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SEQUENCES FROM PYRAMIDS OF INTEGERS

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Abstract: This paper presents a number of sequences based on integers arranged in a square pyramid structure. This approach provides a simple derivation of some well known sequences. In addition, a number of new integer sequences are obtained.

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1. Introduction

In a previous paper Gulliver [1], several well-known sequences (and many new sequences), were derived from two-dimensional arrays of integers. In this paper, we consider the case of a three-dimensional array which is a square (four-sided) pyramid of integers. For example, the number of elements in the pyramid is

$$s_n = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1), \qquad (1)$$

where n is the height of the pyramid. Starting from $n = 1$, we have

$$1, 5, 14, 30, 55, \ldots, \qquad (2)$$

which is sequence A000330 in the Encyclopedia of Integer Sequences maintained by Sloane [2], and appropriately called the square pyramidal numbers. Additional sequences based on this pyramid are given in the next section.

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2. Pyramids of Integers

A pyramidal array of integers has the following structure for $n = 1$ to 4

$$1\ ,\quad \begin{array}{c} 1 \\ \hline 2\ 3 \\ 4\ 5 \end{array}\ ,\quad \begin{array}{c} 1 \\ \hline 2\ 3 \\ 4\ 5 \\ \hline \begin{array}{ccc} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{array} \end{array}\ ,\quad \begin{array}{c} 1 \\ \hline 2\ 3 \\ 4\ 5 \\ \hline \begin{array}{ccc} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{array} \\ \hline \begin{array}{cccc} 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & 26 \\ 27 & 28 & 29 & 30 \end{array} \end{array}$$

In addition to (2), the following simple sequences are obtained from the corners of the pyramid

$$\begin{array}{llllll} 1, & 2, & 6, & 15, & 31, & \ldots\ , \\ 1, & 3, & 8, & 18, & 35, & \ldots\ ,\ . \\ 1, & 4, & 12, & 27, & 51, & \ldots\ . \end{array}$$

The first of these is sequence A056520, and is given by

$$s_n = \frac{1}{6}(n+1)(2n^2 - 5n + 6).$$

It can also be defined as a recurrence relation

$$a(n) = a(n-1) + n^2,$$

with $a(0) = 1$. The second is sequence A081489 given by

$$s_n = \frac{1}{6}n(2n^2 - 3n + 7),$$

while the third is sequence A047732 given by

$$s_n = \frac{2n^3 + 3n^2 - 5n + 6}{6}.$$

The rows on the faces of the pyramid give the sequences

$$\begin{array}{llllll} 1, & 5, & 21, & 66, & 165, & \ldots, \\ 1, & 6, & 27, & 84, & 205, & \ldots, \\ 1, & 8, & 33, & 96, & 225, & \ldots, \\ 1, & 9, & 39, & 114, & 265, & \ldots, \end{array} \tag{3}$$

with

$$s_n = \frac{1}{6}n(2n+1)(n^2-2n+3)\,, \tag{4}$$

$$s_n = \frac{1}{3}n(n^3-n+3)\,, \tag{5}$$

$$s_n = \frac{1}{3}n^2(n^2+2)\,, \tag{6}$$

$$s_n = \frac{1}{6}n(2n^3+3n^2-2n+3)\,, \tag{7}$$

respectively. The second sequence is A100188, as it is also obtained from a polar structured n-gonal anti-diamond sequence. The third sequence is A014820, and it also appears in Gulliver [1]. Note that (6) provides a simple proof that a square or a square + 2 is always divisible by 3. Sequences (4) and (7) do not appear in the database and so are new. All subsequent sequences in this paper are also new, unless otherwise noted.

The sums of the elements on the faces of the pyramid (the sums of the elements in (3)), give the sequences

$$\begin{array}{llllll} 1, & 6, & 27, & 93, & 258, & \ldots, \\ 1, & 7, & 34, & 118, & 323, & \ldots, \\ 1, & 9, & 42, & 138, & 363, & \ldots, \\ 1, & 10, & 49, & 163, & 428, & \ldots, \end{array} \tag{8}$$

with

$$s_n = \frac{1}{6}n(2n+1)(n^2-2n+3)\,, \tag{9}$$

$$s_n = \frac{1}{3}n(n^3-n+3)\,, \tag{10}$$

$$s_n = \frac{1}{3}n^2(n^2+2)\,, \tag{11}$$

$$s_n = \frac{1}{6}n(2n^3+3n^2-2n+3)\,, \tag{12}$$

respectively. Cutting through the pyramid diagonally results in the sequence

$$1, \quad 7, \quad 30, \quad 90, \quad 215, \quad \ldots,$$

with terms

$$s_n = \frac{1}{6}n(n+1)(2n^2 + -2n + 3).$$

Adding the elements in the four corners of each level gives the new sequence

$$4, \quad 14, \quad 40, \quad 90, \quad 172, \quad \ldots,$$

with terms

$$s_n = \frac{2}{3}(n+1)(2n^2 - 2n + 3).$$

Note that the apex is counted once for each corner. Summing the terms in the above sequence gives the sum of the elements on the edges

$$4, \quad 18, \quad 58, \quad 148, \quad 320, \quad \ldots,$$

with

$$s_n = \frac{1}{3}(n^3 + 2n^2 + 2n + 7).$$

Considering those elements at the center of the pyramid gives

$$1, \quad 10, \quad 43, \quad 116, \quad 245, \quad \ldots,$$

with

$$s_n = \frac{1}{3}n(8n^2 - 12n + 7).$$

Extending this idea, one can take the central elements on each face of the pyramid to obtain

$$\begin{array}{cccccc} 1, & 7, & 33, & 95, & 209, & \ldots, \\ 1, & 9, & 41, & 113, & 241, & \ldots, \\ 1, & 11, & 45, & 119, & 249, & \ldots, \\ 1, & 13, & 53, & 137, & 281, & \ldots, \end{array} \tag{13}$$

with

$$s_n = \frac{1}{3}(4n-1)(2n^2 - 4n + 3), \tag{14}$$

$$s_n = \frac{1}{3}(8n^3 - 12n^2 + 4n + 3), \tag{15}$$

$$s_n = \frac{1}{3}(2n-1)(4n^2-4n+3)\,, \tag{16}$$

$$s_n = \frac{1}{3}(6n^3-6n^2-2n+3), \tag{17}$$

respectively. The third sequence is A057813, obtained from the shells of atoms.

If one considers the sum of the elements in the pyramid, we obtain sequence A076767

$$1, \quad 15, \quad 105, \quad 465, \quad 1540, \quad \ldots, \tag{18}$$

with

$$s_n = \frac{1}{72}n(n+1)(n+2)(2n+1)(2n^2-n+3)\,.$$

Note that the elements of this sequence are the triangular numbers (A000217) with square pyramidal indices, i.e. the indices are given by (1). Then the sum of the elements on each level can be calculated by taking the differences of successive terms

$$1, \quad 14, \quad 90, \quad 360, \quad 1075, \quad \ldots,$$

with

$$s_n = \frac{1}{6}n^2(n+1)(2n^2-2n+3).$$

This is sequence A077538, and is just the sums of a partition of the positive integers into groups of n^2 integers without repeating or skipping any. It is also the first differences of (18).

References

[1] T.A. Gulliver, Sequences from arrays of integers, *Int. Math. Journal*, **1** (2002), 323-332.

[2] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/index.html.