International Journal of Pure and Applied Mathematics
Volume 36 No. 2 2007, 167-173

2-COASSOCIATIVE QED HOPF ALGEBRA OF PLANAR BINARY TREES

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Abstract: In this paper the QED Hopf algebra of planar binary trees is endowed with a structure of 2-coassociative Hopf algebra.

AMS Subject Classification: 16W30, 17A30, 81R99, 05C05
Key Words: bialgebra, Hopf algebra, 2-coassociative, planar binary trees

1. Introduction

It has been demonstrated in [2], that it is possible to renormalize the Feynman propagators of quantum electrodynamics using planar binary trees, where a few Hopf algebra structures, like the photon and electron propagator Hopf algebras and the charge Hopf algebra are constructed. Motivated by structures of 2-associative algebras and 2-coassociative coalgebras introduced in [3] and [5], we will upgrade before mentioned structures of planar binary trees. Although physical interpretation of those structures is not known currently, we believe that this possibility exists.

The paper is organized as follows. In the second section we recall the notions of the bialgebra structures and we construct the 2-coassociative structure on the charge Hopf algebra of planar binary trees. In the third section we state and prove the theorem about upgrading of Hopf algebra to 2-coassociative Hopf algebra, and apply this theorem to the propagators Hopf algebras of planar binary trees. Finally, in the fourth section we endow the QED Hopf algebra of planar binary trees with a structure of 2-coassociative Hopf algebra.

Received: February 13, 2006

2. A 2-Coassociative Structure on the charge Hopf Algebra of Planar Binary Trees

2.1. Basic Definitions

Let us recall the definitions of Hopf algebra, 2-associative bialgebra, 2-coassociative bialgebra and unital infinitesimal bialgebra.

Definition 1. A Hopf algebra $(\mathcal{H}, *, \Delta, \eta, \varepsilon, S)$ is a vector space $\mathcal{H}$ equipped with an associative product $* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a unit $\eta : K \to \mathcal{H}$, and a coassociative coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a counit $\varepsilon : \mathcal{H} \to K$ such that $*$ and η are morphisms of coalgebras or, equivalently, Δ and ε are morphisms of algebras. An antipode for $\mathcal{H}$ is a linear map $S : \mathcal{H} \to \mathcal{H}$ such that $S \star \mathrm{Id} = \eta \circ \varepsilon = \mathrm{Id} \star S$, where $\star$ is convolution, for two given linear maps $f, g : \mathcal{H} \to \mathcal{H}$ defined by $f \star g : * \circ (f \otimes g) \circ \Delta$.

Definition 2. A *unital infinitesimal bialgebra* $(\mathcal{H}, \cdot, \Delta)$ is a vector space $\mathcal{H}$ equipped with a unital associative product $\cdot$ and a counital coassociative coproduct Δ which are related by the *unital infinitesimal relation*:

$$\Delta(x \cdot y) = (x \otimes 1) \cdot \Delta(y) + \Delta(x) \cdot (1 \otimes y) - x \otimes y,$$

where the product $\cdot$ on $\mathcal{H} \otimes \mathcal{H}$ is given by

$$(x \otimes y) \cdot (x' \otimes y') := x \cdot x' \otimes y \cdot y'.$$

Definition 3. A *2-associative bialgebra* (resp. 2-associative Hopf algebra) $(\mathcal{H}, *, \cdot, \Delta)$ is a vector space $\mathcal{H}$ equipped with two operations $*$ and $\cdot$ and one cooperation Δ, such that:
— $(\mathcal{H}, *, \Delta)$ is a bialgebra (resp. Hopf algebra),
— $(\mathcal{H}, \cdot, \Delta)$ is a unital infinitesimal bialgebra.

Definition 4. A *2-coassociative bialgebra* (resp. 2-coassociative Hopf algebra) $(\mathcal{H}, \cdot, \Delta, \delta)$ is a vector space $\mathcal{H}$ equipped with two cooperations Δ and δ and one operation $\cdot$, such that:
— $(\mathcal{H}, \cdot, \delta)$ is a bialgebra (resp. Hopf algebra),
— $(\mathcal{H}, \cdot, \Delta)$ is a unital infinitesimal bialgebra.

