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FINITE ELEMENT ANALYSIS OF A BEAM ON A WINKLER FOUNDATION

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Abstract: In this paper we consider an equation which models the displacement of an elastic beam resting on a Winkler foundation which reacts only in compression. We use linear elasticity to model the beam. However, due to the assumption that the foundation reacts only in compression, the problem is nonlinear. The equation is fourth order so it is natural to use a mixed finite element method with piecewise linear elements to obtain approximate solutions. We present error estimates for the solution obtained by the finite element method and illustrate our proceedure with an example.

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Key Words: displacement of an elastic beam resting, linear elasticity, error estimates, finite element method

1. Introduction

In this paper we consider the static behavior of an elastic beam resting on a tensionless Winkler foundation. See [1] (which treats the dynamic case) and the references cited therein. We use linear elasticity to model the beam. Since the foundation is assumed to react only in compression, gaps between the beam and the foundation may occur. Thus the problem may be considered as a free boundary value problem, the free boundary consisting of those points at which the beam separates from the foundation. Of course, these points are not known

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in advance. However, our treatment does not take this point of view. We treat the reaction as a nonlinear force. We thus formulate the problem as a nonlinear boundary value problem. Fairly standard techniques are used to prove existence and uniqueness of a solution.

The main purpose of this paper is to obtain approximate solutions to the problem using the finite element method. Since the equation is fourth order, it seems natural to use a mixed method, which in our case simply means that we introduce the second derivative of the displacement (which is proportional to the bending moment) as a new variable. This enables us to use piecewise linear elements. The main result of this paper is the error estimate for the finite element solution. A slight complication is due to the fact that we use an approximate rather than an exact integration method in the finite element equations. We present an iterative method for solving the finite element equations. We then illustrate the method on a test problem. The test problem, which was suggested by [2], was chosen because the exact solution (found by a combination of analysis and numerics) exhibits a region in which the beam separates from the foundation.

In Section 2, we formulate the problem and prove existence and uniqueness. In Section 3 we will introduce the mixed finite element method for the approximate solution of the problem. In Section 4 we will apply our method to the test problem.

2. Formulation of the Problem

We consider an elastic beam of length $2L$ pressed against an elastic foundation by a load $P(x)$. The elastic foundation is assumed to react in compression only. The governing equation is

$$EIy'''' + ky^+ = P(x), \qquad -L < x < L\,. \tag{2.1}$$

In (2.1) EI is the flexural rigidity of the beam, k is the coefficient of stiffness of the foundation and y is the deflection of the beam (taken positive in the downward direction). We assume that the ends of the beam are fixed at $y = 0$ but the ends of r the beam are free to rotate. Thus the boundary conditions are

$$y(\pm L) = y''(\pm L) = 0\,. \tag{2.2}$$

After rescaling, we can rewrite the problem as

$$\frac{1}{4}u'''' + u^+ = \tilde{F}(x), \qquad -L < x < L, \tag{2.3}$$

$$u(\pm L) = u''(\pm L) = 0\,. \tag{2.4}$$

We will consider a more general version of (2.3). Let $\phi : \mathbf{R} \to \mathbf{R}$ be Lipschitz continuous and monotone increasing. The problem we will study is

$$u'''' + \phi(u) = F(x)\,, \quad -L < x < L\,, \tag{2.5}$$

with the boundary conditions (2.4).

We give the weak formulation of the problem. Let $X = H^2(-L,L) \cap H_0^1(-L,L)$ equipped with the norm of H^2. Let X^* be the dual space of X. Then the weak formulation of the problem is: Given $F \in X^*$ find $u \in X$ such that

$$(u'', v'') + (\phi(u), v) = \langle F, v\rangle, \quad \text{for all } v \in X \tag{2.6}$$

In (2.6) round brackets denote the L_2 inner product while the brackets denote the pairing between X and its dual.

