ON THE HYPERFACTORIAL FUNCTION, 
HYPERTRIANGULAR FUNCTION, AND 
THE DISCRIMINANTS OF CERTAIN POLYNOMIALS

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Abstract: For any natural number \( n \), let \( H_f(n) = 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdots n^n \) be the hyperfactorial function of \( n \), and let \( H_t(n) = 1^1 + 2^2 + 3^3 + 4^4 + \cdots + n^n \), be the hypertriangular function of \( n \). Also, let \( r_1, r_2, r_3, \ldots, r_k \), represent the roots of the polynomial equation \( p_k(z) = z^k - 1 = 0 \) (for a fixed \( k \geq 2 \)). First, we show that \( k^k \) (for any integer \( k \geq 2 \)) can be written in terms of the discriminant of \( p_k(z) \). Then, we use this result to show that \( H_f(n) \) and \( H_t(n) \) can be written as a product of discriminants of certain polynomials, and as a sum of discriminants of these polynomials, respectively. Also, we use a known result to present an upper and a lower bound for \( H_t(n) \). Finally, we pose two questions for the reader.

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1. Preliminaries

Motivated by J. Kinkelin’s paper [8], E.W. Barnes [2] reintroduced and stud-
ied the Barnes \( G \)-function, which is also, known as the \( G \)-function. Barnes \( G \)-function which is a generalization of the Euler gamma function is defined as
\[
G(z + 1) = \Gamma(z)G(z),
\]
where \( z \) is a complex number, \( G(1) = 1 \), and \( \Gamma \) is the well known gamma Euler function. In this article we make use of a special case of the Barnes \( G \)-function; namely, when \( z \) is a positive integer. In this particular case, the Barnes \( G \)-function simply will be a product of factorials. That is,
\[
G(n + 1) = \prod_{m=1}^{n-1} m!.
\]

For a natural number \( n \), the \( K \)-function, denoted by \( K(n) \), can be written in terms of Barnes \( G \)-function, and the Euler gamma function as
\[
K(n) = 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot ... \cdot (n - 1)^{n-1} = \frac{(\Gamma(n))^{n-1}}{G(n)}.
\]

Also, we recall that for any natural number \( n \),
\[
H_f(n) = 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot ... \cdot n^n
\]
is called the hyperfactorial function of \( n \), see [11]. From the definitions of hyperfactorial function and the \( K \)-function, we can immediately see that \( K(n + 1) = H_f(n) \). Moreover, the discriminant of a polynomial
\[
p_n(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0
\]
of degree \( n \) \((n \geq 2)\) with the leading coefficient \( a_n \) and the roots \( r_1, r_2, r_3, ..., r_n \) is defined as
\[
D(p_n) = a_n^{2(n-1)} \prod_{1 \leq i < j \leq n} (r_i - r_j)^2.
\]
It is easy to see that \( D(p_n) = 0 \) if and only if \( p_n(z) \) has a repeated root, and we note that there are \((n - 1)!\) terms of the form \((r_i - r_j)^2\) in the above product. Finally, we define the hypertriangular function of \( n \) as
\[
H_t(n) = 1^1 + 2^2 + 3^3 + 4^4 + ... + n^n.
\]

Our goal in this article is to make use of the discriminants of polynomial equations \( p_k(z) = z^k - 1 = 0 \) \((k \geq 2)\) as well as the above definitions and a known result to show that:

\[
(i) \quad |D(p_k(z))| = \left| \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 \right| = k^k \quad (k \geq 2),
\]
\[ H_f(n) = \prod_{k=2}^{n} \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 \quad (n \geq k \geq 2), \]

\[ n^{\frac{4n-3}{4n-4}} \leq H_t(n) < n^n \left[ \frac{2}{e(n-1)} \right], \quad n \geq 2. \]

2. Results

In a problem proposed by G.W. Wishard in 1945 in the American Mathematical Monthly and solved in 1946 by F. Underwood [14], an upper and a lower bound were given for the sum \(1^1 + 2^2 + 3^3 + 4^4 + \ldots + n^n\). Using limit process, Underwood has shown that for \(n > 2\)

\[ n^n \left[ 1 + \frac{1}{4(n-1)} \right] < 1^1 + 2^2 + 3^3 + 4^4 + \ldots + n^n < n^n \left[ 1 + \frac{2}{e(n-1)} \right]. \]

It is clear that \(n\) cannot be 1. However, for \(n = 2\), the sum will be smaller than its upper limit, but equal to its lower limit. With this observation we restate this problem here as our Proposition 2.1 for \(n \geq 1\).

**Proposition 2.1.** If \(n\) is any natural number greater than 1 and \(e\) is the base of the natural (Napierian) logarithm, then

\[ n^n \left[ \frac{4n-3}{4n-4} \right] \leq 1^1 + 2^2 + 3^3 + 4^4 + \ldots + n^n < n^n \left[ \frac{2}{e(n-1)} \right]. \]

**Proposition 2.2.** For any natural number \(n\)

\[ 1^1.2^2.3^3.4^4\ldots n^n = \frac{(n!)^n}{\prod_{m=1}^{n-1} m!}. \]

The proof of this proposition is a direct consequence of the definitions of the \(G\)-function, \(K\)-function, and the gamma function.

