5\}$. To cover c_1 and c_2 the word $y \in S_4$. Consequently, as before, we can write $I(y) = \{c_1, c_2, c_3, c_4\}$ for some $c_3, c_4 \in S_3$. Moreover, $I(\beta) = \{c_3, c_4, c_6, c_7\}$ or we are done. The pattern 3 now eliminates this situation.

This completes the proof of Theorem 2. $\square$