Theorem 1. *Equation (2.6) has a unique solution*

Proof. We define a mapping $A : X \to X^*$ by

$$\langle Au, v\rangle = (u'', v'') + (\phi(u), v), \quad v \in X,$$

so that

$$\langle Au_1 - Au_2, u_1 - u_2\rangle = \|u_1'' - u_2''\|_{L_2}^2 + (\phi(u_1) - \phi(u_2), u_1 - u_2).$$

By the Poincaré and generalized Poincaré inequalities, on X, $\|u''\|_{L_2}$ is equivalent to $\|u\|_{H^2}$. From this we see that the mapping A is strongly monotone and Lipschitz continuous. Thus we may apply [3, Theorem 25.B] to conclude that (2.6) has a unique solution. □

We now write (2.5), (2.4) as a system. If we let $v = -u''$, (2.5), (2.4) becomes

$$-u'' = v, \quad -v'' + \phi(u) = F, \tag{2.7}$$

$$u(\pm L) = v(\pm L) = 0. \tag{2.8}$$

We will assume that $F \in H^{-1}(-L,L)$. Then the weak formulation of (2.7), (2.8) is: Find $(u,v) \in H_0^1 \times H_0^1$ such that

$$(u', w') = (v, w) \quad \text{for all } w \in H_0^1\,, \tag{2.9a}$$

$$(v', z') + (\phi(u), z) = \langle F, z\rangle \quad \text{for all } z \in H_0^1\,, \tag{2.9b}$$

where now the brackets denote the pairing between H_0^1 and H^{-1}. In the next section we will study the finite element approximation of (2.9).

3. The Finite Element Approximation

In this section we consider the finite element approximation of (2.6) which, since we have transformed the problem into (2.9) is a mixed finite element approximation.

We let $V = H_0^1(-L, L)$ equipped with the Dirichlet norm and inner product

$$D(u,v) = \int_{-L}^{L} u'(t)v'(t)\,\mathrm{d}t, \quad \|u\|_1 = D(u,u)^{1/2}. \tag{3.1}$$

For a positive integer N, let $h = \frac{2L}{N+1}$, $x_i = -L + ih$, $i = 0,\ldots,N+1$. We define $V_h \subset V$ to be the space of continuous, piecewise linear functions with breakpoints $\{x_i\}_{i=1}^N$. The set $\{\psi_i\}_{i=1}^N$ of "hat functions" form a basis for V_h. ($\phi_i(x_j) = \delta_{ij}$, $i = 1,\ldots,N$, $j = 0,\ldots,N+1$.) Our finite element approximation to (u,v) is the solution of the following problem: Find $(u_h, v_h) \in V_h \times V_h$ such that

$$D(u_h, w_h) = (v_h, w_h), \quad \text{for all } w_h \in V_h, \tag{3.2a}$$

$$D(v_h, z_h) + T(\phi(u_h), z_h) = \langle F, z_h\rangle \quad \text{for all } z_h \in V_h. \tag{3.2b}$$

In (3.2b) the term $T(\phi(u_h), z_h)$ means that we approximate the integral $(\phi(u_h), z_h)$ by means of the trapezoid rule. Thus the finite dimensional system which must be solved is

$$D(u_h, \psi_j) = (v_h, \psi_j), \quad j = 1,\ldots,N, \tag{3.3a}$$

$$D(v_h, \psi_j) + h\phi(u_h(x_j)) = \langle F, \psi_j\rangle, \quad j = 1,\ldots,N. \tag{3.3b}$$

We write

$$u_h = \sum_{i=1}^N \alpha_i\psi_i, \quad v_h = \sum_{i=1}^N \beta_i\psi_i. \tag{3.4}$$

When we insert (3.4) into (3.3) we get a system of nonlinear equations in $\mathbf{R^{2N}}$. We can solve (3.3a) for the β's in terms of the α's. When this is inserted into (3.3b) we get an equation in $\mathbf{R^N}$. By applying the argument of Theorem 1 we see that (3.3) has a unique solution.