2.2. A 2-Coassociative Structure on the charge Hopf Algebra of Planar Binary Trees

In this section the structure of 2-coassociative charge Hopf algebra of planar binary trees is constructed.

Proposition 5. *The quadruple ($\mathcal{H}^\alpha$, $\cdot$, Δ^α, δ) is a 2-coassociative Hopf algebra.*

Proof. It is shown in [2] that $\mathcal{H}^\alpha := \mathbb{C}[V(t), t \in \mathbb{Y}]$, the polynomial algebra generated by all trees of the form $V(t) = | \vee t$ with a coproduct $\Delta^\alpha : \mathcal{H}^\alpha \to \mathcal{H}^\alpha \otimes \mathcal{H}^\alpha$ and a coaction $\delta^\alpha : \mathcal{H}^\alpha \to \mathcal{H}^\alpha \otimes \mathcal{H}^\alpha$, defined with the following recursive relations:

$$\Delta^\alpha | = | \otimes |, \quad \Delta^\alpha V(t) = | \otimes V(t) + \delta^\alpha V(t), \quad \Delta^\alpha (t \vee s) = \Delta^\alpha t / \Delta^\alpha V(s),$$

and

$$\delta^\alpha | = | \otimes |, \quad \delta^\alpha V(t) = (V \otimes \mathrm{Id})\delta^\alpha(t), \quad \delta^\alpha(t \vee s) = \Delta^\alpha t / \delta^\alpha (V(s)),$$

form a structure of Hopf algebra, called the charge Hopf algebra of planar binary trees.

Let us remind that we can identify $\mathcal{H}^\alpha$ with $(\mathbb{C}\mathbb{Y}, /)_{ab}$, and represent the unit 1 as the root bitree $|$. Because of the fact that each tree can be uniquely decomposed as $t = (t^l/(| \vee t^r))$, the map $V(t) \mapsto V(t)$ and $1 \mapsto |$ is an algebra isomorphism from $\mathcal{H}^\alpha$ to the abelianization of $(\mathbb{C}\mathbb{Y}, /)$.

Under the inverse of this isomorphism, the natural homogenous component $\mathbb{C}\mathbb{Y}_n$ of degree n corresponds to the subspace

$$\mathcal{H}^\alpha_n = \bigoplus_{n_1 \leq \ldots \leq n_k} \mathbb{C}V(\mathbb{Y}_{n_1}) \otimes \cdots \otimes \mathbb{C}V(\mathbb{Y}_{n_k})$$

of total degree $n = n_1 + \ldots + n_k + k$ in $\mathcal{H}^\alpha$.

Furthermore, we put the second coalgebra structure on $\mathcal{H}^\alpha$ by defining deconcatenation coproduct

$$\delta(t_1 \cdots t_n) = \sum_{i=0}^{n} t_1 \cdots t_i \otimes t_{i+1} \cdots t_n.$$

Consequently, we only need to prove that ($\mathcal{H}^\alpha$, $\cdot$, δ) is a unital infinitesimal bialgebra.

Let us compute $\delta(x \cdot y)$ for $x = t_1 \ldots t_k$ and $y = t_{k+1} \ldots t_n$. We have:

$$\delta(x \cdot y) = \delta(t_1 \ldots t_n) = \sum_{i=0}^{n} t_1 \ldots t_i \otimes t_{i+1} \ldots t_n$$
$$= \sum_{i=0}^{k} t_1 \ldots t_i \otimes t_{i+1} \ldots t_n - t_1 \ldots t_k \otimes t_{k+1} \ldots t_n + \sum_{i=k}^{n} t_1 \ldots t_i \otimes t_{i+1} \ldots t_n$$

$$= \delta(t_1 \dots t_k) \cdot (1 \otimes t_{k+1} \dots t_n) - t_1 \dots t_k \otimes t_{k+1} \dots t_n$$
$$+ (t_1 \dots t_k \otimes 1) \cdot \delta(t_{k+1} \dots t_n) = (x \otimes 1) \cdot \delta(y) - x \otimes y + \delta(x) \cdot (1 \otimes y) . \quad \square$$

Remark 6. The constructed bialgebra is cofree, i.e. as a coalgebra, it is isomorphic to tensor coalgebra over the space of primitive elements of corresponding vector space.

