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ON THE HYPERFACTORIAL FUNCTION, HYPERTRIANGULAR FUNCTION, AND THE DISCRIMINANTS OF CERTAIN POLYNOMIALS

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Abstract: For any natural number n, let $H_f(n) = 1^1.2^2.3^3.4^4...n^n$ be the hyperfactorial function of n, and let $H_t(n) = 1^1 + 2^2 + 3^3 + 4^4 + ... + n^n$, be the hypertriangular function of n. Also, let $r_1, r_2, r_3, ..., r_k$, represent the roots of the polynomial equation $p_k(z) = z^k - 1 = 0$ (for a fixed $k \geq 2$). First, we show that k^k (for any integer $k \geq 2$) can be written in terms of the discriminant of $p_k(z)$. Then, we use this result to show that $H_f(n)$ and $H_t(n)$ can be written as a product of discriminants of certain polynomials, and as a sum of discriminants of these polynomials, respectively. Also, we use a known result to present an upper and a lower bound for $H_t(n)$. Finally, we pose two questions for the reader.

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1. Preliminaries

Motivated by J. Kinkelin's paper [8], E.W. Barnes [2] reintroduced and stud-

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ied the Barnes $G-$function, which is also, known as the $G-$function. Barnes $G-$function which is a generalization of the Euler gamma function is defined as $G(z+1) = \Gamma(z)G(z)$, where z is a complex number, $G(1) = 1$, and Γ is the well known gamma Euler function. In this article we make use of a special case of the Barnes $G-$function; namely, when z is a positive integer. In this particular case, the Barnes $G-$function simply will be a product of factorials. That is,

$$G(n+1) = \prod_{m=1}^{n-1} m!.$$

For a natural number n, the $K-$function, denoted by $K(n)$, can be written in terms of Barnes $G-$function, and the Euler gamma function as

$$K(n) = 1^1.2^2.3^3.4^4...(n-1)^{n-1} = \frac{(\Gamma(n))^{n-1}}{G(n)}.$$

Also, we recall that for any natural number n,

$$H_f(n) = 1^1.2^2.3^3.4^4...n^n$$

is called the *hyperfactorial* function of n, see [11]. From the definitions of hyperfactorial function and the $K-$function, we can immediately see that $K(n+1) = H_f(n)$. Moreover, the *discriminant* of a polynomial

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

of degree n ($n \geq 2$) with the leading coefficient a_n and the roots $r_1, r_2, r_3, ..., r_n$ is defined as

$$D(p_n) = a_n^{2(n-1)} \prod_{1 \leq i < j \leq n} (r_i - r_j)^2.$$

It is easy to see that $D(p_n) = 0$ if and only if $p_n(z)$ has a repeated root, and we note that there are $(n-1)!$ terms of the form $(r_i - r_j)^2$ in the above product. Finally, we define the *hypertriangular* function of n as

$$H_t(n) = 1^1 + 2^2 + 3^3 + 4^4 + ... + n^n.$$

Our goal in this article is to make use of the discriminants of polynomial equations $p_k(z) = z^k - 1 = 0$ ($k \geq 2$) as well as the above definitions and a known result to show that:

$$(i) \qquad |D(p_k(z))| = |\prod_{1 \leq i < j \leq k} (r_i - r_j)^2| = k^k \qquad (k \geq 2)\,,$$

$$(ii) \qquad H_f(n) = \prod_{k=2}^{n} |\prod_{1\le i<j\le k} (r_i - r_j)^2| \quad (n \ge k \ge 2)\,,$$

$$(iii) \qquad n^n[\frac{4n-3}{4n-4}] \le H_t(n) < n^n[\frac{2+e(n-1)}{e(n-1)}]\,, \quad n \ge 2.$$

2. Results

In a problem proposed by G.W. Wishard in 1945 in the American Mathematical Monthly and solved in 1946 by F. Underwood [14], an upper and a lower bound were given for the sum $1^1+2^2+3^3+4^4+...+n^n$. Using limit process, Underwood has shown that for $n > 2$

$$n^n[1+\frac{1}{4(n-1)}] < 1^1+2^2+3^3+4^4+...+n^n < n^n[1+\frac{2}{e(n-1)}].$$

It is clear that n cannot be 1. However, for $n = 2$, the sum will be smaller than its upper limit, but equal to its lower limit. With this observation we restate this problem here as our Proposition 2.1 for $n \ge 1$.

Proposition 2.1. *If n is any natural number greater than 1 and e is the base of the natural (Napierian) logarithm, then*

$$n^n[\frac{4n-3}{4n-4)}] \le 1^1+2^2+3^3+4^4+...+n^n < n^n[\frac{2+e(n-1)}{e(n-1)}].$$

Proposition 2.2. *For any natural number n*

$$1^1.2^2.3^3.4^4...n^n = \frac{(n!)^n}{\prod_{m=1}^{n-1} m!}.$$

The proof of this proposition is a direct consequence of the definitions of the $G-$function, $K-$function, and the gamma function.