We let $\|\cdot\|$ denote the L_2 norm. We wish to estimate $\|u - u_h\|$ and $\|v - v_h\|$. For the application in [2] we do not want to assume that $F \in L_2$. Thus we cannot get estimates in the Dirichlet norm. We define

$$e_u = u - u_h, \quad e_v = v - v_h. \tag{3.5}$$

These quantities satisfy

$$D(e_u, w) = (e_v, w), \quad \text{for all } w \in V_h\,, \tag{3.6a}$$

$$D(e_v, z) + (\phi(u) - \phi(u_h), z) = E(u_h, z) \quad \text{for all } z \in V_h\,. \tag{3.6b}$$

In (3.6b) $E(u_h, z)$ is defined as

$$E(u_h, z) = T(\phi(u_h), z) - (\phi(u_h), z), \quad z \in V_h\,. \tag{3.7}$$

We introduce u^* and v^*, the orthogonal projections of u and v respectively in the Dirichlet norm onto V_h. As is well known, u^* and v^* are simply the piecewise linear interpolants of u and v. We have

$$\|u - u^*\| \le h\|u\|_1, \quad \|v - v^*\| \le h\|v\|_1\,. \tag{3.8}$$

By elliptic regularity $u \in H^3(-L, L) \cap H_0^1(-L, L)$ and so

$$\|u - u^*\| \le h^2\|u\|_{H^2}\,. \tag{3.9}$$

Since we are only assuming $F \in H^{-1}$ we do not have the corresponding inequality for $\|v - v^*\|$. Now

$$\begin{aligned}
\|v^* - v_h\|^2 &= (v^* - v, v^* - v_h) + (v - v_h, v^* - v_h) \\
&= (v^* - v, v^* - v_h) + D(u - u_h, v^* - v_h) \\
&= (v^* - v, v^* - v_h) + D(u - u^*, v^* - v_h) \\
&+ D(u^* - u_h, v^* - v_h)\,.
\end{aligned}$$

The second term on the right is zero by definition of u^*. Thus

$$\begin{aligned}
\|v^* - v_h\|^2 &= (v^* - v, v^* - v_h) + D(u^* - u_h, v^* - v_h) \\
&= (v^* - v, v^* - v_h) + D(u^* - u_h, v^* - v) \\
&+ D(u^* - u_h, v - v_h)\,.
\end{aligned}$$

Again, the second term on the right is zero and so

$$\begin{aligned}
\|v^* - v_h\|^2 &= (v^* - v, v^* - v_h) + D(u^* - u_h, v - v_h) \\
&= (v^* - v, v^* - v_h) + E(u_h, u^* - u_h) \\
&- (\phi(u) - \phi(u_h), u^* - u_h) \\
&= (v^* - v, v^* - v_h) - (\phi(u) - \phi(u_h), u - u_h) \\
&- (\phi(u) - \phi(u_h), u^* - u) + E(u_h, u^* - u_h)\,,
\end{aligned}$$

and so

$$\|v^* - v_h\|^2 + (\phi(u) - \phi(u_h), u - u_h) =$$

$$(v^* - v, v^* - v_h) - (\phi(u) - \phi(u_h), u^* - u) + E(u_h, u^* - u_h)\,. \quad (3.10)$$

Now let $y \in H_0^1(-L, L) \cap H^2(-L, L)$ be the weak solution of

$$y'''' + \frac{\phi(u) - \phi(u_h)}{e_u} y = e_u, \quad y(\pm L) = y''(\pm L) = 0.$$

Then

$$\|e_u\|^2 = -D(y'', e_u) + (\phi(u) - \phi(u_h), y).$$

By elliptic regularity, $y'' \in H_0^1 \cap H^2$. Let p^* be the projection of y'' onto V_h. Then

$$\begin{aligned}
\|e_u\|^2 &= -D(p^*, e_u) + D(p^* - y'', e_u) + (\phi(u) - \phi(u_h), y) \\
&= -(p^*, e_v) + D(p^* - y'', e_u) + (\phi(u) - \phi(u_h), y) \\
&= (y'' - p^*, e_v) + D(p^* - y'', e_u) - (y'', e_v) + (\phi(u) - \phi(u_h), y) \\
&= (y'' - p^*, e_v) + D(p^* - y'', e_u) + D(y, e_v) + (\phi(u) - \phi(u_h), y) \\
&= (y'' - p^*, e_v) + D(p^* - y'', e_u) + D(y - y^*, e_v) + D(y^*, e_v) \\
&+ (\phi(u) - \phi(u_h), y^*) + (\phi(u) - \phi(u_h), y - y^*)\,,
\end{aligned}$$