Remark 7. Since we have never used the commutativity of the product in $\mathcal{H}^\alpha$, all proved is also true in noncommutative case. The commutative case is considered because of its importance for renormalization procedure. Further, in noncommutative case, $\tilde{\mathcal{H}}^\alpha$ would be isomorphic to $\mathbb{CY}$ as a vector space.

3. A 2-Coassociative Structure on the Free Hopf Algebra

In this section the result from previous subsection is generalized and applied on the propagators Hopf algebras.

3.1. A 2-Coassociative Structure on the Free Hopf Algebra

Theorem 8. *For any Hopf algebra $(\mathcal{H}, \cdot, \Delta)$ which corresponding algebra is free, there exists a coproduct δ such that $(\mathcal{H}, \cdot, \Delta, \delta)$ is connected 2-coassociative Hopf algebra.*

Proof. Since we already have the Hopf algebra structure, we only need to construct a structure of unital infinitesimal bialgebra on existing free associative algebra. It is known that there exists isomorphism between free associative algebra and corresponding tensor algebra (see [4], p. 34) and in [5, Proposition 2.3] is proved that every tensor module equipped with the concatenation product and the deconcatination coproduct form a structure of unital infinitesimal bialgebra. Therefore, we define the second coproduct as the deconcatenation coproduct. The constructed bialgebra is cofree (as a coalgebra, it is isomorphic to corresponding tensor coalgebra), that implies its connectedness. $\square$

3.2. A 2-Coassociative Structure on the propagators Hopf Algebras of Planar Binary Trees

In this part the objective is to endow the photon and electron propagator Hopf algebras to the structure of 2-coassociative Hopf algebras. We will concentrate on the electron propagator Hopf algebra because of their importance for renormalization procedure, but in the case of the photon propagator Hopf algebra, all proofs can be done on a similar way.

The structure of the electron propagator Hopf algebra was introduced in [2] in the following way. The free associative algebra on the set of trees, $\mathcal{H}^e := \mathbb{C}\langle\mathbb{Y}\rangle/(1-|)$, equipped with the graded coassocitive coproduct

$$\Delta_e^p(t) = \sum_{t=t_1/t_2} t_1 \otimes t_2$$

(extended by multyplicatively on tensor products of trees), gives Hopf algebra structure, which is neither commutative nor cocommutative. If we apply the theorem from previous section we have the following corollary.

Corrolary 9. *The quadruple ($\mathcal{H}^e$, $\cdot$, Δ_e^p, δ) is a 2-coassociative Hopf algebra.*

Remark 10. Observe that there is connection between coproducts of constructed 2-coassociative structures, precisely the maps δ and Δ_e^p are isomorphic on the vector space $\mathbb{C}\mathbb{Y}$. This isomorphism was explained in proof of Proposition 2.5.

4. 2-Coassociative QED Hopf Algebra of Planar Binary Trees

In this section the structures on trees seen in Sections 2 and 3 are assembled. The result is 2-coassociative structure of QED Hopf algebra on trees.

In [2] it was explained that if we have two Hopf algebras $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ with multiplications $m^{(1)}$, $m^{(2)}$ and coproducts $\Delta^{(1)}$, $\Delta^{(2)}$, and if $\mathcal{H}^{(1)}$ coacts on $\mathcal{H}^{(2)}$ from the right and the coaction satisfies some conditions, then the semidirect product $\mathcal{H}^{(1)} \ltimes \mathcal{H}^{(2)}$ is a tensor algebra, and in the same time a coalgebra.