Proposition 2.3. *If $r_1, r_2, r_3, ..., r_k$, represent the roots of the equation $p_k(z) = z^k - 1 = 0$ (for a fixed $k \ge 2$), then*

$$|D(p_k(z))| = |\prod_{1\le i<j\le k} (r_i - r_j)^2| = k^k.$$

Remark 2.4. First we note that the above product is always greater than zero. For $r_1, r_2, r_3, ..., r_k$, represent the k-th roots of unity which are distinct and hence, $r_i - r_j \ne 0$, for all $i \ne j$.

Proof of Proposition 2.3. Since $r_1, r_2, r_3, ..., r_k$, represent the roots of the equation $p_k(z) = z^k - 1 = 0$, we can write it as

$$z^k - 1 = (z - r_1)(z - r_2)...(z - r_k). \tag{1}$$

Next, we differentiate (1) to get

$$kz^{k-1} = \sum_{m=1}^{k} (z - r_1)...(z - r_{m-1})(z - r_{m+1})...(z - r_k). \tag{2}$$

Then, we evaluate (2) for $z = r_1, r_2, r_3, ..., r_k$, to obtain the following k equations:

$$\begin{aligned} kr_1^{k-1} &= (r_1 - r_2)(r_1 - r_3)...(r_1 - r_{k-1})(r_1 - r_k)\,, \\ kr_2^{k-1} &= (r_2 - r_1)(r_2 - r_3)...(r_2 - r_{k-1})(r_2 - r_k)\,, \\ &\vdots \\ kr_k^{k-1} &= (r_k - r_1)(r_k - r_2)...(r_k - r_{k-1})\,. \end{aligned}$$

Now, if we multiply the corresponding sides of the above k equations we obtain

$$\begin{aligned} k^k(r_1.r_2.r_3...r_k)^{k-1} &= \prod_{1\le i<j\le k} [-(r_i - r_j)^2] \\ &= (-1)^{\frac{k(k-1)}{2}} \prod_{1\le i<j\le k} (r_i - r_j)^2. \end{aligned} \tag{3}$$

Also, since $r_1, r_2, r_3, ..., r_k$, represent the k-th roots of unity we have

$$\prod_{m=1}^{k} r_m = e^{\sum\limits_{m=1}^{k} \frac{2m\pi}{k} i} = (-1)^{k-1}. \tag{4}$$

Consequently, from (3) and (4) we have

$$k^k((-1)^{k-1})^{k-1} = (-1)^{\frac{k(k-1)}{2}} \prod_{1\le i<j\le k} (r_i - r_j)^2. \tag{5}$$

Finally, we take absolute values of both sides of (5) to obtain the desired equality. □

Remark 2.5. We note that if r_i and r_j are two different non-real roots of $p_k(z)$, then $(r_i - r_j)^2 < 0$. Therefore, for $k > 2$, the absolute value sign around $\prod_{1\leq i<j\leq k}(r_i - r_j)^2$ in Proposition 2.4 is necessary. However, it is easy to verify that if the number of non-real roots of $p_k(z)$ is a multiple of 4, then $\prod_{1\leq i<j\leq k}(r_i - r_j)^2 > 0$, and hence the absolute value sign around $\prod_{1\leq i<j\leq k}(r_i - r_j)^2$ is not needed.

Now, as a direct consequence of Proposition 2.2, Proposition 2.3, and the definition of $H_f(n)$, we obtain the following theorem.

Theorem 2.6. *Let $p_k(z) = z^k - 1$, for $k = 2, 3, ..., n$. If $r_1, r_2, r_3, ..., r_k$, represent the roots of the equation $p_k(z) = z^k - 1 = 0$, then*

$$H_f(n) = \prod_{k=2}^{n} |\prod_{1\leq i<j\leq k}(r_i - r_j)^2|.$$

Theorem 2.7. *If $r_1, r_2, r_3, ..., r_k$, represent the roots of the equation $p_k(z) = z^k - 1 = 0$, for $k = 2, 3, 4, ..., n$, then*

$$n^n[\frac{4n-3}{4n-4}] \leq H_t(n) = 1 + \sum_{k=2}^{n} |\prod_{1\leq i<j\leq k}(r_i - r_j)^2| < n^n[\frac{2+e(n-1)}{e(n-1)}]$$

The proof immediately follows from the definition of $H_t(n)$, Proposition 2.1, and Proposition 2.3.

3. Questions

Question 3.1. What can be said about Proposition 2.3, and hence Theorem 2.6 and Theorem 2.7 if $p_k(z) = z^k - 1$ is replaced by any arbitrary polynomial of order k with distinct roots? (One could start with the determinant of the $(2n-1) \times (2n-1)$ Sylvester matrix associated with the coefficients of $p_k(z)$ and $p_k'(z)$; that is, the resultant of $p_k(z)$ and $p_k'(z)$.)

Question 3.2. What can be said about Question 3.1 as well as the results that we have obtained in this paper, provided the polynomial $p_k(z)$ is defined over an arbitrary field, rather than the complex field? (Note that the product formula involving the roots of $p_k(z)$ still will be valid, provided the roots are taken from some splitting field of the polynomial $p_k(z)$.)

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