where y^* is the projection of y on V_h. Thus

$$\begin{aligned}
\|e_u\|^2 &= (y'' - p^*, e_v) + D(p^* - y'', e_u) + D(y - y^*, e_v) \\
&+ (\phi(u) - \phi(u_h), y - y^*) + E(u_h, y^*) \\
&= (y'' - p, e_v) + D(p^* - y'', u) + D(y - y^*, v) \qquad (3.11) \\
&+ (\phi(u) - \phi(u_h), y - y^*) + E(u_h, y^*)\,.
\end{aligned}$$

In order to proceed we must estimate E.

$$E(u_h, z) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (I_i(\phi(u_h)z) - \phi(u_h)z)\,\mathrm{d}x, \quad z \in V_h,$$

where $I_i(w)$ is the linear interpolent of w on $[x_{i-1}, x_i]$. Recall that $z \in V_h$ so is linear on $[x_{i-1}, x_i]$. We have

$$\begin{aligned}
E(u_h, z) &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (I_i(I_i(\phi(u_h))z) - I_i(\phi(u_h))z)\,\mathrm{d}x \\
&+ \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (I_i(\phi(u_h))z - \phi(u_h)z)\,\mathrm{d}x
\end{aligned}$$

$$= \quad E_1 + E_2 \,.$$

Now

$$\int_{x_{i-1}}^{x_i} (I_i(I_i(\phi(u_h))z) - I_i(\phi(u_h))z)\,\mathrm{d}x = \frac{h^3}{12}(I_i(\phi(u_h))z)'' \\ = \frac{h}{6}(\phi(u_h(x_i)) - \phi(u_h(x_{i-1})))(z(x_i) - z(x_{i-1}))$$

Thus

$$|\int_{x_{i-1}}^{x_i} (I_i(I_i(\phi(u_h))z) - I_i(\phi(u_h))z)\,\mathrm{d}x| \\ \le \frac{Kh}{6}|u_h(x_i) - u_h(x_{i-1})|\,|z(x_i) - z(x_{i-1})| = \frac{Kh}{6}|\int_{x_{i-1}}^{x_i} u_h'(t)\,\mathrm{d}t||\int_{x_{i-1}}^{x_i} z'(t)\,\mathrm{d}t| \\ \le \frac{Kh}{12}\left(\left[\int_{x_{i-1}}^{x_i} u_h'(t)\,\mathrm{d}t\right]^2 + \left[\int_{x_{i-1}}^{x_i} z'(t)\,\mathrm{d}t\right]^2\right) \\ \le \frac{Kh^2}{12}\left[\int_{x_{i-1}}^{x_i} u_h'(t)^2\,\mathrm{d}t + \int_{x_{i-1}}^{x_i} z'(t)^2\,\mathrm{d}t\right],$$

where K is a Lipschitz constant for ϕ. So

$$|E_1| \le Ch^2(\|u_h'\|^2 + \|z'\|^2). \tag{3.12}$$

In (3.12) and below, C denotes a generic constant depending on u and v but not on h. We now estimate E_2.

$$|\int_{x_{i-1}}^{x_i} (I_i(\phi(u_h)) - \phi(u_h))z\,\mathrm{d}x|^2 \le \int_{x_{i-1}}^{x_i} |I_i(\phi(u_h)) - \phi(u_h))|^2\,\mathrm{d}x \int_{x_{i-1}}^{x_i} z^2\,\mathrm{d}x\,,$$

$$I_i(\phi(u_h)) - \phi(u_h) \quad = \quad \frac{(\phi(u_h(x_i)) - \phi(u_h(x))(x - x_{i-1})}{h} \\ + \quad \frac{(\phi(u_h(x_{i-1})) - \phi(u_h(x)))(x_i - x)}{h}\,.$$