Moreover, if $\mathcal{H}^{(1)}$ is commutative, then $\mathcal{H}^{(1)} \ltimes \mathcal{H}^{(2)}$ is a Hopf algebra.

So, it was concluded that the semidirect product $\mathcal{H}^{\mathrm{qed}} := \mathcal{H}^{\alpha} \ltimes \mathcal{H}^{\mathrm{e}}$ is graduated connected Hopf algebra, which is neither commutative nor co-commutative. The grading is given by the sum of the orders of all bitrees appearing in a monomial. The coproduct $\Delta^{qed} : \mathcal{H}^{\mathrm{qed}} \to \mathcal{H}^{\mathrm{qed}} \otimes \mathcal{H}^{\mathrm{qed}}$ was explicitly given by:

$$\Delta^{qed}(t \otimes s_1 \dots s_n) = \Delta^{\alpha}(t)[(\delta^e \otimes \mathrm{Id})\Delta_e^p(s_1 \dots s_n)].$$

If we define the second coproduct (generalized deconcatenation coproduct) as

$$\delta(t_1 \dots t_n \otimes s_1 \dots s_m) = \sum_{i=0}^{n} \sum_{j=0}^{m} t_1 \dots t_i \otimes s_1 \dots s_j \otimes t_{i+1} \dots t_n \otimes s_{j+1} \dots s_m,$$

we obtain a 2-coassociative Hopf algebra structure.

Theorem 11. *The quadruple* ($\mathcal{H}^{\mathrm{qed}}$, $\cdot$, Δ^{qed}, δ) *is a 2-coassociative Hopf algebra.*

Proof. We have already explained the construction of Hopf algebra structure which is semidirect product of Hopf algebras. We recall that product in this algebra is given by

$$m_{\mathcal{H}^{(1)} \ltimes \mathcal{H}^{(2)}} = (m_{\mathcal{H}^{(1)}} \otimes m_{\mathcal{H}^{(2)}}) \circ (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}),$$

where τ is the flip operator.

Furthermore, we only need to prove that ($\mathcal{H}^{\mathrm{qed}}$, $\cdot$, δ) is a unital infinitesimal bialgebra.

Let us compute $\delta(x \cdot y)$ for $x = t_1 \dots t_k \otimes s_1 \dots s_l$ and $y = t_{k+1} \dots t_n \otimes s_{l+1} \dots s_m$. We have:

$$\begin{aligned}
\delta(x \cdot y) &= \delta(t_1 \dots t_n \otimes s_1 \dots s_m) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{m} t_1 \dots t_i \otimes s_1 \dots s_j \otimes t_{i+1} \dots t_n \otimes s_{j+1} \dots s_m \\
&= \sum_{i=0}^{k} \sum_{j=0}^{l} t_1 \dots t_i \otimes s_1 \dots s_j \otimes t_{i+1} \dots t_n \otimes s_{j+1} \dots s_m \\
&\quad - t_1 \dots t_k \otimes s_1 \dots s_l \otimes t_{k+1} \dots t_n \otimes s_{l+1} \dots s_m \\
&\quad + \sum_{i=k}^{n} \sum_{j=l}^{m} t_1 \dots t_i \otimes s_1 \dots s_j \otimes t_{i+1} \dots t_n \otimes s_{j+1} \dots s_m \\
&= \delta(t_1 \dots t_k \otimes s_1 \dots s_l) \cdot (1 \otimes 1 \otimes t_{k+1} \dots t_n \otimes s_{l+1} \dots s_m) \\
&\quad - t_1 \dots t_k \otimes s_1 \dots s_l \otimes t_{k+1} \dots t_n \otimes s_{l+1} \dots s_m \\
&\quad + (t_1 \dots t_k \otimes s_1 \dots s_l \otimes 1 \otimes 1) \cdot \delta(t_{k+1} \dots t_n \otimes s_{l+1} \dots s_m) \\
&= \delta(x) \cdot (1 \otimes y) - x \otimes y + (x \otimes 1) \cdot \delta(y) .
\end{aligned}$$

□

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