**Proposition 2.3.** If \(r_1, r_2, r_3, \ldots, r_k\), represent the roots of the equation \(p_k(z) = z^k - 1 = 0\) (for a fixed \(k \geq 2\)), then

\[ |D(p_k(z))| = \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 = k^k. \]

**Remark 2.4.** First we note that the above product is always greater than zero. For \(r_1, r_2, r_3, \ldots, r_k\), represent the \(k\)-th roots of unity which are distinct and hence, \(r_i - r_j \neq 0\), for all \(i \neq j\).
Proof of Proposition 2.3. Since \( r_1, r_2, r_3, \ldots, r_k \), represent the roots of the equation \( p_k(z) = z^k - 1 = 0 \), we can write it as
\[
z^k - 1 = (z - r_1)(z - r_2)\ldots(z - r_k). \tag{1}
\]

Next, we differentiate (1) to get
\[
kz^{k-1} = \sum_{m=1}^{k} (z - r_1)\ldots(z - r_{m-1})(z - r_{m+1})\ldots(z - r_k). \tag{2}
\]

Then, we evaluate (2) for \( z = r_1, r_2, r_3, \ldots, r_k \), to obtain the following \( k \) equations:
\[
k r_1^{k-1} = (r_1 - r_2)(r_1 - r_3)\ldots(r_1 - r_{k-1})(r_1 - r_k),
\]
\[
k r_2^{k-1} = (r_2 - r_1)(r_2 - r_3)\ldots(r_2 - r_{k-1})(r_2 - r_k),
\]
\[
\vdots
\]
\[
k r_k^{k-1} = (r_k - r_1)(r_k - r_2)\ldots(r_k - r_{k-1}).
\]

Now, if we multiply the corresponding sides of the above \( k \) equations we obtain
\[
k^k (r_1, r_2, r_3, \ldots, r_k)^{k-1} = \prod_{1\leq i<j\leq k} [- (r_i - r_j)^2] = \prod_{1\leq i<j\leq k} (r_i - r_j)^2. \tag{3}
\]

Also, since \( r_1, r_2, r_3, \ldots, r_k \), represent the \( k \)-th roots of unity we have
\[
\prod_{m=1}^{k} r_m = e^{\sum_{m=1}^{\frac{2\pi m}{k}}} = (-1)^{k-1}. \tag{4}
\]

Consequently, from (3) and (4) we have
\[
k^k((-1)^{k-1})^{k-1} = (-1)^{\frac{k(k-1)}{2}} \prod_{1\leq i<j\leq k} (r_i - r_j)^2. \tag{5}
\]

Finally, we take absolute values of both sides of (5) to obtain the desired equality.
Remark 2.5. We note that if \( r_i \) and \( r_j \) are two different non-real roots of \( p_k(z) \), then \((r_i - r_j)^2 < 0\). Therefore, for \( k > 2 \), the absolute value sign around \( \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 \) in Proposition 2.4 is necessary. However, it is easy to verify that if the number of non-real roots of \( p_k(z) \) is a multiple of 4, then \( \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 > 0 \), and hence the absolute value sign around \( \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 \) is not needed.

Now, as a direct consequence of Proposition 2.2, Proposition 2.3, and the definition of \( H_f(n) \), we obtain the following theorem.

**Theorem 2.6.** Let \( p_k(z) = z^k - 1 \), for \( k = 2, 3, ..., n \). If \( r_1, r_2, r_3, ..., r_k \), represent the roots of the equation \( p_k(z) = z^k - 1 = 0 \), then

\[
H_f(n) = \frac{n}{\prod_{k=2}^{n} \prod_{1 \leq i < j \leq k} (r_i - r_j)^2}.
\]

**Theorem 2.7.** If \( r_1, r_2, r_3, ..., r_k \), represent the roots of the equation \( p_k(z) = z^k - 1 = 0 \), for \( k = 2, 3, 4, ..., n \), then

\[
n^4 \left[ 4n - 3 \over 4n - 4 \right] \leq H_t(n) = 1 + \sum_{k=2}^{n} \prod_{1 \leq i < j \leq k} (r_i - r_j)^2 < n^4 \left[ 2 + e(n - 1) \over e(n - 1) \right].
\]

The proof immediately follows from the definition of \( H_t(n) \), Proposition 2.1, and Proposition 2.3.

3. Questions

**Question 3.1.** What can be said about Proposition 2.3, and hence Theorem 2.6 and Theorem 2.7 if \( p_k(z) = z^k - 1 \) is replaced by any arbitrary polynomial of order \( k \) with distinct roots? (One could start with the determinant of the \( (2n - 1) \times (2n - 1) \) Sylvester matrix associated with the coefficients of \( p_k(z) \) and \( p_k'(z) \); that is, the resultant of \( p_k(z) \) and \( p_k'(z) \).)

**Question 3.2.** What can be said about Question 3.1 as well as the results that we have obtained in this paper, provided the polynomial \( p_k(z) \) is defined over an arbitrary field, rather than the complex field? (Note that the product formula involving the roots of \( p_k(z) \) still will be valid, provided the roots are taken from some splitting field of the polynomial \( p_k(z) \).)
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References