So

$$(I_i(\phi(u_h)) - \phi(u_h))^2 \le 2K^2(u_h(x_i) - u_h(x))^2 + 2K^2(u_h(x_{i-1}) - u_h(x))^2 \\ = 2K^2\left(\int_x^{x_i} u_h'(t)\,\mathrm{d}t\right)^2 + 2K^2\left(\int_{x_{i-1}}^{x} u_h'(t)dt\right)^2$$

$$\leq 2K^2 \left((x_i - x) \int_x^{x_i} u_h'(t)^2 \,\mathrm{d}t + (x - x_{i-1}) \int_{x_{i-1}}^{x} u_h'(t)^2 \,\mathrm{d}t \right) ,$$

$$\int_{x_{i-1}}^{x_i} |I_i(\phi(u_h)) - \phi(u_h))|^2 \,\mathrm{d}x \leq Ch^2 \int_{x_{i-1}}^{x_i} u_h'(t)^2 \,\mathrm{d}t \,,$$

and so

$$\begin{aligned}
&| \int_{x_{i-1}}^{x_i} (I_i(\phi(u_h)) - \phi(u_h)) z \,\mathrm{d}x| \\
&\qquad \leq Ch \left(\int_{x_{i-1}}^{x_i} u_h'(t)^2 \,\mathrm{d}t \right)^{1/2} \left(\int_{x_{i-1}}^{x_i} z'(t)^2 \,\mathrm{d}t \right)^{1/2} \\
&\qquad\qquad \leq Ch \left(\int_{x_{i-1}}^{x_i} u_h'(t)^2 \,\mathrm{d}t + \int_{x_{i-1}}^{x_i} z'(t)^2 \,\mathrm{d}t \right) .
\end{aligned}$$

Thus

$$|E_2| \leq Ch(\|u_h'\|^2 + \|z'\|^2). \tag{3.13}$$

So from (3.12) and (3.13)

$$|E(u_h, z)| \leq C(h^2 + h)(\|u_h'\|^2 + \|z'\|^2). \tag{3.14}$$

We can transform (3.14) by noting that by (3.2a) with $w_h = u_h$

$$\|u_h'\|^2 = (v_h, u_h) \,. \tag{3.15}$$

Thus we can write (for $h < 1$)

$$|E(u_h, z)| \leq Ch(\|v_h\|^2 + \|u_h\|^2 + \|z'\|^2) \,. \tag{3.16}$$

We now use (3.10), (3.11) and (3.16) to estimate $\|e_u\|$ and $\|e_v\|$. We start by estimating the terms on the right hand side of (3.10).

$$|(v^* - v, v^* - v_h)| \leq \|v^* - v\| \|v^* - v_h\| \leq Ch\|v\|_1 \|v^* - v_h\| \,, \tag{3.17}$$

$$|(\phi(u) - \phi(u_h), u^* - u)| \leq Ch^2 \|u\|_{H^2} \|e_u\| \,. \tag{3.18}$$

From (3.16)

$$\begin{aligned}
|E(u_h, u^* - u_h)| &\leq Ch(\|v_h\|^2 + \|u_h\|^2 + \|u^* - u_h\|^2) \qquad (3.19)\\
&\leq Ch(\|v\|^2 + \|e_v\|^2 + \|u\|^2 + 2\|e_u\|^2 + \|u^* - u\|^2) \\
&\leq Ch(\|v\|^2 + \|u\|^2 + \|e_u\|^2 + \|e_v\|^2 + h^4 \|u\|_{H^2}^2) \,.
\end{aligned}$$

So from (3.10), (3.17), (3.18), (3.19)

$$\|e_v\|^2 \le 2\|v - v^*\|^2 + 2\|v^* - v_h\|^2 \le Ch(\|e_v\|^2 + \|e_u\|^2 + C_1)\,, \tag{3.20}$$

where C_1 is independent of h. Now, turning to (3.11), since

$$\|y'' - p^*\| \le Ch^2\|y\|_{H^4} \le Ch^2\|e_u\|$$

$$|(y'' - p^*, e_v)| \le Ch^2\|e_u\|\,\|e_v\|\,, \tag{3.21}$$

$$|D(y'' - p^*, u)| \le Ch\|e_u\|\,\|u\|_1\,, \tag{3.22}$$

$$|D(y - y^*, v)| \le Ch\|e_u\|\,\|v\|_1\,, \tag{3.23}$$

$$|(\phi(u) - \phi(u_h), y - y^*)| \le Ch^2\|e_u\|^2\,. \tag{3.24}$$

By (3.16)

$$|E(u_h, y^*)| \le Ch(\|v_h\|^2 + \|u_h\|^2 + \|(y^*)'\|^2),$$

$$\|(y^*)'\|^2 \le 2(\|y'\|^2 + \|y' - y^*\|^2 \le 2(\|e_u\|^2 + Ch^2\|e_u\|^2)\,,$$

so that

$$|E(u_h, y^*)| \le Ch(\|e_u\|^2 + \|e_v\|^2 + C_2)\,. \tag{3.25}$$

From (3.11), (3.21)-(3.25)

$$\|e_u\|^2 \le Ch(\|e_u\|^2 + \|e_v\|^2 + C_3)\,. \tag{3.26}$$

If we add (3.20) and (3.26) we find

$$\|e_u\|^2 + \|e_v\|^2 \le Ch(\|e_u\|^2 + \|e_v\|^2 + C_3)\,,$$

so that if h is sufficiently small

$$\|e_u\|^2 + \|e_v\|^2 \le Ch\,, \tag{3.27}$$

proving convergence of the finite element approximation.

4. Implimentation

In this section we will consider a solution method for (3.3). We will apply the results to a test problem.

If we insert (3.4) into (3.3) we obtain

$$\frac{2}{h}\alpha_j - \frac{1}{h}\alpha_{j-1} - \frac{1}{h}\alpha_{j+1} = \frac{2}{3}h\beta_j + \frac{h}{6}\beta_{j-1} + \frac{h}{6}\beta_{j+1}\,, \tag{4.1a}$$

$$\frac{2}{h}\beta_j - \frac{1}{h}\beta_{j-1} - \frac{1}{h}\beta_{j+1} + h\phi(\alpha_j) = F(\psi_j)\,. \tag{4.1b}$$

In (4.1) $j = 1, \dots, N$ with $\alpha_0 = \alpha_{N+1} = \beta_0 = \beta_{N+1} = 0$. We solve (4.1) by an iteration scheme

$$(2+a)\beta_j^{(n+1)} - \beta_{j-1}^{(n+1)} - \beta_{j+1}^{(n+1)} = hF(\psi_j) - h^2\phi(\alpha_j^{(n)}) + a\beta_j^{(n)}\,, \tag{4.2a}$$

$$\begin{aligned}(2+a)\alpha_j^{(n+1)} - \alpha_{j-1}^{(n+1)} - \alpha_{j+1}^{(n+1)} = \frac{2}{3}h^2\beta_j^{(n+1)} &+ \frac{h^2}{6}\beta_{j-1}^{(n+1)} \\ &+ \frac{h^2}{6}\beta_{j+1}^{(n+1)} + a\alpha_j^{(n)}\,, \end{aligned} \tag{4.2b}$$

where a is a relaxation parameter.

Our test problem will be

$$\frac{1}{4}u'''' + u^+ = F\delta(x) + \sigma, \qquad -L < 0 < L\,. \tag{4.3}$$

We will take $F = 100$, $\sigma = 1.5$, $L = 6$.

The exact solution to (4.3) with the boundary condition (2.4) is given by

$$u(x) = \begin{cases} \sigma + a_1 \sinh x \sin x + b_1 \cosh x \cos x & \\ +\frac{1}{2}F(\cosh x \sin|x| - \sinh|x| \cos x, & |x| \le x_0\,, \\ \frac{\sigma}{6}(|x|-x_0)^4 + a_2(|x|-x_0)^3 + b_2(|x|-x_0)^2 & \\ +c_2(|x|-x_0), & x_0 < |x| < x_1\,, \\ \sigma + a_3\sinh(|x|-x_1)\sin(|x|-x_1) & \\ -\sigma\cosh(|x|-x_1)\cos(|x|-x_1) & \\ +c_3\sinh(|x|-x_1)\cos(|x|-x_1) & \\ +d_3\cosh(|x|-x_1)\sin(|x|-x_1), & x_1 \le |x| \le L\,, \end{cases} \tag{4.4}$$

where

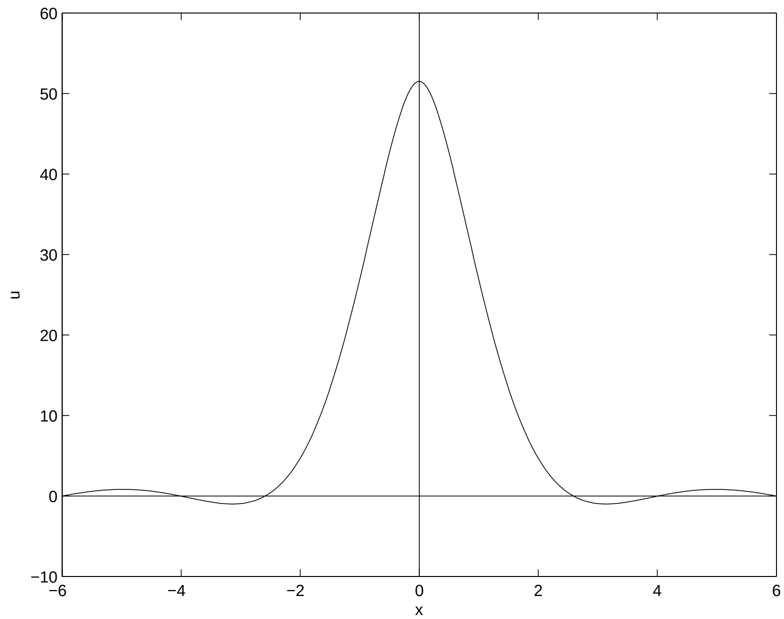

Figure 1: The solution of the test problem

$$\begin{aligned} x_0 &= 2.58903946\,, & b_2 &= 5.25599288\,, \\ x_1 &= 4.01260924\,, & c_2 &= -4.12931770\,, \\ a_1 &= -50.03287169\,, & a_3 &= -0.29003817\,, \\ b_1 &= 50.02816804\,, & c_3 &= 1.62915204\,, \\ a_2 &= -2.01040547\,, & d_3 &= -0.13145503\,. \end{aligned}$$

In this case, the beam is in contact with the foundation for $|x| \leq x_0$ and $x_1 \leq |x| \leq L$ and is detached from it for $x_0 < |x| < x_1$. The solution is shown in Figure 1.

If $\alpha^{(\mathbf{k})} = (\alpha_j^{(k)})$, $\beta^{(\mathbf{k})} = (\beta_j^{(k)})$ we ran the iterations (4.2) until

$$\|\alpha^{(\mathbf{k+1})} - \alpha^{(\mathbf{k})}\|_2^2 + \|\beta^{(\mathbf{k+1})} - \beta^{(\mathbf{k})}\|_2^2 < 10^{-12}.$$

We chose the relaxation parameter a which seemed to give the fastest convergence. The choice of a affects the rate of convergence dramatically. The iteration scheme will not converge if a is too small. We computed the quantities $\|u_h - u\|$ and $\|v_h - v\|$ using the *Matlab* integrator quadl. The results are summarized in Table 1.

$N+1$	h	$\|u-u_h\|$	$\|v-v_h\|$	a	No. of iter.
50	.24	.3665	.5288	.055	147
100	.12	.0917	.1318	.015	161
200	.06	.0234	.0329	.004	162
400	.03	..0058	.0080	.001	167
800	.015	.0014	.0021	.00024	162
1600	.0075	.000345	.000495	.00006	165
3200	.00375	.000090	.000123	.000013	160

Table 1: Results for the sample problem (4.3), (2.4)

It appears from Table 1 that v_h converges to v at a rate of h^2 which is not predicted by our analysis. This rate may be explained by the precise nature of the nonlinear term and the forcing term and by the fact that the exact solution is globally $\mathcal{C}^3$ and piecewise $\mathcal{C}^\infty$.

In future work we will explore nonlinear versions of this problem as well as two-dimensional models. We will also consider dynamic problems.

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